Sum of independent exponentials

Lemma 1. *Let* $(X_i)_{i=1\dots n}$, $n \geq 2$, *be independent exponential random variables with pairwise distinct respective parameters* λ_i. *Then the density of their sum is*

$$f_{X_1+X_2+\dots+X_n}(x) = \left[\prod_{i=1}^{n} \lambda_i\right] \sum_{j=1}^{n} \frac{\mathrm{e}^{-\lambda_j x}}{\prod\limits_{\substack{k\neq j\\k=1}}^{n} (\lambda_k - \lambda_j)}, \qquad x > 0. \tag{1}$$

Proof. First we compute the convolutions needed in the proof.

$$\mathrm{e}^{-ax} * \mathrm{e}^{-bx} = \int_0^x \mathrm{e}^{-a(x-u)}\mathrm{e}^{-bu}\,\mathrm{d}u = \mathrm{e}^{-ax}\frac{\mathrm{e}^{(a-b)x}-1}{a-b} = \frac{\mathrm{e}^{-bx}-\mathrm{e}^{-ax}}{a-b}.$$

For $n = 2$,

$$f_{X_1+X_2}(x) = f_{X_1}(x) * f_{X_2}(x) = \lambda_1\lambda_2\frac{\mathrm{e}^{-\lambda_2 x}-\mathrm{e}^{-\lambda_1 x}}{\lambda_1-\lambda_2} = \lambda_1\lambda_2\left[\frac{\mathrm{e}^{-\lambda_1 x}}{\lambda_2-\lambda_1} + \frac{\mathrm{e}^{-\lambda_2 x}}{\lambda_1-\lambda_2}\right],$$

in accordance to (1). Now inductively, fix $n \geq 3$, and assume the statement is true for $n-1$. Then

$$f_{X_1+X_2+\dots+X_n}(x) = f_{X_1+X_2+\dots+X_{n-1}}(x) * f_{X_n}(x) = \left[\prod_{i=1}^{n-1} \lambda_i\right] \sum_{j=1}^{n-1} \frac{\mathrm{e}^{-\lambda_j x}}{\prod\limits_{\substack{k\neq j\\k=1}}^{n-1} (\lambda_k - \lambda_j)} * f_{X_n}(x)$$

$$= \left[\prod_{i=1}^{n} \lambda_i\right] \sum_{j=1}^{n-1} \frac{\mathrm{e}^{-\lambda_n x}-\mathrm{e}^{-\lambda_j x}}{(\lambda_j-\lambda_n)\prod\limits_{\substack{k\neq j\\k=1}}^{n-1} (\lambda_k - \lambda_j)} = \left[\prod_{i=1}^{n} \lambda_i\right] \left[\sum_{j=1}^{n-1} \frac{\mathrm{e}^{-\lambda_j x}}{\prod\limits_{\substack{k\neq j\\k=1}}^{n} (\lambda_k - \lambda_j)} - \sum_{j=1}^{n} \frac{\mathrm{e}^{-\lambda_n x}}{\prod\limits_{\substack{k\neq j\\k=1}}^{n} (\lambda_k - \lambda_j)}\right].$$

The proof is done as soon as we show that the coefficient of $\mathrm{e}^{-\lambda_n x}$ fits the coefficients seen in the sum of (1), i.e.

$$-\sum_{j=1}^{n-1} \frac{1}{\prod\limits_{\substack{k\neq j\\k=1}}^{n} (\lambda_k - \lambda_j)} = \frac{1}{\prod\limits_{k=1}^{n-1} (\lambda_k - \lambda_n)} \tag{2}$$

or, equivalently,

$$\sum_{j=1}^{n} \frac{1}{\prod\limits_{\substack{k\neq j\\k=1}}^{n} (\lambda_k - \lambda_j)} = 0.$$

To this order, we write

$$\sum_{j=1}^{n} \frac{1}{\prod\limits_{\substack{k\neq j\\k=1}}^{n} (\lambda_k - \lambda_j)} = \sum_{j=1}^{n} \frac{\prod\limits_{\substack{k\neq l\neq j\\k,\,l=1}}^{n} (\lambda_k - \lambda_l)}{\prod\limits_{\substack{k\neq l\\k,\,l=1}}^{n} (\lambda_k - \lambda_l)}$$

which is zero if and only if

$$\sum_{j=1}^{n} \prod_{\substack{k\neq l\neq j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)$$

is zero. We transform the latter in the following display. The nontrivial steps are changing orders of λ's and thus signs in the factors of the products.

$$\begin{aligned}
\sum_{j=1}^{n} \prod_{\substack{k\neq l\neq j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l) &= \sum_{j=1}^{n} \prod_{\substack{j\neq k\neq l\neq j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l) \prod_{\substack{k=j\neq l\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)\\
&= \pm \sum_{j=1}^{n} \prod_{\substack{j\neq k> l\neq j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)^2 \prod_{\substack{k=j> l\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l) \prod_{\substack{k=j< l\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)\\
&= \pm \sum_{j=1}^{n} \prod_{\substack{j\neq k> l\neq j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)^2 \prod_{\substack{j=k> l\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l) \prod_{\substack{k>l=j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)\,(-1)^{n-j} =\\
&= \pm \prod_{\substack{k>l\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l) \sum_{j=1}^{n} \prod_{\substack{j\neq k> l\neq j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)\,(-1)^{n-j},
\end{aligned}$$

which is zero if and only if

$$\sum_{j=1}^{n} \prod_{\substack{j\neq k> l\neq j\\ k,\, l=1}}^{n} (\lambda_k - \lambda_l)\,(-1)^{j} \tag{3}$$

is zero. Notice that the product here is a Vandermonde determinant of the form

$$\begin{vmatrix}
1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-2}\\
1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-2}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & \lambda_{j-1} & \lambda_{j-1}^2 & \cdots & \lambda_{j-1}^{n-2}\\
1 & \lambda_{j+1} & \lambda_{j+1}^2 & \cdots & \lambda_{j+1}^{n-2}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-2}
\end{vmatrix},$$

and hence (3) is nothing else but the expansion of the determinant

$$\begin{vmatrix}
1 & 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-2}\\
1 & 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-2}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
1 & 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-2}
\end{vmatrix}$$

w.r.t. its second column. As this determinant is zero, so is (3) and thus (2) is proven. □