Bálint Tóth:

Scaling limits for self-interacting random walks and diffusions, 2

(lecture notes for Berlin-Zürich Summer School on Stochastic Models of Complex processes

31 Aug-4 Sept 2009

Limit theorems for the "Myopic" (or "True") self avoiding RW with edge repulsion on $\mathbb{Z}$ (based on BT '95)

$X(n)$ nearest neighbour walk on $\mathbb{Z}$

$$U(n,x) = \#\{m \in [0,n) : X(m) = x, X(m+1) = x+1\}$$

= no. of upcrossings $(x) \to (x+1)$

$$D(n,x) = \#\{m \in [0,n) : X(m) = x+1, X(m+1) = x\}$$

= no. of downcrossings $(x+1) \to (x)$

$$H(n,x) = U(n,x) + D(n,x)$$

= total no of jumps $(x) \leftrightarrow (x+1)$

($\ell(n, x+\frac{1}{2})$ in the notation of previous lecture)

$$Z(n,x) = -H(n,x) + H(n,x-1)$$

= negative gradient of $H(n,\cdot)$

law:

$$P(X(n+1) = x \pm 1 \mid \text{past}) = \frac{r(\pm Z(n,x))}{r(Z(n,x)) + r(-Z(n,x))}$$

where $r : \mathbb{Z} \to (0, \infty)$, nondecreasing, non-constant

* interpretation as deposition / domain growth model (at blackboard)

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Stopping times ("inverse local times"):

$$T^U_{x,r} := \inf\{n : U(n,x) = r\}$$

$$T^D_{x,r} := \inf\{n : D(n,x) = r\}$$

Local time profile at inverse local times

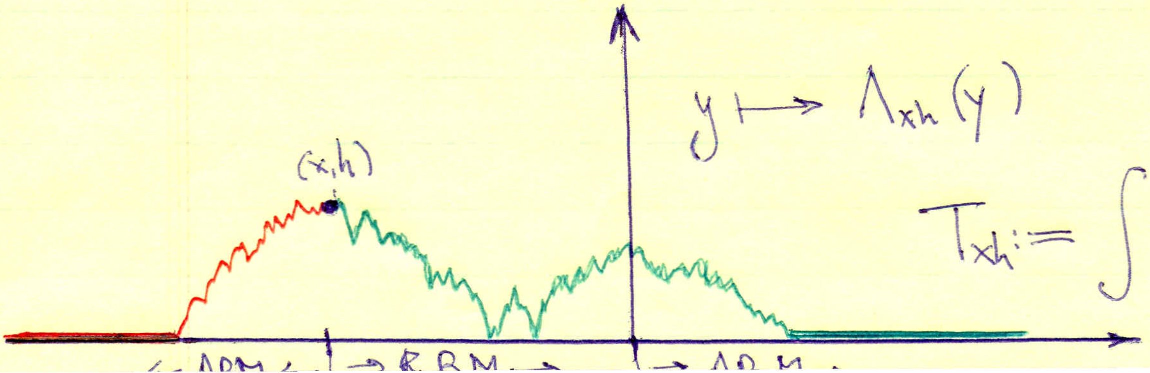
$$\Lambda^*_{x,r}(y) := H(T^*_{x,r}, y)$$

$$\left.\begin{array}{l} * = U, D \\ x \in \mathbb{Z},\ r \in \mathbb{N} \end{array}\right\} \text{fixed}$$

$$y \in \mathbb{Z}.$$

RAB (= reflecting / absorbed Brownian motion):

