# BAIRE PROPERTY IMPLIES CONTINUITY FOR SOLUTIONS OF FUNCTIONAL EQUATIONS — EVEN WITH FEW VARIABLES

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Abstract. It is proved that — under certain conditions — solutions $f$ of the functional equation

$$f(x) = h(x, y, f(g_1(x,y)), \dots, f(g_n(x,y))), \quad (x,y) \in D \subset \mathbb{R}^n \times \mathbb{R}^l$$

having Baire property are continuous, even if $1 \le l \le n$. As a tool we introduce new function classes which — roughly speaking — interpolate between Baire property and continuity.

## 1. Introduction

In connection with his fifth problem Hilbert [6] suggested that although the method of reduction to differential equations makes it possible to solve functional equations in an elegant way, the inherent differentiability assumptions are typically unnatural (see [2]). Such shortcomings can be overcome by appealing to regularity theorems.

In this spirit the following general regularity problem for functional equations with two variables and without iteration was formulated by the author and included by Aczél among the most important open problems on functional equations (see Aczél [1] and Járai [8]):

**1.1. Problem.** *Let $X$ and $Z$ be open subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let $D$ be an open subset of $X \times X$. Let $f : X \to Z$, $g_i : D \to X$ $(i = 1, 2, \dots, n)$ and $h : D \times Z^{n+1} \to Z$ be functions. Suppose that*

*(1)*

$$f(x) = h(x, y, f(y), f(g_1(x,y)), \dots, f(g_n(x,y))) \text{ whenever } (x,y) \in D;$$

*(2) $h$ is analytic;*

*(3) $g_i$ is analytic and for each $x \in X$ there exists a $y$ for which $(x,y) \in D$ and $\dfrac{\partial g_i}{\partial y}(x,y)$ has rank $s$ $(i = 1, 2, \dots, n)$.*

*Is it true that every $f$ which is measurable or has the Baire property is analytic?*

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The following steps can be used:

(I) Measurability implies continuity.

(II) Baire property implies continuity.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All $p$ times continuously differentiable solutions are $p+1$ times continuously differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

The complete answer to the problem above is not known. The author discussed this problem in several papers and solved problems corresponding to (I), (II), (IV) and (V) (see [8]), and under some additional compactness condition (III) (see [9]). References can be found in the survey paper [16]. There are some partial results in connection with (VI). Moreover, other properties of solutions such as having locally bounded variation or local Hölder continuity are also discussed (see [13] and references in [16]). It is possible to extend these results to manifolds, and the $\mathcal{C}^\infty$-part of the problem is completely solved on compact manifolds [12]. The most applicable results are treated in the booklet [11].

Regularity theorems of the type "locally integrable solutions are infinitely many times differentiable" can be obtained using distributions. The essence of the method is to prove that solutions in the distribution sense satisfy a differential equation having only infinitely many times differentiable solutions. This idea was used by Światak [18] to prove general regularity results for the functional equation

$$\sum_{i=1}^{n} h_i(x,y) f(g_i(x,y)) = h(x, f(g_{n+1}(x)), \dots, f(g_m(x))) + h_0(x,y),$$

where $f$ is the only unknown function. Roughly speaking, she applies a partial differential operator in $y$ to the equation in the distribution sense. Of course, the nonlinear term on the right hand side disappears. If, after substituting a fixed $y_0$, we are fortunate enough to obtain a hypoelliptic partial differential equation, then by the regularity theory of partial differential equations all distribution solutions are in $\mathcal{C}^\infty$. For the exact details of how to overcome the difficulties and for applications see her paper [18].

Further references about regularity theorems for functional equations can be found in the survey paper [16]. Some other papers concerning the distribution method are also referred to there.

The above equation of Światak is "almost linear", so, formally, it is much less general than equation (1). However her theorems can be applied even if the rank of $\frac{\partial g_i}{\partial y}$ is much smaller than the dimension of the domain of the unknown function $f$. Roughly speaking, the present author's results, quoted above, may be applied to prove regularity of a solution $f$ having $m$ variables, only if there are at least $2m$ variables in the functional equation. The method of Światak may be applied even if there are only $m+1$ variables. This is the minimal number of variables: in Hilbert's paper [6] there is an example that for "one variable" functional equations (this may mean an $m$-dimensional vector variable) no regularity theorem holds. So the results of Światak suggest that the rank condition in the problem above is too strong, and the results concerning the above problem can be extended for a much more general case.

