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Scaling limits for self-interacting random walks and diffusions, 5

lecture notes for Berlin-Zürich Summer School on Stochastic Models of Complex processes

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CLT / invariance principle for additive functionals of Markov processes – a la "Kipnis–Varadhan" (and more ...)

(Ω, π) probability space, $\mathcal{H} = L^2(\Omega, \pi)$

$t \mapsto \eta(t)$ stationary & ergodic on (Ω, π)

G : the infinitesimal generator of the semigroup $P_t f = \mathbb{E}\left(f(\eta(t)) \mid \mathcal{F}_0\right)$ on $L^2(\Omega, \pi)$

G^* : adjoint generator

$S := -\frac{1}{2}(G + G^*)$; $A = \frac{1}{2}(G - G^*)$

assumption: S & A well defined ...

$$\boxed{\begin{array}{l}\underline{Q}: \text{ let } f \in L^2(\Omega, \pi), \quad \int f \, d\pi = 0 \\ \text{? CLT for } t^{-1/2} \int_0^t f(\eta(s)) \, ds \text{ ?}\end{array}}$$

<u>The variance:</u>

$$c(s) := \mathbb{E}\big(f(\eta(t))\, f(\eta(t+s))\big) = (f, e^{sG} f)$$

$$\tilde{D}(t) := \mathrm{Var}\Big(t^{-1/2}\int_0^t f(\eta(s))\,ds\Big)$$

$$= 2\int_0^t \frac{t-s}{t}\, c(s)\,ds$$

$$\hat{D}(\lambda) := 2\int_0^\infty e^{-\lambda s} c(s)\,ds = (f, (\lambda - G)^{-1} f)$$

$$\hat{D}(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda s}\, s\, \tilde{D}(s)\,ds$$

Hence:

$$\underline{\lim}_{t\to\infty} \tilde{D}(t) \le \underline{\lim}_{\lambda\to 0} \hat{D}(\lambda) \le \overline{\lim}_{\lambda\to 0} \hat{D}(\lambda) \le \overline{\lim}_{t\to\infty} \tilde{D}(t)$$

Wanted: $\underline{\lim}_{t\to\infty} \tilde{D}(t) = \overline{\lim}_{t\to\infty} \tilde{D}(t) =: \sigma^2 \in (0,\infty)$

<u>Remark</u>: If $G = G^*$, then $c(s) > 0$ and

$\exists\ \sigma^2 \in (0, \infty]$ granted a priori

Variational formula for the variance:

$$(f,(\lambda-G)^{-1}f) =$$

$$\sup_{g\in\mathcal{H}}\left\{2(f,g)-(g,(\lambda+S)g)-(Ag,(\lambda+S)^{-1}Ag)\right\}$$

notation: $(\lambda-G)^{-1} =: R_\lambda$

Proof:

$$(f,R_\lambda f) = (R_\lambda f,(\lambda-G)R_\lambda f)$$

$$= (R_\lambda f,(\lambda+S)R_\lambda f)$$

$$= \sup_{g\in\mathcal{H}}\left\{2(g,R_\lambda f)-(g,(\lambda+S)^{-1}g)\right\}$$

$$= \sup_{g\in\mathcal{H}}\left\{2(g,f)-((\lambda-G^*)g,(\lambda+S)^{-1}(\lambda-G^*)g)\right\}$$

$$= \sup_{g\in\mathcal{H}}\left\{2(g,f)-(g,(\lambda+S)g)-(Ag,(\lambda+S)^{-1}Ag)\right\} \quad \square$$

In particular:

$$0<\left\{\begin{array}{l}\text{lower bound from}\\ \text{cleverly chosen}\\ g\in\mathcal{H}\end{array}\right\}\leq (f,(\lambda-G)^{-1}f)\leq (f,(\lambda+S)^{-1}f)$$

The $\boxed{\text{simplest}}$ case, Gordin-Lifshitz '78 :

