Bálint Tóth:

Scaling limits for self-interacting random walks and diffusions, 1

lecture notes for Berlin-Zürich Summer School on Stochastic Models of Complex processes

31 Aug-4 Sept 2009

Part 1:

- motivation
- phenomenological argument
- models
- conjectures
- preview of results and major open problems

① Motivation:

What is Brownian motion?

For the probabilist: Wiener process

For the physicist: Motion of a tagged particle suspended in a fluid, interacting with the background — properly scaled

Are they the same? (in some proper sense).

The physical phenomenon is enormously complicated $\longrightarrow$ simplified models needed.

An instructive phenomenological "handwaving":

$X(t)$: position of the tagged particle

$$X(t) = X(0) + \int_0^t V(s)\,ds + M(t) - M(0)$$

where:

$V(s)$: instantaneous "velocity" or the compensator of the displacement

$M(s)$: a martingale

<u>Assume</u>: $\left.\begin{array}{l}\text{stationarity}\\ \text{ergodicity}\end{array}\right\}$ from the point of view of the particle

zero mean

the martingale part is diffusive:

$$\mathbb{E}\left(\left(M(t)-M(0)\right)^2\right) = c\cdot t$$

by stationarity:

$$\mathbb{E}\left(\left(\int_0^t V(s)\,ds\right)^2\right) = 2t\int_0^t \left(1-\frac{s}{t}\right)\mathbb{E}\left(V(0)V(s)\right)ds$$

this is the difficult part.

The $\boxed{\text{diffusivity}}$:

$$D(t) := t^{-1}\,\mathbb{E}\left(\left(X(t)-X(0)\right)^2\right)$$

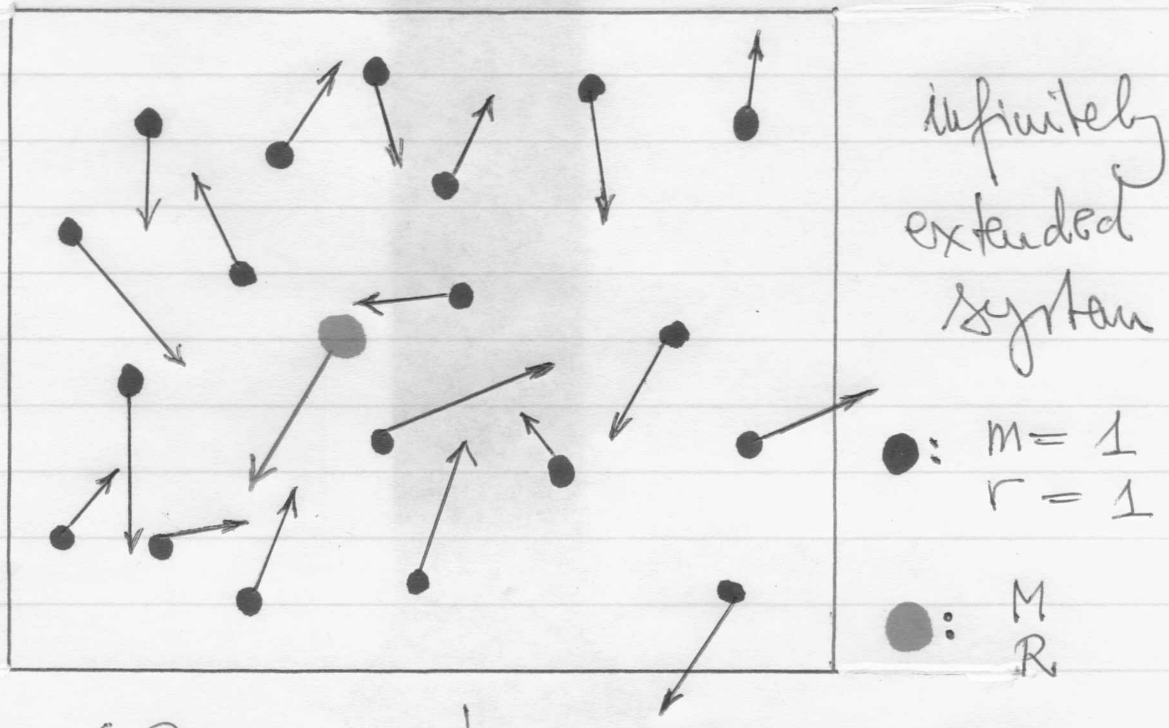
$D(t) \ll 1$: subdiffusive

$D(t) \asymp 1$: diffusive

$D(t) \gg 1$: superdiffusive

Example: tagged particle in a gas of hard balls in $\mathbb{R}^d$ —

(a mathematical challange for the next centuries):



Distribution: { Poisson points | nonoverlapping balls
independent Gaussian velocities of variance $1/m$ resp. $1/M$

Dynamics: uniform motion + elastic collisions

Assume : - a.s. existence of the dynamics
- stationarity + ergodicity from the point of view of the tagged particle.

Note : deterministic dynamics
randomness comes only from the initial conditions.

Numerical experiments : (1960-ies.)
BJ Alder, TE Wainwright - molecular dynamics

$d=1$: $D(t) \asymp 1$ - diffusive

$d=2$: $D(t) \asymp (\log t)^{\alpha}$ - superdiffusive

$d=3$: $D(t) \asymp 1$ - diffusive

$d=1$ very special for some reason ...

$d=2$ $\boxed{\text{very surprising}}$ why?

Rigorously : essentially nothing known in $d \geq 2$.

Phenomenological argument for the "Alder - Wainwright effect":

⊛ Assume that $X(t)$ scales with $\underline{\alpha(t)}$

$$\mathbb{P}\left(X(t) \in dx\right) \sim \alpha(t)^{-d}\, \varphi\left(\frac{x}{\alpha(t)}\right) dx$$

⊛ Condition on initial velocity $\underline{V(0)}$
– this will be averaged out at the end

⊛ Physically reasonable: assume that the initial $\boxed{\text{extra}}$ velocity field

$$V(0)\, \delta\left(x - X(0)\right) \quad \text{[due to conditioning]}$$

diffuses at rate $\beta(t) \leq \alpha(t)$.

Essential: conservation of "velocity field"

So the (nonequilibrium) velocity field at late time t is

$$\sim V(0)\, \psi\left(\frac{x}{\beta(t)}\right) \cdot \beta(t)^{-d}$$

Then we get:

$$\mathbb{E}\big(V(t)V(0)\big) \sim \int_{\mathbb{R}^d} \varphi\left(\frac{x}{\alpha(t)}\right) \alpha(t)^{-d}\, \psi\left(\frac{x}{\beta(t)}\right) \beta(t)^{-d}\, dx$$

$$= \alpha(t)^{-d} \int_{\mathbb{R}^d} \varphi\left(\frac{\beta(t)}{\alpha(t)}\, x\right) \psi(x)\, dx$$

$$\sim \alpha(t)^{-d}$$

Compute the diffusivity

$$D(t) \asymp \int_0^t E(V(s)V(0))\,ds \asymp \int^t \alpha(s)^{-d}\,ds$$

on the other hand

$$D(t) = t^{-1} E\left(X(t)^2\right) \asymp t^{-1}\,\alpha(t)^2$$

by the scaling assumption

$\boxed{d=1}$ $\qquad \alpha(s) = s^{\nu}$

$$D(t) \asymp \int^t s^{-\nu}\,ds \asymp t^{1-\nu}$$

$$D(t) \asymp t^{-1}\,t^{2\nu} \asymp t^{2\nu-1}$$

$$\boxed{\nu = \frac{2}{3}}$$

robustly super-diffusive scaling

$\boxed{d \geq 3}$

$$\alpha(s) = s^{\nu}$$

$$D(t) \asymp \int^{t} s^{-d\nu}\, ds \asymp \begin{cases} t^{0} & \text{if } \nu > \frac{1}{d} \\ t^{1-d\nu} & \text{if } \nu < \frac{1}{d} \end{cases}$$