Such "measurability implies continuity" type results were treated recently in Járai [14]. Well-known analogies between measurability and Baire property (see Oxtoby [17]) suggest to try to prove analogous results for Baire property. Differences, such as the lack of "$\varepsilon$-technique", convergence in measure and theorems connected with it (for example Riesz theorem), Hausdorff measure, etc., shows that we need a separate treatment.

In this paper we will prove a "Baire property implies continuity" type result for the general explicit nonlinear functional equation (1) without the strong rank condition in (3) to the inner functions. All earlier "Baire property implies continuity" type results that I know of use the strong rank condition in (3) or some abstract version of it. In the spirit of the "bootstrap" method corresponding to steps (I)–(VI) we introduce a sequence of properties, which — roughly speaking — are between Baire property and continuity. This sequence of properties gives a stairway to climb up from Baire property to continuity. First we shall investigate the basic properties of the new notions. Then the regularity theorem will be proved. An example is given how to apply the theorem in nontrivial cases. A refinement of the theorem is also proved. Finally, further properties of the new notions are investigated.

## 2. The new notions

**2.1. Notations.** If $f$ is a function, $\operatorname{rng} f$ denotes the *range* of $f$. All *normed spaces* are supposed to be real; the norm is denoted by $|\ |$. If $f : D \to Y$ is a function mapping an open subset of a normed space into a normed space, then $f'$ shall denote the *derivative* of $f$. If $D \subset X_1 \times X_2 \times \ldots \times X_n$, we shall use the *partial sets*

$$D_{x_i} = \big\{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) : (x_1, ..., x_n) \in D\big\}.$$

The *partial functions* $f_{x_i} : D_{x_i} \to Y$ are defined by

$$f_{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_n)$$

whenever $(x_1, \ldots, x_n) \in D$ (notice that $x_i$ is held constant in $f_{x_i}$). Also $D_{x_{i_1}, \ldots, x_{i_r}}$ and $f_{x_{i_1}, \ldots, x_{i_r}}$ are defined similarly. Now, if $X_i$ and $Y$ are normed spaces and

$$D_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$$

is an open subset of $X_i$ we define the *partial derivative* denoted by

$$\partial_i f, \quad \partial_{x_i} f \quad \text{or} \quad \frac{\partial f}{\partial x_i}$$

as the derivative of $f_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$. Other notions of calculus are used in the usual way.

Concerning topology we follow the terminology and notations of Bourbaki [3]. The most important facts concerning *Baire category* can be found in Bourbaki [3]; see Chapter IX, § 5, and the corresponding exercises, but here we shall use the different (and more usual) terminology of Oxtoby [17]. For clearity we summarise the notions and facts we shall use here.

We will say that a subset $A$ of a topological space $X$ *is of first category*, if $A$ can be represented as a countable union of nowhere dense sets, otherwise $A$ is of *second category*. Let $E$ be a subset of the topological space $X$ and let $D(E)$ denote the set of all points of $X$ such that for each neighbourhood $V$ the set $V \cap E$ is of second category. Then $D(E) = \emptyset$ if and only if $E$ has first category. Moreover $D(E)$ is closed and the set $E \backslash D(E)$ is of first category.

$X$ is called a *Baire space* if every nonvoid open subset of $X$ is of second category. We will say that $E \subset X$ has the Baire property if there exists an open set $V$ such that the symmetric difference $E \triangle V$ is of first category. All subsets of $X$ having Baire property form a $\sigma$-algebra. Of course this $\sigma$-algebra contains Borel sets, the members of the smallest $\sigma$-algebra containing all open sets. A set $E \subset X$ has the Baire property if and only if $E \backslash D(E)$ has first category.

Suppose that $X$ is a topological space, $E \subset X$. The set $E$ has Baire property if and only if each point $x$ of $X$ has an open neighbourhood $U$ such that $U \cap E$ has the Baire property in $X$.

Let $X$ and $Y$ be topological spaces and suppose that the topology of one of them has a countable base. A subset $E \times F$ of $X \times Y$ has the Baire property if and only if one of the sets $E$, $F$ is of first category or both of them have Baire property.

