Bálint Tóth:

Scaling limits for self-interacting random walks and diffusions, 4

lecture notes for Berlin-Zürich Summer School on Stochastic Models of Complex processes

31 Aug-4 Sept 2009

The environment as seen from the moving particle, stationary picture

① Myopic Self-Avoiding walk on $\mathbb{Z}^d$ — continuous time

② The Brownian Polymer model

① <u>MSAW on $\mathbb{Z}^d$</u>:

$X(t)$ nearest neighbour jump walk on $\mathbb{Z}^d$

$$\ell(t,x) = \ell(0,x) + \left| \{ 0 < s < t : X(s) = x \} \right|$$

$$P(X(t+dt) = y \mid \text{past}, X(t) = x) =$$

$$r(\ell(t,x) - \ell(t,y))\,dt + o(dt)$$

$r : \mathbb{R} \to (0, \infty)$

$$s(u) = r(u) + r(-u)$$

$$a(u) = r(u) - r(-u) \; \boxed{\text{non-decreasing}}$$

Natural approach: follow the environment (= local time field) from the position of the random walker.

$$\omega_k(x) = \ell(x) - \ell(x+e_k)$$

unit vector in direction k.

$$\Omega := \Big\{ \underline{\omega} = \big(\omega(x)\big)_{x\in\mathbb{Z}^d} :$$

$$\omega(x) = \big(\omega_1(x), \dots, \omega_d(x)\big) \in \mathbb{R}^d,$$

$$\underbrace{\omega_k(x) + \omega_\ell(x+e_k) = \omega_\ell(x) + \omega_k(x+e_\ell)}_{\text{gradient condition}} \Big\}$$

$\eta(t) \in \Omega$: the environment seen from the position of the walker: <u>Markov pr</u>

infinitesimal generator: later

A relevant probability measure on Ω:

$V : \mathbb{R} \to [0,\infty) \qquad V(u) := \int_0^u (r(v) - r(-v))\,dv$

$V(u) = V(-u)$, increases at least linearly at $\pm\infty$.

"$\pi(d\underline{\omega}) = \exp\{-\sum_{x,k} V(\omega_k(x))\}\, d\underline{\omega}$"

$\{\omega_k(x) : x \in \mathbb{Z}^d,\ k = 1,\dots,d\}$ iid $\dfrac{e^{-V(u)}\,du}{Z}$

conditioned to be gradient

Properly defined as Gibbs measure

<u>Particular case</u>: $r(v) - r(-v) = v$

$V(u) = \dfrac{u^2}{2}$

$\pi(d\omega) =$ gradient of massless free Gaussian field

$$E(\omega_k(x)) = 0 \; ; \; \mathbb{E}(\omega_k(x)\,\omega_\ell(y)) = C_{k\ell}(y-x)$$

$$C_{k\ell}(z) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{(1-e^{i\theta\cdot e_k})(1-e^{-i\theta e_\ell})}{D(\theta)} \, d\theta$$

$$D(\theta) = \frac{1}{d} \cdot \sum_{j=1}^{d} (1-\cos\theta_j).$$

Remark: we don't consider now the tilted fields.

Shifts:

$$\tau_k : \Omega \to \Omega \; ; \; (\tau_k(\underline{\omega}))(x) = \omega(x+e_k)$$

$$(\Omega, \mathbb{P}, \tau_k)_{k=1,\dots,d} \text{ ergodic.}$$

We put ourselves in $L^2(\Omega, \pi)$:

The infinitesimal generator and its adjoint:

$$(Gf)(\underline{\omega}) = \sum_{k=1}^{d} \left(\frac{\partial f}{\partial \omega_k(0)}(\underline{\omega}) - \frac{\partial f}{\partial \omega_k(-e_k)}(\underline{\omega}) \right)$$

$$+ \sum_{k=1}^{d} \Big\{ r(\omega_k(0)) \left(f(\tau_k \underline{\omega}) - f(\underline{\omega}) \right) +$$

$$r(-\omega_k(-e_k)) \left(f(\tau_k^{-1} \omega) - f(\underline{\omega}) \right) .$$

$$(G^* f)(\underline{\omega}) = \sum_{k=1}^{d} \left(-\frac{\partial f}{\partial \omega_k(0)}(\underline{\omega}) + \frac{\partial f}{\partial \omega_k(-e_k)}(\underline{\omega}) \right)$$

$$+ \sum_{k=1}^{d} \Big\{ r(-\omega_k(0)) \left(f(\tau_k \underline{\omega}) - f(\underline{\omega}) \right) +$$

$$r(\omega_k(-e_k)) \left(f(\tau_k^{-1} \omega) - f(\underline{\omega}) \right) \Big\}$$

[Computation of the adjoint follows from the integration by parts formula:

$$\left(\frac{\partial}{\partial \omega_t(x)}\right)^* = -\frac{\partial}{\partial \omega_t(x)} + \left(r(\omega_t(x)) - r(-\omega_t(x))\right).$$
]

First consequences:

① π is stationary & ergodic measure for the Markov process $\eta(t)$

② Yaglom (skew) reversibility:

$$(\mathcal{J}f)(\underline{\omega}) := f(-\omega)$$

$$\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$$

$$G^* = \mathcal{J} G \mathcal{J}$$

Probabilistic meaning: $\eta^*(t) := -\eta(-t)$

then $\eta^*(t) \stackrel{\text{law}}{=} \eta(t)$.

<u>The displacement</u>:

$$X(t)-X(s) = M(t)-M(s) + \int_s^t \varphi(\eta(u))\,du$$

where $\varphi : \Omega \longrightarrow \mathbb{R}^d$

$$\begin{aligned}
\varphi_k(\underline{\omega}) &:= r(\omega_k(0)) - r(-\omega_k(-e_k)) \\
&= \big(s(\omega_k(0)) - s(\omega_k(-e_k))\big) + \\
&\quad \big(a(\omega_k(0)) + a(\omega_k(-e_k))\big) \\
&= \overline{\varphi}_k(\underline{\omega}) + \tilde{\varphi}_k(\underline{\omega})
\end{aligned}$$

$$X(t)-X(s) = M(t)-M(s)$$

Martingale with stationary & ergodic increments

$$+ \int_s^t \overline{\varphi}(\eta(u))\,du$$

$$+ \int_s^t \hat{\varphi}(\eta(u))\,du$$

to be treated by martingale approximation a la Kipnis Varadhan

② The Brownian Polymer model

History : Norris - Rogers - Williams '87

Durrett - Rogers '92

Cranston - Le Jan '95

Cranston - Mountford '96

Mountford - Tarrès '08. ...

$$t \longmapsto X(t) \in \mathbb{R}^d$$

$$X(t) = \sigma B(t) + \int_0^t \int_0^s F(X(s) - X(u))\, du\, ds$$

$$F(x) = -\operatorname{grad} V(x)$$

$$V(x) = \exp\{-|x|^2/2\}$$ or "something else of this kind"

Remark : qualitatively other choices of $F(\cdot)$ are also possible — but this one is the most interesting ...

$$dX(t) = \sigma\, dB(t) + \left\{ \int_0^t F(X(t) - X(u))\, du \right\} dt$$

Let $L(t, dx)$ be the occupation time measure of $X(\cdot)$

$$L(t, A) = \int_0^t \mathbb{1}(X(u) \in A)\, du$$

then

$$dX(t) = \sigma\, dB(t) + \int_{\mathbb{R}^d} F(z)\, L(t, X(t) + dz)\, dt$$

$$dX(t) = \sigma\, dB(t) - \operatorname{grad}\left(V * L(t, \cdot)\right)(X(t))\, dt$$

This is the same mechanism in continuous space ~~time~~ as MSAW on $\mathbb{Z}^d$.