$f \in \mathrm{Ran}(G)$; $f = -Gg$, $g \in L^2(\Omega, \pi)$

<u>Then</u>:

$$\int_0^t f(\eta(s))\, ds = -\int_0^t Gg(\eta(s))\, ds$$

$$= M(t) - g(\eta(t)) + g(\eta(0))$$

$$M(t) := g(\eta(t)) - g(\eta(0)) - \int_0^t Gg(\eta(s))\, ds$$

martingale with stationary & ergodic increments

$$\frac{M(t)}{\sigma\sqrt{t}} \Rightarrow N(0,1)$$ [actually: invariance principle holds]

by martingale CLT

$$\sigma^2 = -2(g, Gg) = -2(f, G^{-1} f)$$

$$\frac{g(\eta(t)) - g(\eta(0))}{\sqrt{t}} \xrightarrow{L^2} 0$$

done ✓

<u>Kipnis-Varadhan '86:</u> $G = G^* = -S$ [reversible process]

<u>Thm</u> CLT holds iff

$$\sigma^2 := 2 \lim_{\lambda \downarrow 0} \left(f, (\lambda - G)^{-1} f \right) < \infty$$

Equivalently: iff $f \in \mathrm{Ran}(S^{1/2})$

H_{-1} $\qquad \| S^{-1/2} f \|^2 =: \| f \|_{-1}^2 < \infty.$

Proof: martingale approximation + spectral calculus [see later]

Applications to reversible cases:

Tagged particle in symmetric simple exclusion

r.w. in symmetric r.e.

many more...

Non-reversible processes:
resolvent (rather than spectral) calculus

$$R_\lambda := (\lambda I - G)^{-1} = \int_0^\infty e^{-\lambda s} P_s \, ds$$

$$u_\lambda := R_\lambda f \; ; \quad \lambda u_\lambda - G u_\lambda = f$$

resolvent eq.

A priori bounds:

$$R_\lambda = (\lambda + S - A)^{-1} = (\lambda+S)^{-1/2} K_\lambda (\lambda+S)^{-1/2}$$

$$K_\lambda := \Big(I - \underbrace{(\lambda+S)^{-1/2} A (\lambda+S)^{-1/2}}_{=: B_\lambda} \Big)^{-1}$$

$B_\lambda^* = -B_\lambda$ [some attention needed...]

$\Rightarrow \|K_\lambda\| \leq 1$ (actually $= 1$)

Proposition (notation as before)

(1)
$$\lim_{\lambda \searrow 0} \lambda \|u_\lambda\| = 0, \qquad \lim_{\lambda \searrow 0} \|G u_\lambda + f\| = 0$$

(2) if in addition: $f \in \mathrm{Ran}(S^{1/2})$:

(H_{-1}) $\|f\|_-^2 := \|S^{-1/2} f\|^2 < \infty$

then
$$\overline{\lim_{\lambda \searrow 0}}\ \lambda^{1/2} \|u_\lambda\| \le \|S^{-1/2} f\| < \infty$$

Proof:

(1)
$$\lambda u_\lambda = \lambda^{1/2}(\lambda+S)^{-1/2} K_\lambda \lambda^{1/2}(\lambda+S)^{-1/2} f$$

$$\|\lambda^{1/2}(\lambda+S)^{-1/2}\| = 1$$

$$\|K_\lambda\| - 1$$

$$\lambda^{1/2}(\lambda+S)^{-1/2} f \longrightarrow 0.$$

(2)
$$\lambda^{1/2} u_\lambda = \lambda^{1/2}(\lambda+S)^{-1/2} K_\lambda (\lambda+S)^{-1/2} f$$

$$(\lambda+S)^{-1/2} f = (\lambda+S)^{-1/2} S^{1/2} S^{-1/2} f$$

$$\| S^{1/2}(\lambda+S)^{-1/2} \| \leq 1$$

$\square$

Theorem - Martingale approximation:

If

(A) $\lambda^{1/2} u_\lambda \longrightarrow 0$

(B) $\underbrace{S^{1/2} u_\lambda}_{\lambda>0,\text{ is Cauchy}} \longrightarrow v$ in $\mathcal{H}$, as $\lambda \to 0$

Then

$$\sigma^2 := 2 \lim_{\lambda\to 0} (u_\lambda, f) \in (0,\infty)$$

and there exists a martingale $t \mapsto M(t)$ with stationary & ergodic increments (on the same probability space) with

$$\mathbb{E}(M(t)^2) = \sigma^2 t$$

and

$$\mathbb{E}\left(\left(\int_0^t f(\eta(s))ds - M(t)\right)^2\right) = o(t)$$

Corollary: $t^{-1/2} \int_0^t f(\eta(s))\,ds \Rightarrow N(0, \sigma)$.

