Kalkulus 1, 13. hét

Határozatlan integrál, megoldások

I. Számítsuk ki a következő integrálokat.

1. $\int \frac{1+x}{\sqrt{x}}\,\mathrm{d}\,x = \int x^{-\frac{1}{2}} + x^{\frac{1}{2}}\,\mathrm{d}\,x = 2\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C$
2. $\int \frac{1}{x^2}\,\mathrm{d}\,x = -\frac{1}{x} + C$
3. $\int 1 + \mathrm{e}^{x-1}\,\mathrm{d}\,x = x + \mathrm{e}^{x-1} + C$
4. $\int \frac{x^2-1}{x^2+1}\,\mathrm{d}\,x = \int \frac{(x^2+1)-2}{x^2+1}\,\mathrm{d}\,x = x - 2\operatorname{arctg} x + C$
5. $\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}\,x = \operatorname{arsh} x + C$
6. $\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}\,x = \operatorname{arch} x + C$
7. $\int \frac{1}{\sin^2} = -\operatorname{ctg} + C$
8. $\int \frac{1}{\cos^2} = \operatorname{tg} + C$
9. $\int \frac{1}{\operatorname{sh}^2} = -\operatorname{cth} + C$
10. $\int \frac{1}{\operatorname{ch}^2} = \operatorname{th} + C$
11. $\int \operatorname{sh} = \operatorname{ch} + C$
12. $\int \operatorname{ch} = \operatorname{sh} + C$

II. A parciális integrálás segítségével határozzuk meg az alábbi integrálokat, ahol $a, b \in \mathbb{R} \setminus \{0\}$ és $k \in \mathbb{N}$.

1. $\int x e^{ax}\,\mathrm{d}\,x = \frac{1}{a^2}\,\mathrm{e}^{ax}(ax-1) + C$
2. $\int x^2\,\mathrm{e}^{-ax}\,\mathrm{d}\,x = -\frac{1}{a^3}\,\mathrm{e}^{-ax}(a^2x^2 + 2ax + 2) + C$
3. $\int x\sin x\,\mathrm{d}\,x = \sin x - x\cos x + C$
4. $\int \mathrm{e}^{ax}\sin bx\,\mathrm{d}\,x = \frac{\mathrm{e}^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx)) + C$
5. $\int \mathrm{e}^{x}\cos x\,\mathrm{d}\,x = \frac{\mathrm{e}^{x}}{2}(\sin x + \cos x) + C$
6. $\int \sqrt{1-x^2}\,\mathrm{d}\,x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$

7. $\int \sqrt{1+x^2}\,\mathrm{d}\,x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\operatorname{arsh} x + C$

8. $\int \sqrt{x^2-1}\,\mathrm{d}\,x = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\operatorname{arch} x + C$

9. $\int \arcsin x\,\mathrm{d}\,x = x\arcsin x + \sqrt{1-x^2} + C$

10. $\int \operatorname{arctg} x\,\mathrm{d}\,x = x\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + C$

11. $\int x\operatorname{arctg} ax\,\mathrm{d}\,x = \frac{x^2}{2}\operatorname{arctg}(ax) + \frac{1}{2a^2}\operatorname{arctg}(ax) - \frac{x}{2a} + C$

12. $\int x^3\ln^2 x\,\mathrm{d}\,x = \frac{x^4}{4}\ln^2 x - \frac{x^4}{8}\ln x + \frac{x^4}{32} + C$

III. A következőkben a racionális törtfüggvényekre vonatkozó integrálási szabályt alkalmazzuk.

1. $\int \frac{1}{1-x^2}\,\mathrm{d}\,x = \int \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{x-1}\,\mathrm{d}\,x = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$

2. $\int \frac{1}{x^2-2x-3}\,\mathrm{d}\,x = \int \frac{1}{4}\cdot\frac{1}{x-3} - \frac{1}{4}\cdot\frac{1}{x+1}\,\mathrm{d}\,x = \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1| + C$

3. $\int \frac{1}{x^2+2x+6}\,\mathrm{d}\,x = \frac{1}{\sqrt{5}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{5}}\right) + C$

4. $\int \frac{x^2-1}{(x+2)^3}\,\mathrm{d}\,x = \int \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{3}{(x+2)^3}\,\mathrm{d}\,x =$

$= \ln|x+2| + \frac{4}{x+2} - \frac{3}{2(x+2)^2} + C$

5. $\int \frac{1}{x^3+1}\,\mathrm{d}\,x = \int \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{3}\cdot\frac{x-2}{x^2-x+1}\,\mathrm{d}\,x =$

$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|x^2-x+1\right| + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