$x \in \mathbb{R}$, $h \in [0, \infty)$ fixed



$$T_{xh} := \int \Lambda_{xh}(y)\, dy$$

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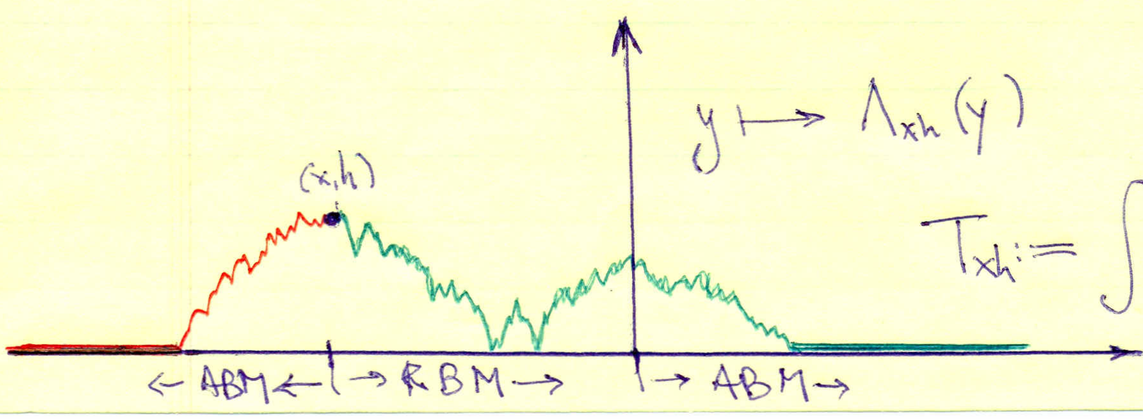


$$y \longmapsto \Lambda_{xh}(y)$$

$$T_{xh} := \int \Lambda_{xh}(y)\, dy$$

Limit thm. for the local time process.

Theorem 1. $x \in \mathbb{R}$, $h \in \mathbb{R}_+$ fixed

$$\frac{\Lambda^*_{[Nx],[\sqrt{N}\sigma h]}([Ny])}{\sigma \cdot \sqrt{N}} \Longrightarrow \Lambda_{x,h}(y)$$

in Skorohod top, as process

σ explicit constant, in terms of the rate function $r(z)$ (later...)

Corollary:

$$\frac{T^*_{[Nx],[N\sigma h]}}{2\sigma N^{3/2}} \Longrightarrow T_{xh}$$

Compare with "classical" Ray-Knight (on blackboard...)

Further notation needed:

$$\rho(t;x,h) := \frac{\partial}{\partial t} P(T_{xh} < t)$$

$$= \text{density of distrib. of } T_{xh}$$

$$\hat{\rho}(s;x\,h) := s\int_0^\infty e^{-st}\rho(t;x,h)\,dt$$

$$= s\cdot \mathbb{E}\left(e^{-sT_{xh}}\right)$$

! Note: (x,h) are parameters of the distribution density $\rho(t;x,h)$

Scaling: $\alpha > 0$

$$\alpha\,\rho(\alpha t;\ \alpha^{2/3}x,\ \alpha^{1/3}h) = \rho(t;x,h)$$

$$\alpha\cdot\hat{\rho}(\alpha^{-1}s;\ \alpha^{2/3}x,\ \alpha^{1/3}h) = \hat{\rho}(s;x\,h)$$

follows from Brownian scaling

Theorem 2: Fix $t > 0$.

$(x, h) \longmapsto \rho(t; x h)$ is a probability distribution density on $\mathbb{R} \times \mathbb{R}_+$.

Remark: Non-trivial, computationally tricky proof involving Airy ...

Look for asymptotic distribution of $X(n)$

$(H(n) := H(n, X(n)))$

let θ_s be independent geometric r.v.

$$\mathbb{P}(\theta_s = n) = (1 - e^{-s}) e^{-s/n}$$

Theorem 3

$$\left(\sigma^{2/3} N^{-2/3} X(\Theta_{s/N}),\ \sigma^{1/3} N^{-1/3} H(\Theta_{s/N})\right) \Longrightarrow \hat{\rho}(s;\cdot,\cdot)$$

