On topological relaxations of chromatic conjectures

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The Lovász–Kneser theorem

For n, k integers (where $0 < 2k \leq n$), the **Kneser graph** $\mathrm{KG}(n, k)$ is the graph whose vertices are all the k-element subsets of an n-element ground set, and two vertices are connected with an edge iff they are disjoint sets.

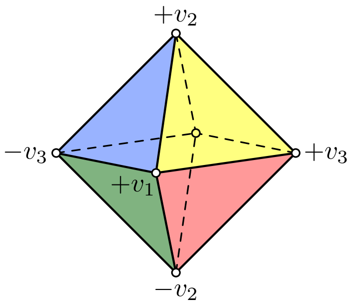
Lovász–Kneser theorem: $\chi(\mathrm{KG}(n, k)) = n - 2k + 2$.

The part that $\chi(\mathrm{KG}(n, k)) \leq n - 2k + 2$ is easy. Proof: for $n - 2k + 1$ points of the ground set, make a color class containing all nodes with that point. The remaining nodes form the last color class.

This theorem was conjectured by M. Kneser in 1955, proved by L. Lovász in 1978. His proof of the hard part starts what we now call the topological method. We start by recapping what this method means.

J. Matoušek, *Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer-Verlag, Heidelberg, 2003.

The B_0 functor makes a $\mathbb{Z}_2$-space from a graph



A **$\mathbb{Z}_2$-space** $(X, -)$ is a topological space X equipped with a continuous map $- : X \to X$ that does not have a fixpoint and where $x = -(-x)$.

From a simple graph G, we create its **box complex** $B_0(G)$, which is a $\mathbb{Z}_2$-space defined as a simplicial complex. Take two vertices $+x, -x$ for each node x of the graph. Put a simplex on a set of vertices $\{+x_1, +x_2, \ldots, +x_k, -y_1, -y_2, \ldots, -y_l\}$ iff each $\{x_i, y_j\}$ edge exists in G (that is, these vertices form a complete bipartite subgraph).

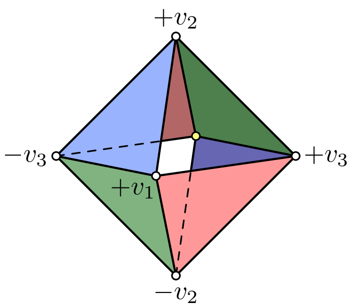
From the complete graph K_t, we get $B_0(K_t)$ which is $\mathbb{Z}_2$-homeomorphic to the sphere $\mathbb{S}^{t-1}$ (where the negation in $\mathbb{S}^{t-1}$ is the reflection around the center of the sphere).

A graph homomorphism $G \to H$ induces a $\mathbb{Z}_2$-map $B_0(G) \to B_0(H)$. (A $\mathbb{Z}_2$-map is a continuous map preserving negation.)

Borsuk–Ulam theorem: there is no $\mathbb{Z}_2$-map from $\mathbb{S}^t \to \mathbb{S}^{t-1}$.

Consequence: no graph homomorphism $K_{t+1} \to K_t$.

Variation: the B functor makes a $\mathbb{Z}_2$-space from a graph



A $\mathbb{Z}_2$**-space** $(X, -)$ is a topological space X equipped with a continuous map $- : X \to X$ that does not have a fixpoint and where $x = -(-x)$.

From a simple graph G, we create its **box complex** $B(G)$, which is a $\mathbb{Z}_2$-space defined as a simplicial complex. Take two vertices $+x, -x$ for each node x of the graph. Put a simplex on a set of vertices $\{+x_1, +x_2, \dots, +x_k, -y_1, -y_2, \dots, -y_l\}$ iff each $\{x_i, y_j\}$ edge exists in G (that is, these vertices form a complete bipartite subgraph), and also $x_1, \dots, x_k$ has a common neighbour and $y_1, \dots, y_l$ has a common neigbour (extra condition matters if $k = 0$ or $l = 0$).

$B(K_t)$ is $\mathbb{Z}_2$-homotopy equivalent to the sphere $\mathbb{S}^{t-2}$.

A graph homomorphism $G \to H$ induces a $\mathbb{Z}_2$-map $B(G) \to B(H)$. (A $\mathbb{Z}_2$-map is a continuous map preserving negation.)

Borsuk–Ulam theorem: there is no $\mathbb{Z}_2$-map from $\mathbb{S}^t \to \mathbb{S}^{t-1}$.