$$D(t) \asymp t^{2\nu-1}$$

$$\boxed{\nu = \frac{1}{2}}$$

diffusive scaling

$\boxed{d = 2}$

$$\alpha(s) = s^{1/2} (\log s)^{\alpha}$$

$$D(t) \asymp \int^{t} s^{-1} (\log s)^{-2\alpha}\, ds \asymp (\log t)^{-2\alpha+1}$$

$$D(t) \asymp t^{-1} \alpha(t)^{2} \asymp (\log t)^{2\alpha}$$

$$\boxed{D(t) \asymp \sqrt{\log t}}$$

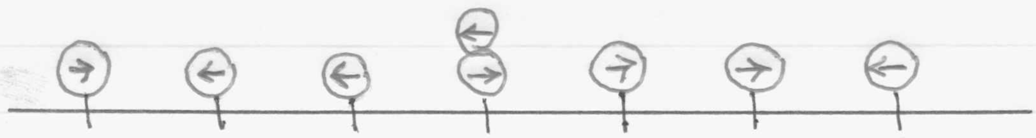
marginally superdiffusive

Can anything of this be done mathematically rigorously?

Note that the argument is robust, relies on few qualitative — and reasonable — assumptions only.

A nontrivial toy model: $d = 1$

On every site of $\mathbb{Z}$ put a marble (particle) $\rightarrow$ or $\leftarrow$ with probab $1/2 - 1/2$ + one extra marble $\rightarrow$ or $\leftarrow$ on the origin



<u>Dynamics</u>: choose at random one marble from the <u>unique</u> doubly occupied site and move it one step in its prescribed direction, ...

$X(t) :=$ the position of the unique doubly occupied site

Q: $\dfrac{X(t)}{t^{\nu}} \Longrightarrow ?$; $\nu = ?$

Obvious extension to $\mathbb{Z}^d$, $d \geq 1$.

② Models of self repelling walks / diffusions considered

The "Myopic" (or "True") SAW on $\mathbb{Z}^d$

(D. Amit, G. Parisi, L. Peliti, 1983)

discrete time:

$X(n)$ nearest neighbour walk on $\mathbb{Z}^d$

$\ell(n,x) := \ell(0,x) + \#\{1 \leq m \leq n : X(m) = x\}$

"local time" (initialization)

law:

$y \sim x$

$$\mathbb{P}\big(X(n+1) = y \mid \text{past}, X(n) = x\big) = \frac{\exp -\beta\big(\ell(n,y) - \ell(n,x)\big)}{\sum_{z \sim x} \exp -\beta\big(\ell(n,z) - \ell(n,x)\big)}$$

more generally:

$$r : \mathbb{Z} \to (0,\infty), \quad \begin{array}{l}\text{non-decreasing}\\ \text{non-constant}\end{array}$$

$$\mathbb{P}\left(X(n+1) = y \mid \text{past}, X(n) = x\right) =$$

$$\frac{r\left(\ell(n,x) - \ell(n,y)\right)}{\sum_{z \sim x} r\left(\ell(n,x) - \ell(n,z)\right)}$$

Guesses, conjectures, — physics "results" —
based on Renormalisation Group arguments
(APP'83, Obukhov-Peliti '83, Peliti-Pietronero '87, ...)

$$D(n) := n^{-1}\,\mathbb{E}\left(X(n)^2\right)$$

$d = 1$: $D(n) \asymp n^{1/3}$; $\left(X(n) \sim n^{2/3}\right)$

$d = 2$: $D(n) \asymp (\log n)^{2\xi}$; $\left(X(n) \sim n^{1/2} (\log n)^{\xi}\right)$

[the value of ξ controversial]

$d \geq 3$: $D(n) \asymp 1$.