In 1d compare also with TSRM

The environment as seen from the position of the moving particle:

<u>1st step</u> separate the evolution of the environment

$$\xi(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$$

$$\begin{cases} \xi(t,x) = \boxed{\xi(0,x)} + \int_0^t F(x - X(u))\,du \\ X(t) = X(0) + \sigma(B(t) - B(0)) + \int_0^t \xi(s, \xi(s))\,ds \end{cases}$$

Then:

$$X(t) = X(0) + \sigma\big(B(t) - B(0)\big) + \int_0^t \Big\{ \boxed{\xi(0, X(s))} + \int_0^s F(X(s) - X(u))\,du \Big\}\,ds$$

we allow initial value for the field ξ.

2nd step: Travel with the moving particle

$$\eta(t,\cdot):\mathbb{R}^d \to \mathbb{R}^d$$

$$\eta(t,x) := \xi(t, X(t)+x)$$

Then $t \mapsto \eta(t,\cdot)$ will be a $\boxed{\text{Markov process on its own}}$, in some function space $\underline{\Omega}$.

<u>Next</u>: we identify a pretty stationary (and ergodic) measure for $\eta(t,\cdot)$:

<u>Ansatz</u>: zero mean, translation-invariant Gaussian with covariances

$$\mathbb{E}(\eta_k(t,x)\,\eta_l(t,y)) = C_{kl}(y-x)$$

<u>Mind</u>: $\boxed{\text{same } t}$. $t \mapsto \eta(t)$ is $\boxed{\text{not}}$ a Gaussian process.

Moment generating functional:

$u : \mathbb{R}^d \to \mathbb{R}^d$ test function

$$\phi(t,u) := \mathbb{E}\left(\exp\{\langle u, \eta(t)\rangle\}\right)$$

Notation:

$$\langle u, v\rangle = \langle u_k, v_k\rangle = \int_{\mathbb{R}^d} u_k(x) v_k(x)\, dx$$

with summation over repeated subscripts

Goal: Find condition on $C_{kl}(z)$

for $\partial_t \phi(t,u) \equiv 0$.

Itô:

$$d\langle u_k, \eta_k(t)\rangle = -\sigma \langle \partial_\ell u_k, \eta_k(t)\rangle\, dB_\ell(t) +$$

$$\left(\frac{\sigma^2}{2}\langle \partial^2_{\ell\ell} u_k, \eta_k(t)\rangle - \langle \partial_\ell u_k, \eta_k(t)\rangle \eta_\ell(t,0) + \langle \partial_k u_k, V\rangle\right) dt$$

Hence:

$$\mathbb{E}\left(d \exp\langle u_k, \eta_k(t)\rangle \,\middle|\, \mathcal{F}_t\right) = \exp\langle u_k, \eta_k(t)\rangle \cdot$$

$$\left(\frac{\sigma^2}{2}\langle \partial^2_{\ell\ell} u_k, \eta_k(t)\rangle + \frac{\sigma^2}{2}\langle \partial_\ell u_k, \eta_k(t)\rangle \langle \partial_\ell u_m, \eta_m(t)\rangle\right.$$

$$\left. - \langle \partial_\ell u_k, \eta_k(t)\rangle \eta_\ell(t,0) + \langle \partial_k u_k, V\rangle\right) dt$$

Use now the Gaussian Ansatz.

If (X, Y, Z) are jointly Gaussian, zero mean, then:

$$\mathbb{E}(Y e^X) = \exp\{\mathbb{E}X^2/2\}\, \mathbb{E}(XY)$$

$$\mathbb{E}(YZ e^X) = \exp\{\mathbb{E}X^2/2\}\left(\mathbb{E}(YZ) + \mathbb{E}(XY)\mathbb{E}(XZ)\right)$$