6. $\int \frac{x^4}{(x-2)(x-3)(x-4)}\,\mathrm{d}\,x = \int x + 9 + \frac{8}{x-2} - \frac{81}{x-3} + \frac{128}{x-4}\,\mathrm{d}\,x =$

$= \frac{x^2}{2} + 9x + 8\ln|x-2| - 81\ln|x-3| + 128\ln|x-4| + C$

7. $\int \frac{16x^2+4x}{x^4+4}\,\mathrm{d}\,x = \int \frac{4x+1}{x^2-2x+2} - \frac{4x+1}{x^2+2x+2}\,\mathrm{d}\,x =$

$= 2\ln\left|x^2-2x+2\right| + 5\operatorname{arctg}(x-1) - 2\ln\left|x^2+2x+5\right| + 3\operatorname{arctg}(x+1) + C$

8. $\int \frac{x^3}{(x^2+1)^2}\,\mathrm{d}\,x = \int \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\,\mathrm{d}\,x = \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C$

9. $\int \frac{x^4+4}{x^3-1}\,\mathrm{d}\,x = \int x + \frac{5}{3}\cdot\frac{1}{x-1} - \frac{5x+7}{3(x^2+x+1)}\,\mathrm{d}\,x =$

$= \frac{x^2}{2} + \frac{5}{3}\ln|x-1| - \frac{5}{6}\ln\left|x^2+x+1\right| - \sqrt{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

10. $\int \frac{1}{(x^2+x+1)^2}\,\mathrm{d}\,x = \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

11. $\displaystyle\int \frac{x}{(x^2+2x+2)^2}\,\mathrm{d}\,x = -\frac{x+2}{2(x^2+2x+2)} - \frac{1}{2}\operatorname{arctg}(x+1)+C$

12. $\displaystyle\int \frac{1}{(x^2+2x+2)^2}\,\mathrm{d}\,x = \frac{x+1}{2(x^2+2x+2)} + \frac{1}{2}\operatorname{arctg}(x+1)+C$

IV. A következő integrálok kiszámításához alkalmazzunk megfelelő helyettesítést. ($k \in \mathbb{N}$)

1. $t = x+2$ $\displaystyle\int \frac{x^3}{(x+2)^4}\,\mathrm{d}\,x = \int \frac{(t-2)^3}{t^4}\,\mathrm{d}\,t = \ln|x+2| + \frac{6}{x+2} - \frac{6}{(x+2)^2} + \frac{8}{3(x+2)^3} + C$
2. $t = \sqrt{1+x}$ $\displaystyle\int \frac{1}{\sqrt{1+x}+(\sqrt{1+x})^3}\,\mathrm{d}\,x = \int \frac{2}{1+t^2}\,\mathrm{d}\,t = 2\operatorname{arctg}\sqrt{1+x}+C$
3. $t = \sqrt[4]{x-1}$ $\displaystyle\int x\sqrt[4]{x-1}\,\mathrm{d}\,x = \int 4t^4+4t^8\,\mathrm{d}\,t = \frac{4}{5}(x-1)^{\frac{5}{4}} + \frac{4}{9}(x-1)^{\frac{9}{4}} + C$
4. $t = \mathrm{e}^x$ $\displaystyle\int \frac{\mathrm{e}^{4x}}{1+\mathrm{e}^x}\,\mathrm{d}\,x = \int \frac{t^3}{1+t}\,\mathrm{d}\,t = \frac{\mathrm{e}^{3x}}{3} - \frac{\mathrm{e}^{2x}}{2} + \mathrm{e}^x - \ln(1+\mathrm{e}^x)+C$
5. $t = \sqrt{\mathrm{e}^x-1}$ $\displaystyle\int \sqrt{\mathrm{e}^x-1}\,\mathrm{d}\,x = \int \frac{2t^2}{t^2+1}\,\mathrm{d}\,t = 2\sqrt{\mathrm{e}^x-1} - 2\operatorname{arctg}(\sqrt{\mathrm{e}^x-1})+C$
6. $t = \sqrt{x}$ $\displaystyle\int \sqrt{x}\,\mathrm{e}^{\sqrt{x}}\,\mathrm{d}\,x = \int 2t^2\,\mathrm{e}^t\,\mathrm{d}\,t = 2\,\mathrm{e}^{\sqrt{x}}(x-2\sqrt{x}+2)+C$
7. $t = \arcsin x$ $\displaystyle\int \sqrt{1-x^2}\,\mathrm{d}\,x = \int \cos^2 t\,\mathrm{d}\,t = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$
8. $t = \sin x$ $\displaystyle\int \sin^k x\cdot\cos x\,\mathrm{d}\,x = \int t^k\,\mathrm{d}\,t = \frac{\sin^{k+1} x}{k+1}+C$
9. $t = \cos x$ $\displaystyle\int \cos^k x\cdot\sin x\,\mathrm{d}\,x = \int -t^k\,\mathrm{d}\,t = -\frac{\cos^{k+1} x}{k+1}+C$
10. $t = \sin x$ $\displaystyle\int \operatorname{ctg} x\,\mathrm{d}\,x = \int \frac{1}{t}\,\mathrm{d}\,t = \ln|\sin x|+C$
11. $t = \cos x$ $\displaystyle\int \operatorname{tg} x\,\mathrm{d}\,x = \int -\frac{1}{t}\,\mathrm{d}\,t = -\ln|\cos x|+C$
12. $t = \operatorname{tg} x$ $\displaystyle\int \operatorname{tg}^2 x\,\mathrm{d}\,x = \int \frac{t^2}{1+t^2}\,\mathrm{d}\,t = \operatorname{tg} x - x + C$
13. $t = \operatorname{tg} x$ $\displaystyle\int \operatorname{tg}^4 x\,\mathrm{d}\,x = \int \frac{t^4}{1+t^2}\,\mathrm{d}\,t = \frac{\operatorname{tg}^3 x}{3} - \operatorname{tg} x + x + C$
14. $t = \mathrm{e}^x$ $\displaystyle\int \frac{2}{\mathrm{e}^{3x}-\mathrm{e}^x}\,\mathrm{d}\,x = \int \frac{2}{t^4-t^2}\,\mathrm{d}\,t = \ln|\mathrm{e}^x-1| - \ln|\mathrm{e}^x+1| + 2\,\mathrm{e}^{-x}+C$
15. $t = \sqrt{x-1}$ $\displaystyle\int \frac{1}{x+\sqrt{x-1}-1}\,\mathrm{d}\,x = \int \frac{2}{t+1}\,\mathrm{d}\,t = 2\ln(\sqrt{x-1}+1)+C$

