Bálint Tóth:

Scaling limits for self-interacting random walks and diffusions, 3

lecture notes for Berlin-Zürich Summer School on Stochastic Models of Complex processes

31 Aug-4 Sept 2009

Construction of the 1d scaling limit:
the "True Self-Repelling Motion" (TSRM)
[based on BT, W. Werner '98]

We have seen the limit theorem for

$$\frac{X([Nt])}{N^{2/3}} \quad \text{with} \quad t \in \mathbb{R}_+ \ \underline{\text{fixed}}.$$

Next: $N^{-2/3} X([Nt]) \overset{?}{\Longrightarrow} \mathcal{X}(t)$ $\boxed{\text{as a process}}$

In particular:

$$\boxed{\text{what could } t \mapsto \mathcal{X}(t) \text{ be?}}$$

Basic properties of the scaling limit <u>process</u> $\mathcal{X}(t)$

① $\mathcal{X}(0) = 0$,

$t \mapsto \mathcal{X}(t)$ a.s. continuous

a.s. recurrent

② Scale invariance

$$\left(X(\alpha t)\right)_{t\geq 0} \overset{\text{law}}{=} \left(\alpha^{2/3} X(t)\right)_{t\geq 0}$$

③ local counterpart:

local variation of power $3/2$:

$$\theta_0^\varepsilon = 0; \quad \theta_{n+1}^\varepsilon = \inf\{t > \theta_n^\varepsilon : |X(t) - X(\theta_n^\varepsilon)| \geq \varepsilon\}$$

$$\varepsilon^{3/2} \sup\{n : \theta_n^\varepsilon < t\} \xrightarrow{P} \frac{2}{\sqrt{\pi}} \cdot t$$

④ existence and regularity of occupation time density

$$\mu_t(A) := \int_0^t \mathbb{1}(X(s) \in A)\, ds$$

$$L(t,x) := \frac{d\mu_t(dx)}{dx}$$

$$(t,x) \longmapsto L(t,x) \quad \text{a.s. continuous}$$

⑤ $t \mapsto (X(t), L(t,\cdot))$ (jointly) Markov

More on this later...

⑥ Dynamics:
phenomenologically:

$$\text{"}\begin{cases} dX(t) = -\partial_x L(t, X(t))\,dt \;(+\text{ noise?}) \\ \partial_t L(t,x) = \delta(x - X(t)) \end{cases}\text{"}$$

comment...

Properly regularized:

$$X(T) + \int_0^T \frac{L(s, X(s)+\varepsilon) - L(s, X(s)-\varepsilon)}{2\varepsilon}\,ds$$

$$- \frac{1}{4}\left\{\sup_{s<T} X(s) + \inf_{s<T} X(s)\right\}$$

$$\xrightarrow{P} 0$$

Construction:

Let $X(t)$ be a process with regular (a.s. continuous) occupation time density $L(t,x)$

$$T(x,h) := \inf\{t \geq 0 : L(t,x) \geq h\}$$ inverse local time

$$\Lambda_{xh}(y) := L(T(x,h), y)$$

Then $\{\Lambda_{xh}(y) : x \in \mathbb{R},\ h \in \mathbb{R}_+,\ y \in \mathbb{R}\}$ contains $\boxed{\text{all}}$ information about the process

$X_\cdot$ can be reconstructed from $\Lambda_{\cdot,\cdot}(\cdot)$

Moreover: $\{\Lambda_{xh}(y) : x \in \mathbb{R},\ h \in \mathbb{R}_+,\ \boxed{y \geq x}\}$ is suff

since, for $y \leq x$:

$$\Lambda_{xh}(y) = \sup\{h' : \Lambda_{yh'}(x) < h\}$$

Thus, we only need:

$$\{ \Lambda_{xh}(y) : x \in \mathbb{R},\ h \geq 0,\ y \geq x \}$$

and go backwards

Look at the joint scaling of

$$\left(\frac{\Lambda^*_{[Nx_i],[\sqrt{N}\sigma h_i]}([Ny_i])}{\sigma\sqrt{N}} ;\ i = 1, 2, \ldots, p \right)$$

on pages 32-33!

$$\{ \Lambda_{xh}(y) : x \in \mathbb{R},\ h \geq 0;\ y \geq x \}$$

must be the $\boxed{\text{Brownian Web}}$

(with reflection / absorption at $h=0$)

Historical remarks, priorities...