Combining these facts with the proof in Oxtoby [17], Chapter 15, we get the following form of a well-known theorem of Kuratowsky and Ulam:

**Theorem.** [Kuratowsky, Ulam] *Let $X$ and $Y$ be topological spaces, and suppose that $Y$ has a countable base. Let $E$ be a subset of $X \times Y$ having the Baire property. Then except for a set of points $x$ of $X$ which is of first category the set $E_x$ has the Baire property. Moreover $E$ is of first category if and only if the set $E_x$ is of first category in $Y$ with the exception of a set of $x$'s of first category.*

The function $f$ has the *Baire property on $E$* if the domain of $f$ contains $E$ except for a set of first category, the range of $f$ is in a topological space $Y$ and $E \cap f^{-1}(W)$ has the Baire property in $X$ for every open subset $W$ of $Y$. We simply say that $f$ has the Baire property, if it has the Baire property on $X$.

This definition is very similar to the definition of a Borel function. A function $f$ mapping some subset of a topological space $X$ into another topological space $Y$ is called a *Borel function*, if for each open subset $V$ of $Y$ the set $f^{-1}(V)$ is a Borel subset of $X$.

The properties of functions having the Baire property are very similar to the properties of measurable functions. We shall use the following statements.

If $f$ is any function defined on a subset of $X$ having the Baire property then all subsets $B$ of $Y$ for which $f^{-1}(B)$ has the Baire property form a $\sigma$-algebra. Hence if $f$ has values in the topological space $Y$ and has the Baire property on a subset of the topological space $X$, moreover $g$ is a Borel function on a subset of the topological space $Y$, then $g \circ f$ has the Baire property on its domain.

Suppose that $Y = \prod_i Y_i$ is a countable product of spaces each having a countable base of topology. A function $f$ mapping a subset of a topological space $X$ into $Y$ has the Baire property if and only if all functions $p_i \circ f$ have the Baire property where $p_i$ is the natural projection of $Y$ onto $Y_i$.

Suppose that $X$ is a topological space, $Y$ a metric space, and the functions $f$, $f_n$ $(n = 1, 2, \dots)$ are defined on $X$ except for a set of first category (depending on the function), have values in $Y$ and have the Baire property. Then the set

$$E = \{x : f_n(x) \to f(x)\}$$

has the Baire property. Indeed, the sets $E_{n,m} = \{x : \operatorname{dist}(f_n(x), f(x)) < 1/m\}$ have the Baire property and $E$ differs from the set

$$\cap_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_{k,m}$$

only in a set of first category.

Suppose that $X$ is a topological space, $Y$ is a metric space, the functions $f$, $f_n$ $(n = 1, 2, \dots)$ are mappings from $X$ into $Y$, the functions $f_n$ have the Baire property for all $n$, and $f_n(x) \to f(x)$ for all $x \in X$ except for a set of first category. Then $f$ has the Baire property, too. Indeed, if $V$ is an open subset of $Y$ and $V_i$ is the open subset of $Y$ containing those points of $Y$ having distance larger than $1/i$ from $Y \setminus V$, then $f^{-1}(V)$ differs from the set

$$\cup_{i=1}^{\infty} \cup_{j=1}^{\infty} \cap_{k=j}^{\infty} f_k^{-1}(V_i)$$

only in a set of first category.

The following theorem is the analogue of Luzin's theorem:

Let $f$ be a mapping of the topological space $X$ into the topological space $Y$ and suppose that there exists a subset $F$ of $X$ of first category such that the restriction of $f$ to the set $X \setminus F$ is continuous. Then $f$ has the Baire property. Conversely, this condition is necessary, if the topology of $Y$ has a countable base.

From the Kuratowsky-Ulam theorem the following analogue of Fubini's theorem follows directly:

If $X$, $Y$ and $Z$ are topological spaces, $Y$ and $Z$ have countable bases, $f : X \times Y \to Z$ has the Baire property, then except for a set of points $x$ which is of first category, the function $f_x$ has the Baire property.