V. A következő integrálok kiszámításához gondoljunk az elemi függvények (sin, tg, exp, arctg, arcsin,...) deriváltjára és a deriválásnál megismert láncszabályra.

1. $\displaystyle\int x\,\mathrm{e}^{-x^2}\,\mathrm{d}\,x = -\frac{1}{2}\,\mathrm{e}^{-x^2}+C$
2. $\displaystyle\int \frac{3^x}{1+9^x}\,\mathrm{d}\,x = \frac{1}{\ln 3}\operatorname{arctg} 3^x + C$

3. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,\mathrm{d}\,x = \arcsin e^x + C$

4. $\displaystyle\int \frac{e^x}{\sqrt[3]{1+e^x}}\,\mathrm{d}\,x = \frac{3}{2}(1+e^x)^{\frac{2}{3}} + C$

5. $\displaystyle\int \frac{\cos\ln x}{x}\,\mathrm{d}\,x = \sin\ln|x| + C$

6. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}\,x = \operatorname{arctg}\sin x + C$

VI. Trigonometrikus azonosságok segítségével számoljuk ki az alábbi határozatlan integrálokat.

1. $\displaystyle\int \cos 2x\cos 5x\,\mathrm{d}\,x = \int \frac{1}{2}\cos(3x) + \frac{1}{2}\cos(7x)\,\mathrm{d}\,x = \frac{1}{6}\sin(3x) + \frac{1}{14}\sin(7x) + C$

2. $\displaystyle\int \cos^5 x\sin^2 x\,\mathrm{d}\,x = \int \cos x(\sin^2 x - 2\sin^4 x + \sin^6 x)\,\mathrm{d}\,x = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

3. $\displaystyle\int \cos^5 x\,\mathrm{d}\,x = \int \cos x(1 - 2\sin^2 x + \sin^4 x)\,\mathrm{d}\,x = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

4. $\displaystyle\int \cos^5 x\sin^3 x\,\mathrm{d}\,x = \int \cos x(\sin^3 x - 2\sin^5 x + \sin^7 x)\,\mathrm{d}\,x = \frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C$

5. $\displaystyle\int \sin^4 x\,\mathrm{d}\,x = \int \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\,\mathrm{d}\,x = \frac{3}{8}x - \frac{1}{4}\cos(2x) + \frac{1}{32}\sin(4x) + C$

6. $\displaystyle\int \cos^5 x\sin^5 x\,\mathrm{d}\,x = \int \cos x(\sin^5 x - 2\sin^7 x + \sin^9 x)\,\mathrm{d}\,x = \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C$