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The lower bound to the chromatic number for a general graph

If X is a $\mathbb{Z}_2$-space, then its **coindex** $\mathrm{coind}(X)$ is the greatest t for which there exists a $\mathbb{Z}_2$-map from $\mathbb{S}^t$ to X.

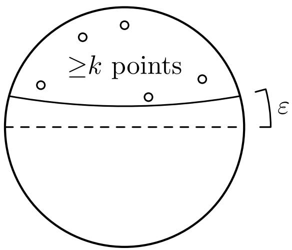
(Recall that we consider only $\mathbb{Z}_2$-spaces where the negation does not have a fixpoint. If the space X was not like this, then there would be a map from any $\mathbb{Z}_2$-space to X, such as a constant function to a fixpoint. This also explains why we are using $\mathbb{Z}_2$-spaces instead of ordinary topological spaces.)

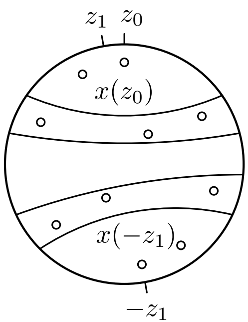
Lemma. For any simple graph G,

$$1 + \mathrm{coind}(B_0(G)) \leq \chi(G) \quad \text{and} \quad 2 + \mathrm{coind}(B(G)) \leq \chi(G).$$

Proof. A coloring of G with $n = \chi(G)$ colors is a graph homomorphism $f : G \to K_n$, this induces a map $F : B_0(G) \to B_0(K_n)$. We've seen that $B_0(K_n)$ is topologically the same as $\mathbb{S}^{n-1}$. But $t = \mathrm{coind}(B_0(G))$ means a map $G : \mathbb{S}^t \to B_0(G)$. The composition is $(F \circ G) : \mathbb{S}^t \to \mathbb{S}^{n-1}$, so the Borsuk–Ulam theorem gives $t \leq n - 1$.

Gale's lemma





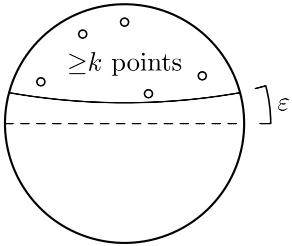
Choose n, k integers where $0 < 2k \leq n$. Recall that the vertices of the **Kneser graph** $G := \mathrm{KG}(n, k)$ are the k-element subsets of an n-element ground set P, and two vertices are joined with an edge iff they are disjoint sets.

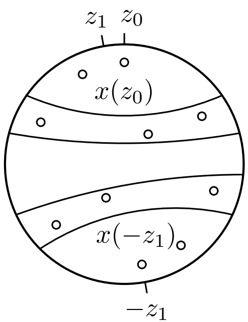
Gale's lemma asserts that there exists a set of n points on the sphere $\mathbb{S}^{n-2k}$ such that any open hemisphere contains at least k points.

Let us choose such a set as the ground set P. Let's assume that in fact every hemisphere with an ε wide ring removed from its border still has at least k points.

For a point z of the sphere, let $x(z)$ denote the set of the k points of P closest to z. Then $x(z)$ is a vertex of G. Now $x(z)$ and $x(-z)$ are disjoint sets, so they are joint with an edge in G. Furthermore, if z_0 and z_1 are very close to each other (distance at most 2ε), then $x(z_0)$ and $x(-z_1)$ are still disjoint sets (and so joined with an edge in G).

Proof of the Lovász–Kneser theorem





For every point z of the sphere we defined $x(z)$ as a vertex of the Kneser graph G, namely the one made of the k points of P closest to z. We found that if $\rho(z_0, z_1) \leq 2\varepsilon$ then $\{x(z_0), x(-z_1)\}$ is an edge.

Now regard $+x(z_0)$ and $-x(-z_1)$ as vertices in the simplicial complex $B(G)$. Define $F(z)$ as the average of $((+x(z_0)) + (-x(-z_0)))/2$, where z_0 is taken uniformly from a ball of radius ε around z. This is in the box complex $B(G)$ because for each $+x(z_0)$ term and each $-x(-z_1)$ term, $\{x(z_0), x(-z_1)\}$ is an edge of the graph.