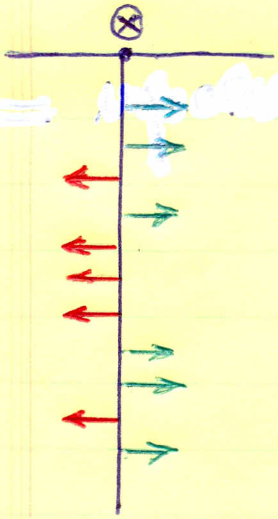
local limit thm for $(X(n), H(n))$ stopped at independent geometric stopping times

Morally:

$$\left(\sigma^{2/3} N^{-2/3} X(N),\ \sigma^{1/3} N^{-1/3} H(N)\right) \to \rho(t;\cdot,\cdot)$$

— Tauberian inversion missing.

Main idea of proof of Theorem 1:



sequence of consecutive left-right jumps from site $\otimes$. Only the order matters, not the time spans between consecutive jumps:

$\boxed{\text{generalized Pólya urn schemes}}$

actually: $\{\#(\rightarrow) - \#(\leftarrow)\}$ is a birth-death process

$\boxed{\text{independent for } x \in \mathbb{Z}}$!!!

with law:

$$\mathbb{P}(z \to z+1) = p(z) \qquad \mathbb{P}(z \to z-1) = q(z)$$

	$x<0$	$x=0$	$x>0$
$p(z) \sim$	$r(-2z-1)$	$r(-2z)$	$r(-2z+1)$
$q(z) \sim$	$r(2z+1)$	$r(2z)$	$r(2z-1)$

Yet another Markov chain: the previous b-d chain $\boxed{\text{stopped at down-steps}}$

Markov chain on $\mathbb{Z}$ with transition matrix

$$P_{xy} = \begin{cases} \prod_{z=x}^{y} p(z) \cdot q(y+1) & y \geq x-1 \\ 0 & y < x-1 \end{cases}$$

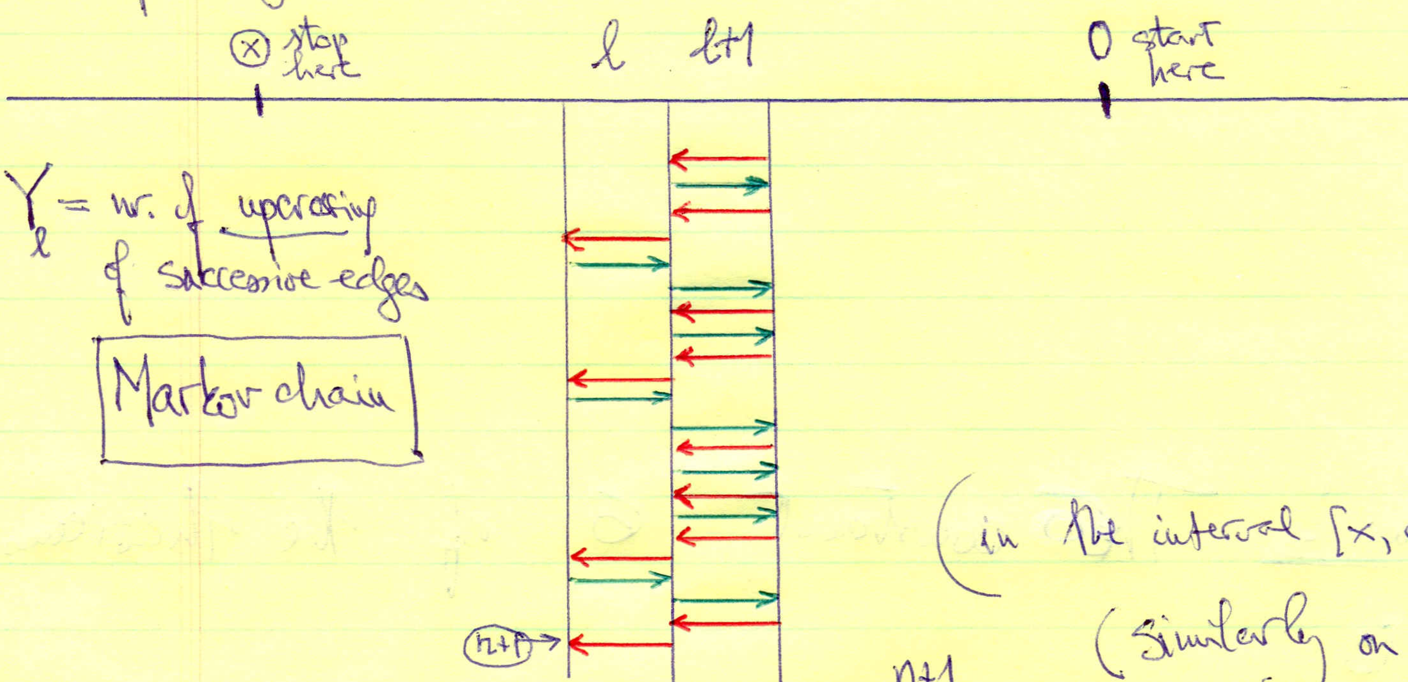
geometrically ergodic with explicitly computed stationary distribution