R. Arratia 1979 (unpublished manuscript)

BT, WW 1998 (full construction

Ch Newman et al (200 (re discover and coin term BW)

Theorem

$\exists!$ a process $\{\Lambda_{xh}(y) : x \in \mathbb{R},\ h \geq 0,\ y \geq x\}$ with the following properties:

① for $(x_1\, h_1) \dots (x_p\, h_p)$ fixed

$$\left(\Lambda_{x_1h_1}(\cdot), \dots, \Lambda_{x_ph_p}(\cdot)\right) \text{ are ICRAB}$$

(see page 32)

② a.s. $\forall (x,h)$: $\Lambda_{xh}(x) = h$

③ a.s. $\forall (x_1,h_1),(x_2,h_2)$, $z \geq y \geq \max\{x_1,x_2\}$

$$\{\Lambda_{x_1h_1}(y) < \Lambda_{x_2h_2}(y)\} \Rightarrow \{\Lambda_{x_1h_1}(z) \leq \Lambda_{x_2h_2}(z)\}$$

no crossing of lines

④ regularity

a.s. $\forall\, x \leq y$: $h \mapsto \Lambda_{xh}(y)$ is $\boxed{\text{left}}$ continuous

Remark: other regularity conditions are also possible (see Arratia, Newman et al, ...)

Construction of the process $\Lambda_{\cdot,\cdot}(\cdot)$:

let $(\tilde{x}_i, \tilde{h}_i)_{i\in\mathbb{N}}$ be dense in $\mathbb{R}\times\mathbb{R}_+$

step 1: $\{\Lambda_{\tilde{x}_i\tilde{h}_i}(\cdot) : i=1,2,3,\dots\}$ easily constructed sequentially

step 2: the law of $\{\Lambda_{\tilde{x}_i,\tilde{h}_i}(\cdot) : i\in\mathbb{N}\}$ will not depend on the order of the sequential construction

step 3: extend by left continuity in h

$$\Lambda_{xh}(y) := \sup\{\Lambda_{\tilde{x}_i\tilde{h}_i}(y) : x_i \le x,\ \Lambda_{\tilde{x}_i\tilde{h}_i}(x) < h\}$$

step 4: properties (1)(2)(3)(4) drop out

step 5: check uniqueness — relies on fine estimates on BM.

First properties of the forward lines:

[to be completed, ...]

done on the blackboard...

The backward lines:

$y \leq x: \quad \Lambda^*_{xh}(y) := \sup\{h' : \Lambda_{yh'}(x) < h\}$

reverted:

$y \geq x: \quad \Lambda_{xh}(y) = \sup\{h' : \Lambda^*_{yh'}(x) < h\}$

Duality: $\quad \Lambda^*_{-x,h}(-y) \stackrel{\text{law}}{=} \Lambda_{xh}(y)$

Topology:

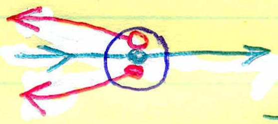
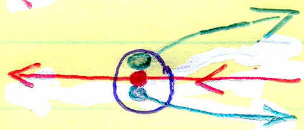
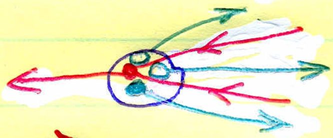
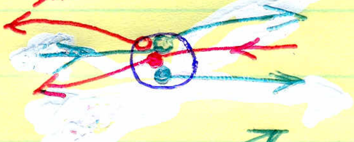
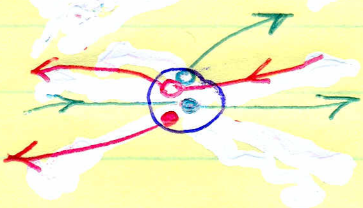
$I(x,h) := \lim_{y \nearrow x} \sup \#\{p \in \mathbb{N} : \exists (x_1 h_1), \dots, (x_p h_p) : x_i \leq x'$
$\forall y \in (y, x) : \Lambda_{x_1 h_1}(y) < \dots < \Lambda_{x_p h_p}(y),$
and $\Lambda_{x_i h_i}(x) = h\}$ – the number of lives coalescing exactly at (xh)

$O(x,h) := \lim_{y \downarrow x} \lim_{\varepsilon \downarrow 0} \#\{\Lambda_{x'h'}(y) : \operatorname{dist}((x'h'),(xh)) < \varepsilon\}$

– number of lives emerging from (x,h).