Clearly F is also continuous and $-F(z) = F(-z)$. Thus, we found a $\mathbb{Z}_2$-map $F : \mathbb{S}^{n-2k} \to B(G)$, that is, $n - 2k \leq \mathrm{coind}(B(G))$. Thus, like we've seen earlier, the Borsuk–Ulam theorem implies there is no map $B(G) \to B(K_{n-2k+1})$, thus no $G \to K_{n-2k+1}$ homomorphism.

(This simple variant of Lovász' proof is by I. Bárány.)

Chromatic conjectures such as Hadwiger's conjecture

Some conjectures of combinatorics can be phrased as an upper bound to the chromatic number of some graph.

The most famous of these is **Hadwiger's conjecture:** if a graph G does not contain K_{t+1} as a graph minor, then $\chi(G) \leq t$.

Some others are the **Behzad–Vizing conjecture** and **Hedetniemi's conjecture**, both of which I'll state later. (The Erdős-Faber-Lovász conjecture is also like this, and so was the four color theorem before it was proved, but these two won't come up in this talk.)

Topological relaxations

Hadwiger's conjecture: if a graph G does not contain K_{t+1} as a graph minor, then $\chi(G) \leq t$.

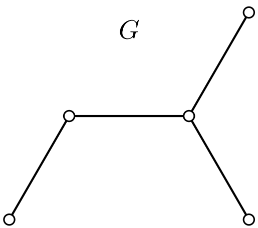
If we can't prove Hadwiger's conjecture, it's natural to try to prove a weakening where we replace the chromatic number with an other graph invariant that's always lower, such as the fractional chromatic number.

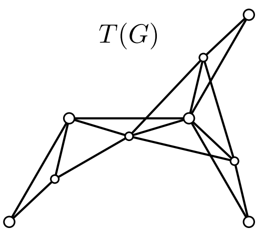
If a graph G does not contain K_{t+1} as a graph minor, then $\chi^*(G) \leq 2t$ (χ^* is the fractional chromatic number). (B. Reed and P. D. Seymour, 1998)

If a graph G does not contain K_{t+1} as a graph minor, then $1 + \operatorname{coind}(B_0(G)) \leq 2t$. (G. Simonyi and G. Tardos, 2006)

(Sadly these contain an extra factor of two, but the original Hadwiger conjecture is not yet solved even up to such a constant factor.)

The Behzad–Vizing conjecture





If G is a graph, let us define its **total graph** $T(G)$ that has both the nodes and edges of G as its nodes. The edges of $T(G)$ are pairs of adjacent nodes or adjacent edges in G, and also pairs of a node of G with one of its endpoints.

The **Behzad–Vizing** conjecture states that for any simple graph G with maximum degree $\Delta(G)$, $\chi(T(G)) \leq 2 + \Delta(G)$.

A relaxation using the fractional chromatic number is known (K. Kilakos and B. Reed, 1993): $\chi^*(T(G)) \leq 2 + \Delta(G)$.

Our first main result is a topological relaxation of the conjecture: $1 + \mathrm{coind}(B_0(T(G))) \leq 2 + \Delta(G)$.

Our tool: the $K_{l,m}$ theorem

For any G simple graph, $1 + \mathrm{coind}(B_0(T(G))) \leq 2 + \Delta(G)$, where $T(G)$ is the total graph of G and $\Delta(G)$ is the maximum degree. How did we prove this? We used the following theorem.

$K_{l,m}$ theorem (P. Csorba, C. Lange, I. Schurr, A. Waßmer, 2004). If $t \leq 1 + \mathrm{coind}(B_0(G))$, then the graph G contains a $K_{k,l}$ complete bipartite graph as a subgraph for every natural numbers k, l such that $t = k + l$.

We used elementary arguments to show that $T(G)$ does not contain $K_{2,\Delta+1}$ as a subgraph (except in some exceptional cases).

This theorem is, by the way, also what Simonyi and Tardos use to prove the previously mentioned fractional relaxation of Hadwiger's conjecture.

Hedetniemi's conjecture

If F and G are graphs, their **categorical product** $F \times G$ is the graph with the vertex set $V(F) \times V(G)$, with the edges being all such $\{(u_F, u_G), (v_F, v_G)\}$ that $\{u_F, v_F\}$ is an edge in F and $\{u_G, v_G\}$ is an edge in G.