$$\pi(x) \sim \begin{cases} \prod_{z=0}^{x} \frac{p(z)}{q(z)} & x \geq 0 \\ \prod_{z=x+1}^{-1} \frac{q(z)}{p(z)} & x < 0 \end{cases}$$

$$x \leq y : \quad \pi(y) / \pi(x) = \prod_{z=x+1}^{y} \frac{p(z)}{q(z)}$$

$\pi(x)$ decays fast (at least exponentially) at $x \longrightarrow \pm\infty$

the Markov chain has very strong ergodic properties.



Y_ℓ = nr. of upcrossings of successive edges

$\boxed{\text{Markov chain}}$

(in the interval $[x, 0]$)

(similarly on the wings...)

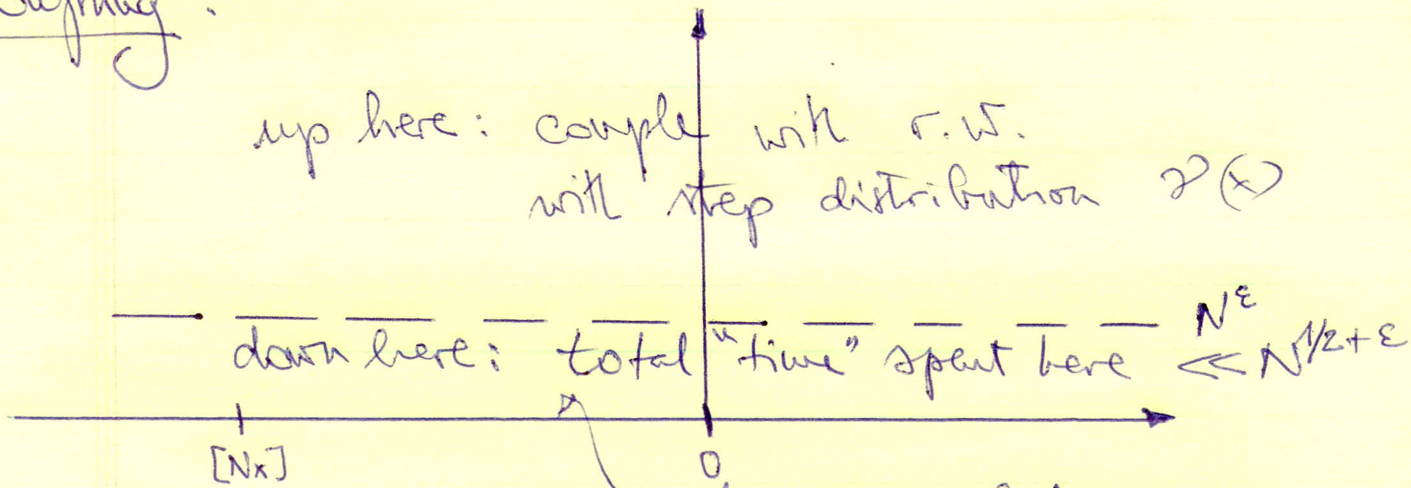
$$\mathbb{P}\left(Y_{\ell+1} - Y_\ell = z \;\middle|\; Y_\ell = n\right) = P^{n+1}_{0,z-1}$$