Hence:

$$\exp\{-\langle u_k, C_{kl} * u_l\rangle\}\, \mathbb{E}\left(d \exp\{\langle u_k, \eta_k(t)\rangle\}\right)/dt =$$

$$\frac{\sigma^2}{2}\langle \partial_{ll}^2 u_k, C_{km} * u_m\rangle + \frac{\sigma^2}{2}\langle \partial_l u_k, C_{km} * \partial_l u_m\rangle$$

cancel by integr. by parts

$$+ \frac{\sigma^2}{2}\underbrace{\langle \partial_l u_k, C_{km} * u_m\rangle}_{=0}\langle \partial_l u_i, C_{ij} * u_j\rangle$$

$$- \overbrace{\langle \partial_l u_k, C_{km} * u_m\rangle}^{0=}\langle u_i, C_{il}\rangle$$

$$+ \langle \partial_k u_k, V\rangle - \langle \partial_l u_k, C_{k,l}\rangle$$

Condition looked after

$$\partial_l C_{kl} = \partial_k V$$

<u>or</u>

$$C_{kl} = \partial_{kl}^2 \Delta^{-1} V$$

$$\hat{C}_{kl}(p) = \frac{p_k p_l}{p^2}\hat{V}(p)$$

$$\Omega := \left\{ \omega \in C^\infty(\mathbb{R}^d \to \mathbb{R}^d) : \omega \text{ gradient}, \ \|\omega\|_{k,m,\ell} < \infty \right\}$$

$$\|\omega\|_{k,m,\ell} := \sup_{x \in \mathbb{R}^d} \left(1 + |x|\right)^{-1/\ell} \left| \partial^{|m|}_{m_1,\dots,m_d} \omega_k(x) \right|$$

$\ell \geq 1$; $k \in \{1,\dots,d\}$; $m = (m_1,\dots,m_d)$, $m_j \geq 0$.

$\pi(d\omega)$ the Gaussian measure with

$$\int_\Omega \omega_k(x)\, \pi(d\omega) = 0 \ ; \quad \int_\Omega \omega_k(x)\, \omega_\ell(y)\, \pi(d\omega) = C_{k\ell}(y - x)$$

Spatial translations

$$\tau_z : \Omega \to \Omega \ ; \quad (\tau_z \omega)_k(x) := \omega_k(z + x)$$

$$\left(\Omega, \pi, \tau_z : z \in \mathbb{R}^d\right) \quad \text{ergodic}$$

$$L^2(\Omega, \pi) =: \mathcal{H} \quad \text{Gaussian Hilbert space}$$

Hilbert spaces & operators:

Notation

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^d \to \mathbb{R}) \qquad \text{Schwartz space}$$

$$\hat{\mathcal{S}} = \{ \hat{u} \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}) : \overline{\hat{u}(p)} = \hat{u}(-p) \}$$

Fourier transform:

$$\hat{u}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} u(x)\, dx$$

$$\check{v}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ip\cdot x} v(p)\, dp$$

$$\mathcal{S}_1 := \{ u : \{1,\dots,d\} \times \mathbb{R}^d \to \mathbb{R} : u_k \in \mathcal{S} \}$$

$$\hat{\mathcal{S}}_1 := \{ \hat{u} : \{1,\dots,d\} \times \mathbb{R}^d \to \mathbb{C} : \hat{u}_k \in \hat{\mathcal{S}} \}$$

$$\langle u, v \rangle := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_k(x)\, C_{k\ell}(y-x)\, u_\ell(y)\, dx\, dy$$

$$\langle \hat{u}, \hat{v} \rangle := \int_{\mathbb{R}^d} \hat{u}_k(p)\, \hat{C}_{k\ell}(p)\, \hat{u}_\ell(-p)\, dp$$

Basic Hilbert space:

$$\mathcal{V} := \overline{\{ u \in \mathcal{S}_1 : \|u\|^2 := \langle u, u \rangle \}}$$

identified by Fourier tr.

$$\hat{\mathcal{V}} := \overline{\{ \hat{u} \in \hat{\mathcal{S}}_1 : \|\hat{u}\|^2 := \langle \hat{u}, \hat{u} \rangle \}}$$