Hedetniemi's conjecture: For any two graphs F and G,

$$\min(\chi(F), \chi(G)) = \chi(F \times G).$$

The part that $\chi(F \times G) \leq \min(\chi(F), \chi(G))$ is easy, because there exists graph homomorphisms from $(F \times G)$ to F and to G.

Fractional relaxations of this conjecture are proved by C. Tardif (2001, 2005) and by X. Zhu (2010).

Our second main result will be a **topological relaxation** of Hedetniemi's conjecture. Before ours, another topological relaxation had been given by P. Hell (1977): this uses a lower bound different from the two I've mentioned in this talk.

Why is it called categorical product?

We defined it combinatorially, but the **categorical product** also has a different description.

The categorical product of two graphs F and G is a graph $F \times G$ such that

- there exist graph homomorphisms (projections) $i_1 : (F \times G) \to F$ and $i_2 : (F \times G) \to G$, and
- for any graph H, with homomorphisms $f : H \to F$ and $g : H \to G$, then there exists a unique homomorphism $h : H \to (F \times G)$ such that $i_1 \circ h = f$ and $i_2 \circ h = g$.

As mentioned, the first condition is what makes sure that $\chi(F \times G) \leq \min(\chi(F), \chi(G))$.

Reformulation of Hedetniemi's conjecture

- We have $i_1 : (F \times G) \to F$; $i_2 : (F \times G) \to G$.
- If $f : H \to F$ and $g : H \to G$, then there exists a unique $h : H \to (F \times G)$ such that $h \circ i_1 = f$ and $h \circ i_2 = g$.

The following statement is **equivalent to Hedetniemi's conjecture**. For any t, there exist an infinite sequence of test graphs $T_0, T_1, \ldots, T_k, \ldots$ such that

- there exist homomorphisms $T_0 \leftarrow \cdots \leftarrow T_k \leftarrow T_{k+1} \leftarrow \cdots$, and
- for any graph G, $t \leq \chi(G)$ iff there is a homomorphism $T_k \to G$ for some k.

Proof that this implies Hedetniemi's conjecture. If F and G cannot be $(t-1)$-colored then there exist $f_1 : T_k \to F$ and $g : T_l \to G$. Suppose $k \leq l$. Let $f_0 : T_l \to T_k$. Then $f = f_1 \circ f_0 : T_l \to F$. Thus f and g induce a homomorphism $h : T_l \to (F \times G)$, so $F \times G$ cannot be t-colored either.

The topological relaxation.

The **topological relaxation of Hedetniemi's conjecture** we proved is the following. For any two graphs F and G,

$$\min(2+\mathrm{coind}(B(F)), 2+\mathrm{coind}(B(G))) = 2+\mathrm{coind}(B(F\times G)).$$

We can repeat the previous argument if we replace the chromatic number by the topological bound $(2+\mathrm{coind}(B))$ to get an **equivalent of the topological relaxation** we need to prove, namely the following.

For any t, there exist an infinite sequence of test graphs $T_0, T_1, \ldots, T_k, \ldots$ such that

- there exist homomorphisms $T_0 \leftarrow \cdots \leftarrow T_k \leftarrow T_{k+1} \leftarrow \cdots$, and
- for any graph G, $t \leq 2+\mathrm{coind}(B(G))$ iff there is a homomorphism $T_k \to G$ for some k.

The test graphs we need: Borsuk graphs

Luckily, Simonyi and Tardos gives a suitable sequence of test graphs in a previous article.

The **Borsuk graph** $B(t, \alpha)$ is an infinite graph whose vertices are the points of the unit sphere $\mathbb{S}^{t-1}$, and two vertices are connected iff their Euclidean distance is at least α. (Here, $\alpha < 2$ is a parameter.)

G. Simonyi and G. Tardos (2006) has proved the following: For any graph G, $t \leq 1 + \mathrm{coind}(B(G))$ if and only if there is a graph homomorphism $B(t, \alpha) \to G$ for some α.

Thus we can choose $T_k = B(t, 2 - 1/k)$ as the testgraphs. Clearly, whenever $\alpha_1 < \alpha_2$, $B(t, \alpha_2)$ is a subgraph of $B(t, \alpha_1)$, so the required homomorphisms $T_k \leftarrow T_{k+1}$ exist.

Remark. The Borsuk graphs are infinite, but it is possible to reduce them to a sequence of finite graphs. Also, A. Dochtermann, C. Schultz (2010) have found a different sequence of test graphs.

Thank you for your attention