If $n \gg 1$:

$$P^{n+1}_{0,z-1} \approx \pi(z-1) = \frac{1}{Z} \prod_{k=1}^{|z|} \frac{r(-2k+1)}{r(2k-1)} = M(2z)$$

(turn page) → /.

Coupling:

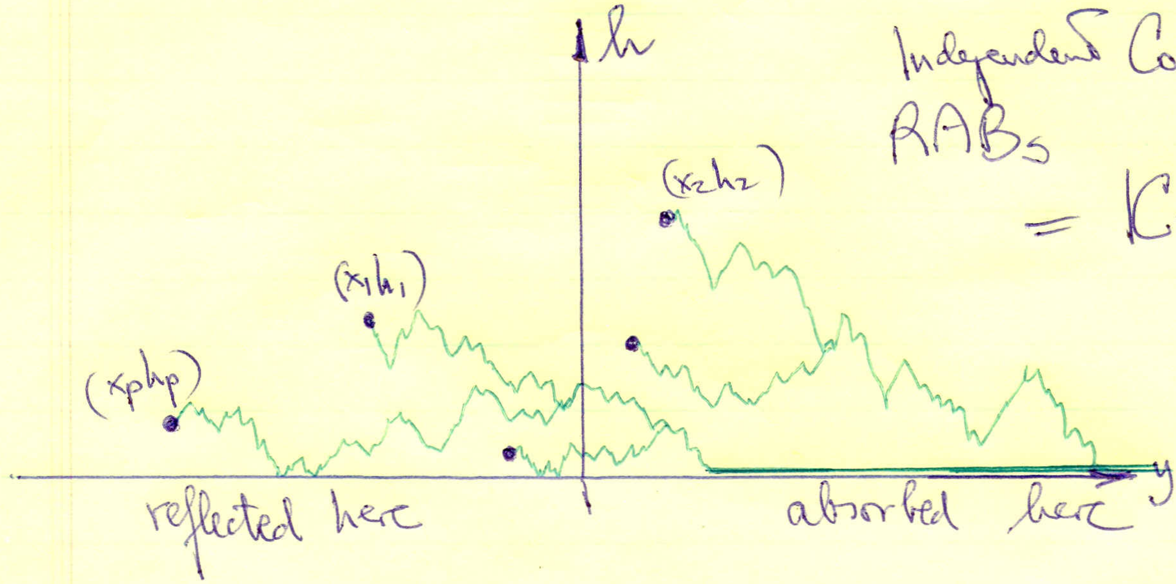


+ boundary behaviour

very much technical (see the paper..)

Joint distribution for stopping at $p \geq 1$ inverse local times:

Independent Coalescing RABs = ICRABs



$$\{ \Lambda_{x_j h_j}(y_j) : j = 1, 2, \dots, p \} \qquad y_j \geq x_j$$

- each one separately is a RAB
- they are independent as long as they stay apart
- they stick together at first encounter

<u>Theorem 4</u>: Fix $(x_i, h_i) \in \mathbb{R} \times \mathbb{R}_+$ $i = 1, 2, \dots, p$

$$\left(\frac{\Lambda^*_{[Nx_i], [\sqrt{N}\sigma h_i]}(Ny_i)}{2\sigma\sqrt{N}} : i = 1, 2, \dots, p \; ; \; y_i \geq x_i \right)$$

$$\Longrightarrow \left(\Lambda_{x_i, h_i}(y_i) : i = 1, 2, \dots, p \; ; \; y_i \geq x_i \right)$$

as processes

independence is surprising!