Still notation:

$$n \geq 1, \quad \underline{k} = (k_1 \ldots k_n), \quad k_j \in \{1, \ldots, d\}$$

$$\underline{x} = (x_1, \ldots, x_n), \quad x_j \in \mathbb{R}^d$$

$$\mathcal{S}_n := \{ u : \{1, \ldots, d\}^n \times \mathbb{R}^{dn} \to \mathbb{R} : u_{\underline{k}} \in \mathcal{S}(\mathbb{R}^{dn} \to \mathbb{R}) ;$$

$$u_{\pi \underline{k}}(\pi \underline{x}) = u_{\underline{k}}(\underline{x}), \quad \pi \in Perm(n) \}$$

$$\hat{\mathcal{S}}_n := \{ \hat{u} : \{1, \ldots, d\}^n \times \mathbb{R}^{dn} \to \mathbb{C} : \hat{u}_{\underline{k}} \in \mathcal{S}(\mathbb{R}^{dn} \to \mathbb{C}), \overline{\hat{u}_{\underline{k}}(p)} = \hat{u}_{\underline{k}}(-p),$$

$$\hat{u}_{\pi \underline{k}}(\pi \underline{x}) = \hat{u}_{\underline{k}}(\underline{x}) ; \quad \pi \in Perm(n) \}$$

$$\langle u, v \rangle := \sum_{\underline{k}} \sum_{\underline{l}} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dn}} u_{\underline{k}}(\underline{x})\, C_{\underline{k}\underline{l}}(\underline{y}-\underline{x})\, v_{\underline{l}}(\underline{y})\, d\underline{x}\, d\underline{y}$$

$$\langle \hat{u}, \hat{v} \rangle := \sum_{\underline{k}} \sum_{\underline{l}} \int_{\mathbb{R}^{dn}} \hat{u}_{\underline{k}}(\underline{p})\, \hat{C}_{\underline{k}\underline{l}}(\underline{p})\, \hat{v}_{\underline{l}}(-\underline{p})\, d\underline{p}$$

$$C_{\underline{k}\underline{l}}(\underline{y}-\underline{x}) := \prod_{m=1}^{n} C_{k_m l_m}(y_m - x_m)$$

$$\hat{C}_{\underline{k}\underline{l}}(\underline{p}) := \prod_{m=1}^{n} \hat{C}_{k_m l_m}(p_m)$$

$$\mathcal{K}_n := \overline{\{ u \in \mathcal{S}_n \;:\; \|u\|^2 := \langle u, u \rangle \}}$$

identified by Fourier tr

$$\hat{\mathcal{K}}_n := \overline{\{ \hat{u} \in \hat{\mathcal{S}}_n \;:\; \|\hat{u}\|^2 := \langle \hat{u}, \hat{u} \rangle \}}$$

$$\mathcal{K}_n = \operatorname{symm}(V^{\otimes n}) ; \quad \hat{\mathcal{K}}_n = \operatorname{symm}(\hat{V}^{\otimes n})$$

$$\mathcal{K} := \overline{\bigoplus_{n=0}^{\infty} \mathcal{K}_n} ; \quad \hat{\mathcal{K}} := \overline{\bigoplus_{n=1}^{\infty} \hat{\mathcal{K}}_n}$$

Fock spaces

Back to $L^2(\Omega, \pi) =: \mathcal{H}$

$\Phi_n : \mathcal{K}_n \to \mathcal{H}$

$$\Phi_n(u) := \frac{1}{n!} \sum_{\underline{k}} \int_{\mathbb{R}^{dn}} u_{\underline{k}}(\underline{x}) :\omega_{k_1}(x_1) \cdots \omega_{k_n}(x_n): d\underline{x}$$

Wick product.