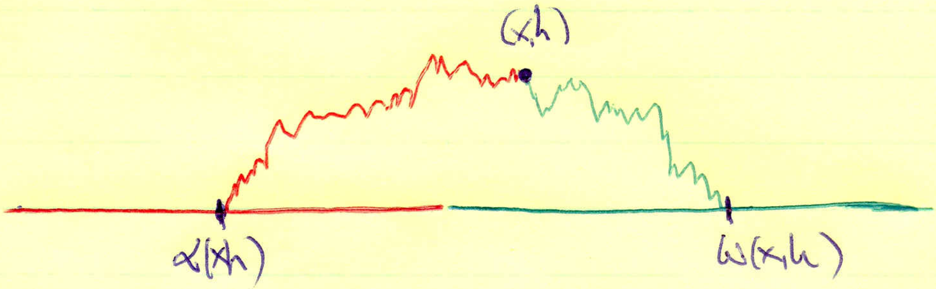
$I^*(x,h)$, $O^*(xh)$ defined similarly for the backward lines

Topological types of points $(xh) \in \mathbb{R} \times \mathbb{R}_+$

$[I, O]$	$[I^*, O^*]$	picture	Hausdorff dimension
$[0,1]$	$[0,1]$		2
$[1,1]$	$[0,2]$		3/2
$[0,2]$	$[1,1]$		3/2
$[0,3]$	$[2,1]$		0 countably many densely
$[2,1]$	$[0,3]$		— " —
$[1,2]$	$[1,2]$		1
			1

$$O - I^* = 1 = O^* - I$$

Construction of the process:

$D(x,h) :=$ 

Almost surely $(x_1 h_1) \neq (x_2 h_2)$

either $D(x_1,h_1) \subsetneq D(x_2 h_2)$ and $\mathrm{int}\left(D(x_1,h_1) \setminus D(x_2 h_2)\right) \neq \emptyset$

or the other way around

$$T_{xh} := \left| D(x,h) \right| = \int_{\alpha(x,h)}^{\omega(x,h)} \lambda_{xh}(y)\, dy$$

$T : \mathbb{R} \times \mathbb{R}_+ \to [0,\infty)$ almost surely injective

This defines almost surely a linear ordering of $\mathbb{R} \times \mathbb{R}_+$!

A. s. properties of $(x,h) \mapsto T(x,h)$:

- injective
- for $x \in \mathbb{R}$ fixed:
  $h \mapsto T(x,h)$ strictly increasing, left continuous
- $(x,h) \mapsto T(x,h)$ lower semicontinuous ($\Rightarrow$ Borel)
- preserves Lebesgue measure from $\mathbb{R} \times \mathbb{R}_+$ to $[0,\infty)$
- $\mathrm{Ran}(T)$ is of full Lebesgue measure in $[0,\infty)$

Though: T is <u>not</u> injective

for $I(x,h)$ <u>not</u> of topological type $[0,1)$

$\lim_{\varepsilon \downarrow 0} T(x, h+\varepsilon) \notin \mathrm{Ran}(T)$!

$$\bigcap_{\varepsilon>0} \overline{\{(x,h) : T(x,h) \in (t-\varepsilon, t+\varepsilon)\}} =: (X(t), H(t))$$

it is a singleton!

First properties of $t \longmapsto (X(t), H(t))$

- a.s. $t \mapsto (X(t), H(t))$ is continuous
- a.s. $t \mapsto X(t)$ is recurrent
- $(X(\alpha t), H(\alpha t)) \stackrel{\text{law}}{=} (\alpha^{2/3} X(t), \alpha^{1/3} H(t))$
- a.s. $t \mapsto (X(t), H(t))$ is plane filling — Peano

points of type $[0,1)/[0,1)$ are visited once

" " $\begin{cases} (1,1]/[0,2] \\ (0,2]/(1,1] \end{cases}$ " " twice

" $\begin{cases} [0,3)/[2,1] \\ [2,1]/(0,3] \\ (1,2]/(1,2] \end{cases}$ " three times

Local time & Dynamics:

$$L(t,y) := \sup\{h>0 : T(y,h) < t\}$$
$$= \inf\{h>0 : T(y,h) \geq t\}$$

since $h \mapsto T(y,h)$ is strictly increasing

$t \mapsto L(t,y)$ is continuous!