Edge version in $d = 1$:

$X(n)$ nearest neighbour walk on $\mathbb{Z}$

$$\ell(n, x+\tfrac{1}{2}) = \ell(0, x+\tfrac{1}{2}) + \#\{1 \leq m \leq n : \{X(m-1), X(m)\} = \{x, x+1\}\}$$

local time on edges = number of jumps across the edge $\langle x, x+1\rangle$ in either direction

$$P\big(X(n+1) = X(n) \pm 1 \mid \text{past}\big) = \frac{r\big(\ell(n, x\mp\tfrac{1}{2}) - \ell(n, x\pm\tfrac{1}{2})\big)}{r\big(\ell(n, x-\tfrac{1}{2}) - \ell(n, x+\tfrac{1}{2})\big) + r\big(\ell(n, x+\tfrac{1}{2}) - \ell(n, x-\tfrac{1}{2})\big)}$$

similar behaviour expected.

The "marbles" model from page 10-11. is a special degenerate case of this one

Continuous time (site) version:

$X(t)$ n.n. jump walk on $\mathbb{Z}^d$

$$\ell(t,x) := \ell(0,x) + \left| \{ 0 < s < t : X(s) = x \} \right|$$

$$P\big(X(t+dt) = y \mid \text{past}, X(t) = x \big) =$$

$$dt \cdot r\big(\ell(t,x) - \ell(t,y)\big) + o(dt)$$

similar asymptotics expected.

Continuous space models:
"the Brownian polymer"

J. Norris, LCG Rogers, D. Williams 1987

RT Durrett, LCG Rogers, 1992

M. Cranston, T Mountford 1996,
M Cranston Y. Le Jan 1995
T. Mountford, P. Tarrès 2008

$$t \longmapsto X(t) \in \mathbb{R}^d$$

$$X(t) = \sigma B(t) + \int_0^t \int_0^s F\big(X(s) - X(u)\big)\, du\, ds$$

or

$$dX(t) = \sigma\, dB(t) + \left\{ \int_0^t F\big(X(t) - X(u)\big)\, du \right\} dt$$

where

$$F(x) = -\operatorname{grad} V(x), \qquad V(x) = e^{-\frac{|x|^2}{2}}$$

(or something else, similar)

In $d = 1$, written in terms of local time (occupation time density)

$$X(t) = \sigma B(t) + \int_0^t \left\{ \int_{-\infty}^{\infty} L(s, X(s) - z) F(z)\, dz \right\} dt$$

$$" = \sigma B(t) - \int_0^t \left\{ \int_{-\infty}^{\infty} L'(s, X(s) + z)\, V(z)\, dz \right\} dt "$$

pushed by the negative gradient of L

Some results / preview of forthcoming lectures:

① $d=1$

edge version   limit thm. for $\frac{X(n)}{n^{2/3}}$

(B. Tóth '95)

continuous time: limit thm for $\frac{X(t)}{t^{2/3}}$

(B.T., B Vető '08)

~~② method~~ [Ray-Knight approach]

② $d=1$ Construction of the scaling limit process

$$\mathcal{X}(t) = \text{sc.lim.}\ \frac{X(nt)}{n^{2/3}}$$

(B.T, W. Werner '98)

$$\text{"}\ d\mathcal{X}(t) = -L'(t, \mathcal{X}(t))\,dt\ \text{"}$$

[compare with Durrett-Rogers...]

[Brownian Web + Ray-Knight]

(3) Bounds on diffusivity $D(t)$
for 1d Brownian Polymer
with various choices of the
self-interaction function F.
(P. Tarrés, BT, B Vetö, 09)
[H_{-1} bounds, variational formulas]

(4) ~~d=3:~~ Diffusive limit & CLT
for 3d self repelling Brownian
polymer & TSAW... (I. Horvath, BT, B. Vetö, 09)

method: environment viewed from
position of the walker
Kipnis-Varadhan-type thm
(non-reversible ...)

(5) Application of "K-V-technology" to self-interacting random walks and diffusions in $d \geq 3$

(Horváth, Tóth, Vető '09)

(6) $d=2$: superdiffusive lower bound – partial results

(Valkó '09).