isometry: $\|\Phi_n(u)\|_{\mathcal{H}} = \|u\|_{\mathcal{K}}$

$$\mathcal{H}_n := \operatorname{Ran} \Phi_n = \{ \Phi_n(u) : u \in \mathcal{K}_n \}$$

<u>Fact</u> $\mathcal{H} = \overline{\bigoplus_{n=0}^{\infty} \mathcal{H}_n}$

$$\bigoplus_{n=0}^{\infty} \mathcal{H}_n = \bigoplus_{n=0}^{\infty} \mathcal{K}_n = \bigoplus_{n=0}^{\infty} \widehat{\mathcal{K}_n}$$

identified by Φ_n

identified by Fourier

we will move freely between these spaces.

Still notation

$$\phi : \mathcal{V} \longrightarrow \mathcal{H}_1 ; \qquad \phi(v) = \int_{\mathbb{R}^d} v_k(x)\,\omega_k(x)\,dx$$

$$v_1, v_2, \dots, v_n \in \mathcal{V} \qquad \left(v_m = \left(v_{m k_m} \right)_{k_m = 1, \dots, d} \right)$$

$$:\phi(v_1) \cdots \phi(v_n):$$

$$= \sum_{\underline{k}} \int_{\mathbb{R}^{dn}} v_{1k_1}(x_1) \cdots v_{nk_n}(x_n) :\omega_{k_1}(x_1) \cdots \omega_{k_n}(x_n): d\underline{x}$$

Operators on $\mathcal{H}$ – generalities:

$A : \mathcal{V} \longrightarrow \mathcal{V}$ linear operator over $\mathcal{V}$

[linear operator = densely defined & closed]

$d\Gamma(A) : \mathcal{H}_n \longrightarrow \mathcal{H}_n$ lin. op. over $\mathcal{H}$

$$d\Gamma(A) :\phi(v_1) \cdots \phi(v_n):$$

$$= \sum_{m=1}^{n} :\phi(v_1) \cdots \phi(A v_m) \cdots \phi(v_n):$$

$u \in \mathcal{V}$

$a^*(u) : \mathcal{H}_n \to \mathcal{H}_{n+1}$, $a(u) : \mathcal{H}_n \to \mathcal{H}_{n-1}$

creation annihilation

$$a^*(u) :\phi(v_1) \cdots \phi(v_n): \; = \; :\phi(u)\phi(v_1) \cdots \phi(v_n):$$

$$a(u) :\phi(v_1) \cdots \phi(v_n): \; =$$

$$\sum_{m=1}^{n} \langle u, v_m \rangle :\phi(v_1) \cdots \cancel{\phi(v_m)} \cdots \phi(v_n):$$

$$a^*(u) = \big(a(u)\big)^*$$

$$J f(\underline{\omega}) = f(-\omega) \; ; \quad J\big|_{\mathcal{H}_n} = (-1)^n I\big|_{\mathcal{H}_n}$$

Commutation relations

$$[a(u), a(v)] = [a^*(u), a^*(v)] = 0 \; ; \; [a(u), a^*(v)] = \langle u, v \rangle I \; ;$$

$$[d\Gamma(A), a^*(u)] = a^*(Au) \; ; \; [d\Gamma(A), a(u)] = -a(A^*u) \; ;$$

$$[J, d\Gamma(A)] = 0 \; ; \; \{J, a^*(u)\} = 0 \; ; \; \{J, a(u)\} = 0$$

multiplication op.:

$u \in \mathcal{V}$; $\big((m(u) f\big)(\omega) := \phi(u)(\omega)\, f(\omega)$

directional differentiation:

$\vartheta \in \Omega$

$$\frac{\delta}{\delta\vartheta} f(\omega) := \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\big(f(\omega+\varepsilon\vartheta) - f(\omega)\big)$$

[All operators defined on Wick monomials and extended by linearity + graph closure]

<u>Two facts:</u> $u \in \mathcal{V}$

① $m(u) = a^*(u) + a(u)$ [straightforward]