Compare with similar thm for ordinary RW.

<u>Proof of Thm 2</u> relies on detailed computations with various objects: Brownian excursions, Itô's intensity measure, Bismut's characterization, Bessel bridges, etc..

<u>Proof of Thm 3</u> straightforward ...

<u>Related results:</u>

① limit theorems for a wide class of self-interacting RW-s with

$$P\left(X(n+1) = x \pm 1 \,\middle|\, \text{past}, X(n) = x\right) = \frac{w\left(\ell(n, x \pm \tfrac{1}{2})\right)}{w\left(\ell(n, x+\tfrac{1}{2})\right) + w\left(\ell(n, x-\tfrac{1}{2})\right)}$$

with various weight functions w BT '96–97 ...

② very similar asymptotics for <u>continuous time</u>, site-repulsion (BT, B. Vető '08)

③ repulsion on $\boxed{\text{oriented}}$ edges:
very different asymptotics (BT, B Vető '08)
Related to rotor-router model. See also
R. Kapri - D. Dhar '09

....

<u>Major open problems:</u>

① This proof is partly "combinatorial", it relies on some details of the model.
Find <u>robust</u> "universal" proof.
Robust superdiffusive lower bound

$$\mathbb{E}(X(t)^2) \geq c \cdot t^{5/8}$$

[P. Tarrés, B.T., B. Valkó '09]

② Relate the distribution $\rho(t; x, h)$ with Airy processes

Preview

Stationary initial conditions – environment seen from the point of view of the moving particle

state space:

$$w_i = l_{i-\frac{1}{2}} - l_{i+\frac{1}{2}}$$

$$\Omega := \{ \underline{\omega} = (\omega_i)_{i\in\mathbb{Z}} : \omega_0 \in 2\mathbb{Z}+1;\ \omega_{i \neq 0} \in 2\mathbb{Z} \}$$

(could be the other way)

$$R : \Omega \to \Omega \qquad (R\underline{\omega})_i = \omega_{i+1} + \delta_{i,0} - \delta_{i,-1}$$

$$L : \Omega \to \Omega \qquad (L\omega)_i = \omega_{i-1} - \delta_{i,0} + \delta_{i,+1}$$

$$\mathcal{J} : \Omega \to \Omega \qquad (\mathcal{J}\omega)_i = -\omega_i$$

$$(R^{-1}\omega)_i = \omega_{i-1} + \delta_{i,0} - \delta_{i,+1}$$

$$(L^{-1}\omega)_i = \omega_{i+1} - \delta_{i,0} + \delta_{i,-1}$$

$$R^{-1} = \mathcal{J} L \mathcal{J}, \qquad L^{-1} = \mathcal{J} R \mathcal{J}$$

$$p(\underline{\omega}) = \frac{r(\omega_0)}{r(\omega_0) + r(-\omega_0)}; \quad q(\underline{\omega}) = \frac{r(-\omega_0)}{r(\omega_0) + r(-\omega_0)}$$

$f : \Omega \to \Omega$,

$$(Pf)(\underline{\omega}) = p(\underline{\omega}) f(R\underline{\omega}) + q(\underline{\omega}) f(L\underline{\omega})$$

this is the transition operator of the process of the "environment seen from the point of view of the moving particle"