Remarks:

① Ⓐ + Ⓑ $\Leftrightarrow$ Ⓒ

Ⓒ: $\lim_{\lambda, \mu \to 0} (\lambda+\mu)(u_\lambda, u_\mu) = 0$

② BT '86 — applied to a RWRE

Varadhan '95 Thm + "sector condition" applied to tagged particle in zero mean asymmetric exclusion

Sethuraman, Varadhan, Yau '00 : Thm + "graded sector condition: tagged particle in asymmetric simple exclusion, $d \geq 3$

③ If $G = G^*$ then Ⓐ $\Leftrightarrow$ Ⓑ $\Leftrightarrow$ $\{\sigma^2 < \infty\}$

so this extends Kipnis-Varadhan

<u>Proof</u> (follows the main lines of K–V)

$$(u_\lambda, f) = (u_\lambda, (\lambda - G) u_\lambda) = \lambda \|u_\lambda\|^2 + \|S^{1/2} u_\lambda\|^2$$

Thus: A + B $\Longrightarrow$ $\sigma^2 = 2\|v\|^2$.

$$M_\lambda(t) := u_\lambda(\eta(t)) - u_\lambda(\eta(0)) - \int_0^t (G u_\lambda)(\eta(u))\, du$$

$$\mathbb{E}\Big(\sup_{0 \le s \le t} \big(M_\lambda(s) - M_\mu(s) \big)^2 \Big) \le 4t \, \| S^{1/2} u_\lambda - S^{1/2} u_\mu \|^2$$

By B $\exists$ $t \longmapsto M(t)$ martingale.

$$\mathbb{E}\Big(\sup_{0 \le s \le t} \big(M_\lambda(s) - M(s) \big)^2 \Big) \le 4t \, \| S^{1/2} u_\lambda - v \|$$

$$\mathbb{E}\big(M(t)^2 \big) = \lim_{\lambda \to 0} \mathbb{E}\big(M_\lambda(t)^2 \big) = 2 \lim_{\lambda \to 0} (u_\lambda, f).$$

The martingale approximation:

$$\int_0^t f(\eta(s))\,ds = M(t) + \big(M_\lambda(t) - M(t)\big)$$

$$- u_\lambda(\eta(t)) + u_\lambda(\eta(0)) + \int_0^t \lambda u_\lambda(\eta(s))\,ds$$

The error terms:

$$\mathbb{E}\Big(\sup_{0\le s\le t}\big(M_\lambda(s) - M(s)\big)^2\Big) \le 4t\,\|S^{1/2}u_\lambda - v\|^2$$

$$\mathbb{E}\big(u_\lambda(\eta(t))^2\big) = t\,(t\lambda)^{-1}\lambda\,\|u_\lambda\|^2$$

$$\mathbb{E}\big(u_\lambda(\eta(0))^2\big) = t\,(t\lambda)^{-1}\lambda\,\|u_\lambda\|^2$$

$$\mathbb{E}\Big(\sup_{0\le s\le t}\Big(\int_0^s \lambda u_\lambda(\eta(r))\,dr\Big)^2\Big) \le t\,\mathbb{E}\Big(\int_0^t \lambda^2 u_\lambda(\eta(r))^2\,dr\Big)$$

$$= t\,(t\lambda)\cdot\lambda\,\|u_\lambda\|^2$$

choose $\lambda = t^{-1}$, then all error terms are $o(t^{1/2})$ in L^2 $\square$

How to check conditions A + B $\Leftrightarrow$ C ?

BT '86 : "persistent RWRE": computational

Varadhan '95 :

The Sector Condition :

$$\forall g, h \in \mathcal{H} :$$

$$|(h, Ag)|^2 \le C\,(h, Sh)(g, Sg)$$

$$\{S.C.\} + \{\|S^{-1/2} f\|^2 < \infty\} \Rightarrow \text{A} + \text{B}$$

H$_{-1}$

Sethuraman, Varadhan, Yau '00 :

The Graded Sector Condition

....