reverted:

$$T(x,h) = \sup\{t \geq 0 : L(t,x) < h\}$$
$$= \inf\{t \geq 0 : L(t,x) \geq h\}$$

Theorem $L(t,y)$ is (indeed!) the occupation time density of $X(t)$:

a.s. $$\int_0^t g(s, X(s))\,ds = \int_{-\infty}^{\infty} \left\{ \int_0^t g(s,x)\,d_s L(s,x) \right\} dx$$

relies on the fact that $T : \mathbb{R} \times \mathbb{R}_+ \longrightarrow (0,\infty)$ preserves Lebesgue measure.

"<u>Ray-Knight Theorems</u>": $L(T(x,h),y) = \Lambda_{xh}(y)$
for free, by construction; $H(t) = L(t, X(t))$

<u>Dynamics</u>: apply the local time identity
to $g(s,y) = L(s, y \pm \varepsilon)$

$$\frac{1}{2\varepsilon}\int_0^{T_{xh}} \big(L(s, X(s)+\varepsilon) - L(s, X(s)-\varepsilon) \big)\, ds =$$

$$\frac{1}{2\varepsilon}\int_{-\infty}^{+\infty} \left\{ \int_0^{T_{xh}} \big(L(s,y+\varepsilon) - L(s,y-\varepsilon) \big)\, d_s L(s,y) \right\} dy =$$

$$\frac{1}{2\varepsilon}\int_{\alpha(x,h)}^{x} \left\{ \int_0^{\Lambda_{xh}(y)} \big(\Lambda_{yu}(y+\varepsilon) - \Lambda_{yu}(y-\varepsilon) \big)\, du \right\} dy +$$

$$\frac{1}{2\varepsilon}\int_{x}^{\omega(xh)} \left\{ \int_0^{\Lambda_{xh}(y)} \big(\Lambda_{yu}(y+\varepsilon) - \Lambda_{yu}(y-\varepsilon) \big)\, du \right\} dy$$

An identity:

$(x,h) \in \mathbb{R} \times \mathbb{R}_+$, $\quad x \leq y \leq z$

$$\circledast \quad \int_0^{\Lambda_{xh}(y)} \Lambda_{yu}(z)\,du + \int_0^{\Lambda_{xh}(z)} \Lambda_{zu}(y)\,du = \Lambda_{xh}(y) \cdot \Lambda_{xh}(z)$$

or, equivalently:

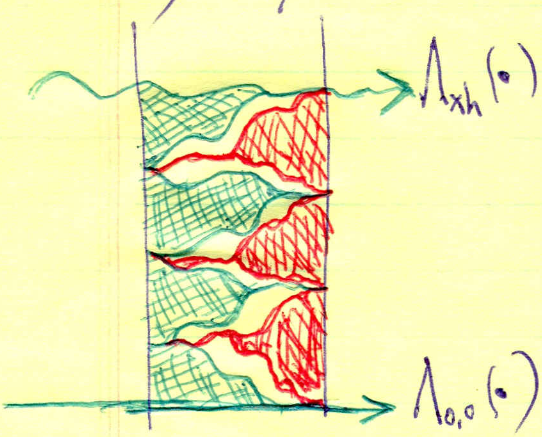
$$\circledast\circledast \quad \int_0^{\Lambda_{xh}(y)} \Lambda_{yu}(z)\,du - \int_0^{\Lambda_{xh}(z)} \Lambda_{xu}(y)\,du =$$

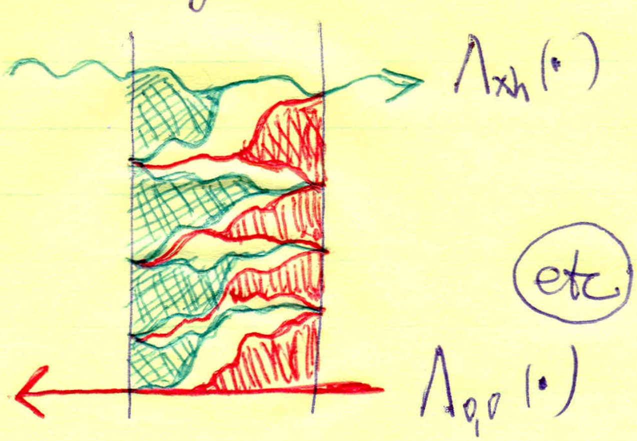
$$\frac{1}{2}\left(\Lambda_{xh}(z) - \Lambda_{xh}(y)\right)^2 - \frac{1}{2}\left(\Lambda_{xh}^2(z) - \Lambda_{xh}^2(y)\right)$$

$$+ 2\int_0^{\Lambda_{xh}(y)} \left(\Lambda_{yu}(z) - u\right)du$$

Proof of $\circledast$:

$x, 0 \leq y \leq z$ $\qquad\qquad\qquad$ $x \leq y \leq z \leq 0$





Contributions of the three terms on r.h.s. of ⊛⊛ (applied with $z = y + \varepsilon$) ....

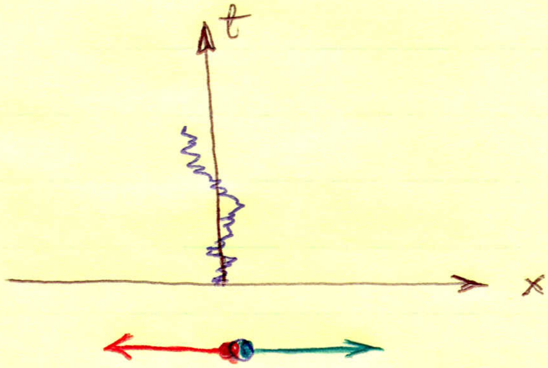
$$\left\{ \frac{1}{4}\left(\omega(x,h) - x \right) + 0 - \frac{1}{2} x \cdot \mathbb{1}(x<0) \right\} +$$

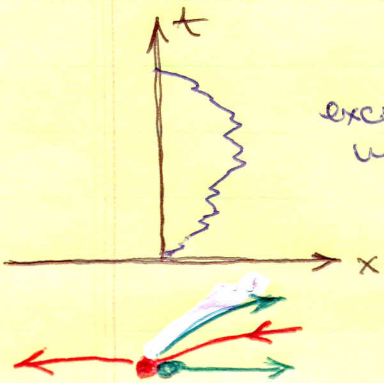
$$\left\{ -\frac{1}{4}\left(x - \alpha(x,h) \right) + 0 - \frac{1}{2} x \cdot \mathbb{1}(x>0) \right\} =$$

$$\frac{1}{4}\left(\alpha(x,h) + \omega(x,h) \right) - x$$

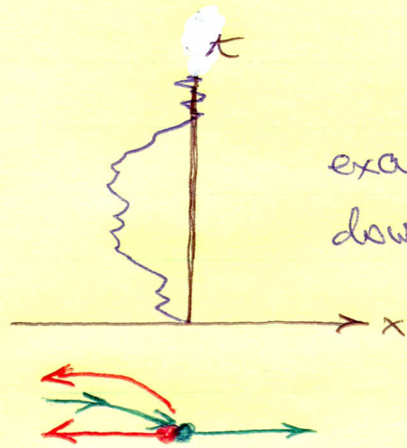
q.e.d.

Topological types revisited:

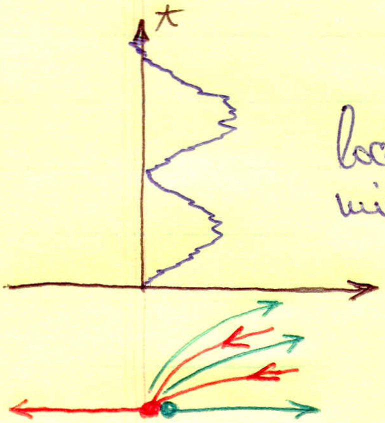




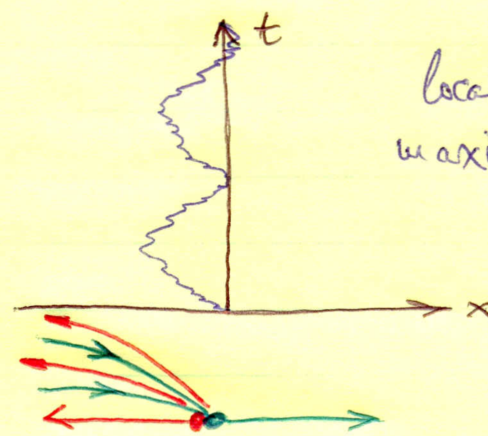
excursion up



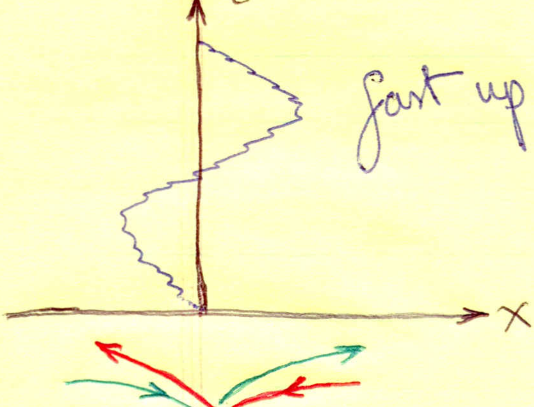
excursion down



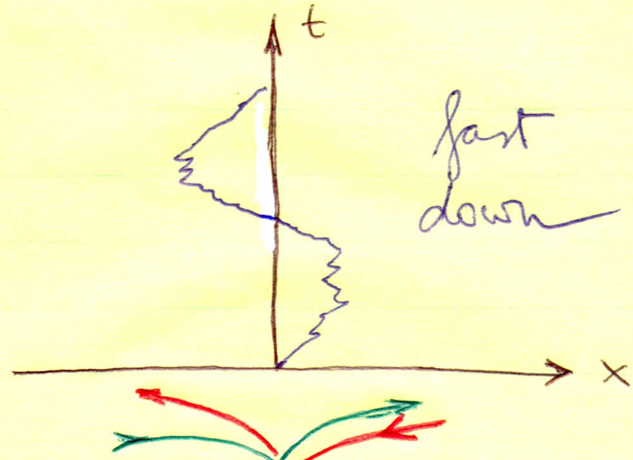
local minimum



local maximum

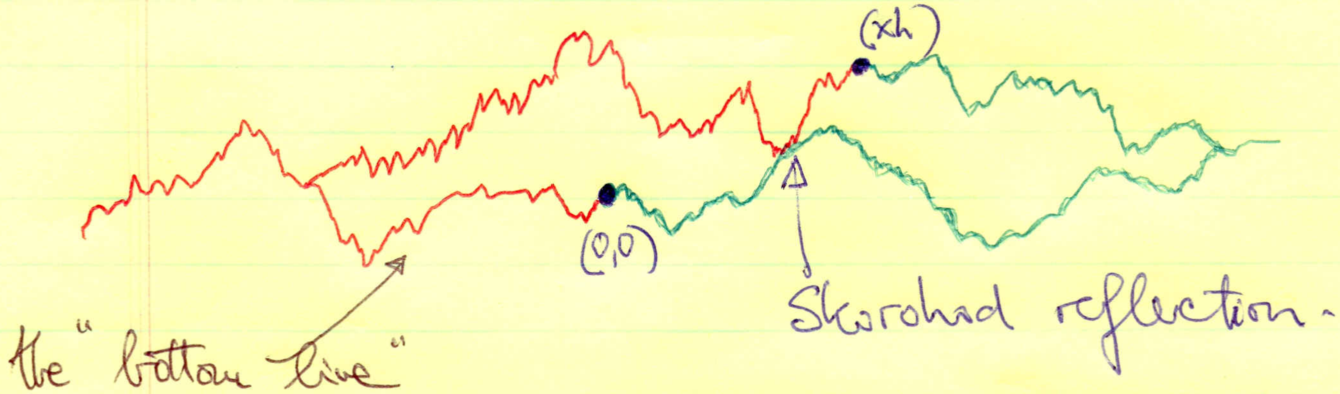


fast up



fast down

The stationary process: start with $\Lambda_{0,0}(y)$ — Brownian (rather than straight line)



$$t \longmapsto \Lambda_{X(t),H(t)}(X(t)+\cdot) - H(t)$$

or

$$t \longmapsto \eta(t,\cdot) := \Lambda'_{X(t),H(t)}(X(t)+\cdot)$$

$$\in \mathcal{S}'(\mathbb{R})$$

Stationary Markov process

Stationary distribution: white noise

Major open problem:

understand better $\eta(t,\cdot)$

e.g. What is its infinitesimal generator acting on

$$L^2(\mathcal{S}', \pi)\ ?$$

white noise

The construction of the process $(X(t), H(t))$ –