Stationary measure : (untilted)

m_{even}, m_{odd} } probab measures on { $2\mathbb{Z}$, $2\mathbb{Z}+1$

$$m_{even}(2z) \sim \prod_{k=1}^{|z|} \frac{r(-2k+1)}{r(2k-1)}$$

$$m_{odd}(2z+1) \sim m_{even}(2z) / r(2z+1) = m_{even}(2z+2) / r(-2z-1)$$

$$\tilde{\mu} = \dots\dots m_{even} \times m_{even} \times m_{odd} \times m_{even} \times m_{even}$$

$$\rho(\underline{\omega}) = \left(r(\omega_0) + r(-\omega_0)\right) \Big/ \left(2 \sum r(2z+1)\, M_{odd}(2z+1)\right)$$

$$\mu(d\underline{\omega}) = \rho(\underline{\omega})\, \tilde{\mu}(d\omega)$$

! is stationary (+ ergodic) for the transition kernel

$$\boxed{\mu P = \mu}$$

Proof:

$$\int (Pf)(\underline{\omega})\, d\mu(\underline{\omega}) =$$

$$\int (Pf)(\omega)\, \rho(\omega)\, d\tilde{\mu}(\omega) =$$

$$\frac{1}{Z} \int \left\{ r(\underline{\omega}) f(R\underline{\omega}) + r(\mathcal{J}\underline{\omega}) f(L\underline{\omega}) \right\} d\tilde{\mu}(\underline{\omega}) =$$

$$\frac{1}{Z} \int \left\{ r(R^{-1}\underline{\omega})\, d\tilde{\mu}(R^{-1}\underline{\omega}) + r(\mathcal{J}L^{-1}\underline{\omega})\, d\tilde{\mu}(L^{-1}\underline{\omega}) \right\} f(\underline{\omega})$$

Main identities:

$$r(R^{-1}\underline{\omega})\, d\tilde{\mu}(R^{-1}\underline{\omega}) = r(\underline{\omega})\, d\tilde{\mu}(\underline{\omega})$$

$$r(\mathcal{J}L^{-1}\underline{\omega})\, d\tilde{\mu}(L^{-1}\underline{\omega}) = r(\mathcal{J}\underline{\omega})\, d\tilde{\mu}(\underline{\omega})$$

$$\frac{d\tilde{\mu}(R^{-1}\underline{\omega})}{d\tilde{\mu}(\underline{\omega})} = \frac{M_{odd}(\omega_{-1}+1)}{M_{even}(\omega_{-1})} \cdot \frac{M_{even}(\omega_0-1)}{M_{odd}(\omega_0)} =$$

$$= \frac{r(\omega_0)}{r(\omega_{-1}+1)} = \frac{r(\underline{\omega})}{r(R^{-1}\underline{\omega})}$$

$$\frac{d\tilde{\mu}(L^{-1}\underline{\omega})}{d\tilde{\mu}(\underline{\omega})} = \frac{M_{odd}(\omega_{+1}-1)}{M_{even}(\omega_{+1})} \cdot \frac{M_{even}(\omega_0+1)}{M_{odd}(\omega_0)}$$

$$= \frac{r(-\omega_0)}{r(-\omega_1+1)} = \frac{r(\mathcal{J}\underline{\omega})}{r(\mathcal{J}L^{-1}\underline{\omega})}$$

$\square$

Reversed chain :

$$\int d\mu(\underline{\omega})\; g(\underline{\omega}) (Pf)(\underline{\omega}) -$$

$$= \int d\mu(\underline{\omega})\, g(\underline{\omega}) \big(p(\underline{\omega}) f(R\underline{\omega}) + q(\underline{\omega}) f(L\underline{\omega}) \big)$$

$= \ldots$ using the same identities ...

$$= \int d\mu(\underline{\omega}) \big(p(\underline{\omega})\, g(R^{-1}\underline{\omega}) + q(\underline{\omega})\, g(L^{-1}\underline{\omega}) \big)\, f(\underline{\omega})$$

$$(P^{*}g)(\underline{\omega}) = p(\underline{\omega})\, g(R^{-1}\underline{\omega}) + q(\underline{\omega})\, g(L^{-1}\underline{\omega})$$

$$P^{*} = \mathcal{J} P \mathcal{J}$$

Yaglom reversibility :

$$\eta^{*}(t) := \mathcal{J}\eta(-t) \overset{\text{law}}{=} \eta(t) \quad !$$

More on this $\boxed{\text{later}}$ ...