$$\{G.S.C\} + \{H_{-1}\} \Rightarrow \text{A} + \text{B}$$

A reasonably weaker sufficient condition:

$$\lambda^{1/2} u_\lambda = \lambda^{1/2}(\lambda+S)^{-1/2} K_\lambda (\lambda+S)^{-1/2} f$$
$$S^{1/2} u_\lambda = S^{1/2}(\lambda+S)^{-1/2} K_\lambda (\lambda+S)^{-1/2} f$$

<u>assume</u> $f \in \mathrm{Ran}(S^{1/2})$; $f = S^{1/2} g$ H_1

equiv: $\| S^{-1/2} f \|^2 < \infty$

Then:

$$\lambda^{1/2} u_\lambda = \lambda^{1/2}(\lambda+S)^{-1/2} K_\lambda S^{1/2}(\lambda+S)^{-1/2} g$$
$$S^{1/2} u_\lambda = S^{1/2}(\lambda+S)^{-1/2} K_\lambda S^{1/2}(\lambda+S)^{-1/2} g$$

$$S^{1/2}(\lambda+S)^{-1/2} \xrightarrow[\text{top}]{\text{st. op.}} \Pi \ , \ \Pi h = h - \int h\, d\pi$$

$$\lambda^{1/2}(\lambda+S)^{-1/2} \xrightarrow[\text{top}]{\text{st. op}} 0 .$$

A sufficient condition:

$$K_\lambda \xrightarrow[\text{top}]{\text{st op}} K \in \mathcal{B}(\mathcal{H}), \text{ as } \lambda \to 0.$$

<u>Proposition</u> $\mathcal{H}$ a Hilbert space. Let B_n, $n \leq \infty$, densely defined closed operators over the Hilbert space $\mathcal{H}$. Assume that

① for some $\mu \in \mathbb{C}$, $\mu \in \bigcap_{n \leq \infty} \mathrm{Res}(B_n)$

$$\sup_{n \leq \infty} \| (\mu - B_n)^{-1} \| \leq C < \infty$$

② There is a dense subspace $\mathcal{C} \subset \mathcal{H}$ which is a core for B_∞ and $\mathcal{C} \subset \mathcal{D}om(B_n)$, $n < \infty$ such that for $\forall g \in \mathcal{C}$:

$$\lim_{n \to \infty} \| B_n g - B_\infty g \| = 0.$$

Then: $$(\mu - B_n)^{-1} \xrightarrow[\text{top}]{\text{st op}} (\mu - B_\infty)^{-1}.$$

Proof: (follows proof of Thm VIII.25 from Reed-Simon)

$$\tilde{\mathcal{C}} := \{ \tilde{g} = (\mu - B_\infty) g \; : \; g \in \mathcal{C} \}$$

$\tilde{\mathcal{C}}$ is dense in $\mathcal{H}$ (since $\mathcal{C}$ is a core for B_∞ and $\mu \in \mathrm{Res}(B_\infty)$).

Hence:

$$\{ (\mu - B_n)^{-1} - (\mu - B_\infty)^{-1} \} \tilde{g} =$$

$$(\mu - B_n)^{-1} (B_n - B_\infty)(\mu - B_\infty)^{-1} \tilde{g} =$$

$$\underbrace{(\mu - B_n)^{-1}}_{\| \cdot \| \le C} \underbrace{(B_n g - B_\infty g)}_{\longrightarrow 0} \qquad \square$$

Condition... (find a good name...)

The operator
$$B := S^{-1/2} A S^{-1/2}$$
makes sense, as densely defined, closed, skew self adjoint operator.

On a common core $\mathcal{C}$ of the skew self-adjoint operators
$$B_\lambda := (\lambda + S)^{-1/2} A (\lambda + S)^{-1/2},$$
for $g \in \mathcal{C}$:
$$\exists \lim_{\lambda \to 0} B_\lambda g =: B_0 g$$

This implies $M_\lambda \xrightarrow[\text{top}]{\text{strong}} M.$

Thm Condition... + H_1
$$\Longrightarrow A + B$$

$\square$

Remarks

① Varadhan's Sector Condition is equivalent to

$$S^{-1/2} A S^{-1/2} \in \mathcal{B}(\mathcal{H})$$

② Varadhan et al.'s Graded Sector Condition is equivalent to

$$S^{-1/2} A S^{-1/2} \in \mathcal{B}(\mathcal{H}_n, \mathcal{H}_{n-1} \oplus \mathcal{H}_{n+1})$$

...