② $C * u \in \Omega$

$(C*u)_k = \sum_\ell C_{k\ell} * u_\ell$ [Malliavin...]

and $\dfrac{\delta}{\delta(C*u)} = a(u)$

[Check on Wick monomials]

Concrete operators:

$$\nabla_\ell := d\Gamma(\partial_\ell) \qquad \ell = 1, \dots, d$$

$$\Delta := \sum_{\ell=1}^{d} \nabla_\ell^2$$

$$\left.\begin{array}{l} a_\ell^* := a^*(e_\ell) \\ a_\ell := a(e_\ell) \end{array}\right\} \left\{\begin{array}{l} e_\ell \in \mathcal{V}, \ (e_\ell)_j(x) = \delta_{\ell j}\,\delta(x) \\ \qquad \ell = 1, \dots, d \end{array}\right.$$

$$\Delta^* = \Delta, \qquad \nabla_\ell^* = -\nabla_\ell$$

$$\exp\{z \cdot \nabla\} =: T_z, \quad T_z f(\omega) = f(\tau_z \omega)$$

[the unitary group of translations

$$\exp\left\{\frac{1}{2} t \sigma^2 \Delta\right\} =: Q_t$$

$$(Q_t f)(\underline{\omega}) = \int_{\mathbb{R}^d} \frac{e^{-z^2/2\sigma^2 t}}{\sqrt{2\pi\sigma^2 t}}\, f(\tau_z \omega)\, dz$$

[the semigroup of BM in random scenery

The infinitesimal generator:

$$\lim_{t \downarrow 0} t^{-1}\, \mathbb{E}\left(f(\eta(t)) - f(\eta(0)) \,\middle|\, \eta(0) = \omega \right) = \left(\frac{\sigma^2}{2} \Delta + \sum_{\ell=1}^{d} m(e_\ell) \nabla_\ell + \frac{\delta}{\delta F} \right) f(\omega)$$

Checked e.g. for

$f = :\phi(v_1) \cdots \phi(v_n):$ sufficiently

with $v_1, \ldots, v_n \in \mathcal{V}$, (smooth

Note: $F = -C * h$

$$h \in \mathcal{V}, \quad h_\ell = \partial_\ell\, \delta$$

We get – using earlier identities

$$G = \frac{\sigma^2}{2} \Delta + \sum_{\ell=1}^{d} \left(a_\ell^* \nabla_\ell + \nabla_\ell a_\ell \right)$$

$$G^* = \frac{\sigma^2}{2} \Delta - \sum_{\ell=1}^{d} \left(a^*_\ell \nabla_\ell + \nabla_\ell a_\ell \right)$$

Yaglom reversibility:

$$G^* = \mathcal{J} G \mathcal{J}$$

$$\eta^*(t) := -\eta(-t), \quad \eta^*(\cdot) \overset{law}{=} \eta(\cdot)$$

Ergodicity:

$$\mathcal{D}(f) = -(f, Gf) = -\frac{\sigma^2}{2}(f, \Delta f)$$

$$\{\mathcal{D}(f) = 0\} \Leftrightarrow \{f = \text{Const}\}.$$

<u>Notation</u> $\quad S := -\frac{1}{2}(G + G^*) = -\frac{\sigma^2}{2}\Delta$

$$A := \frac{1}{2}(G - G^*) = \sum_{l=1}^{d} (a_l^* \nabla_l + \nabla_l a_l) = A_+ + A_-$$

$$S : \mathcal{H}_n \to \mathcal{H}_n \,; \quad A_+ : \mathcal{H}_n \to \mathcal{H}_{n+1} \,; \quad A_- : \mathcal{H}_n \to \mathcal{H}_{n-1}$$

$$S\big|_{\mathcal{H}_0} = 0; \quad A_+\big|_{\mathcal{H}_0} = 0\,; \quad A_-\big|_{\mathcal{H}_0 \oplus \mathcal{H}_1} = 0.$$