**2.2. Definition.** Let $X$ be a set, $Y$ a metric space, and $f : X \to Y$ be a function. Let $U$ be a topological space, and $P$ a topological space, the "parameter space" with a given point $p_0 \in P$. Let $\varphi$ be a function from $U \times P$ into $X$. We shall think of $\varphi$ as a surface $\varphi_p : u \mapsto \varphi(u, p)$ for each $p$, depending on the parameter $p$. The analogue of Luzin's theorem above and generalizations of the theorem of Piccard (see [10]) suggest that the following condition is connected with the Baire property:

(S) *For each sequence $p_m \to p_0$ we have $f(\varphi(u, p_m)) \to f(\varphi(u, p_0))$ except for a set of first category of points $u \in U$.*

For our investigations we need the following property:

(B) *$u \mapsto f(\varphi(u, p_0))$ has the Baire property.*

We shall often check conditions (S) and (B) locally. If for each $u_0 \in U$ there is a neighbourhood $U_0$ of $u_0$ and $P_0$ of $p_0$ such that $\varphi|U_0 \times P_0$ satisfies (S), then $\varphi$ also satisfies (S). This easily follows from the locality of first category mentioned in 2.1. Similarly, if for each $u_0 \in U$ there is a neighbourhood $U_0$ of $u_0$ and $P_0$ of $p_0$ such that $\varphi|U_0 \times P_0$ satisfies (B), then $\varphi$ satisfies (B).

Let $X$ be an open subset of $\mathbb{R}^n$ and $0 \le k \le n$. The class of all functions $f$ for which the condition (S) [(B)] is satisfied whenever $U$ is an open subset of $\mathbb{R}^k$, $P$ is an open subset of some Euclidean space, $p_0 \in P$ and $\varphi : U \times P \to X$ is a $\mathcal{C}^1$-function for which $\varphi_p$ is an immersion of $U$ into $X$ for each $p \in P$, will be denoted by $\mathcal{S}_k(X, Y)$ [$\mathcal{B}_k(X, Y)$] or shortly by $\mathcal{S}_k$ [$\mathcal{B}_k$]. (We take $\mathbb{R}^0 = \{0\}$). It is clear that $f \in \mathcal{B}_k$ if and only if the condition

(B′) *$f \circ \psi$ has the Baire property*

is satisfied whenever $\psi$ is an immersion of some open subset $U$ of $\mathbb{R}^k$ into $X$.

**2.3. Remarks.** (1) Our main results will show that, roughly speaking, solutions $f$ from $\mathcal{S}_{k+1}$ are also in $\mathcal{S}_k$. We shall prove that $\mathcal{S}_0$ is the class of continuous functions, and that all functions $f : X \to Y$ from the open subset $X \subset \mathbb{R}^n$ into some second countable space $Y$ and having the Baire property are in $\mathcal{S}_n$. Hence, step-by-step, Baire property of solutions implies their continuity.

(2) The analogy with the measure theoretical case is remarkable but not complete. About the history of the analogous measure theoretical notions see some references in Járai [14].

(3) Solutions of functional equations having the Baire property were studied by several authors. The generalized Cauchy equation

$$f(g(x, y)) = h(x, y, f_1(x), f_2(y))$$

was studied the most. See the references in Járai [8]. A "sequential approach" was used by Grosse-Erdmann [4] and much earlier by Haupt [5]. The results of Grosse-Erdmann can be applied to prove that for the functional equation

$$f(g(x, y)) = h(y, f_1(x))$$

with unknown functions $f$, $f_1$ — under suitable conditions — Baire property of $f_1$ implies the continuity of $f$. He applies his abstract results for the case where $(x, y) \in D$, where $D$ is some open subset of $\mathbb{R}^n \times \mathbb{R}^n$, $g : D \to \mathbb{R}^n$ and $\det \frac{\partial g}{\partial x}$ and $\det \frac{\partial g}{\partial y}$ are nonzero. His method has the advantage that one only needs the continuity of $h$ with respect to the second variable. Note that substituting $t = g(x, y)$ we have locally

$$f(t) = h(y, f_1(g_1(t, y)));$$

compare this with problem 1.1.

(4) The class $\mathcal{S}_k$ $[\mathcal{B}_k]$ remains the same if we suppose only that (S) [(B)] is satisfied whenever $U$ is an open subset of $\mathbb{R}^k$, $P$ is an open subset of some Euclidean space, $p_0 \in P$ and $\varphi : U \times P \to X$ is a $\mathcal{C}^1$-function for which $\varphi_p$ is an embedding of $U$ into $X$ for each $p \in P$. This easily follows from the locality principle mentioned in the definition. Similarly, supposing only that $\varphi_{p_0}$ is an immersion, the resulting class $\mathcal{S}_k$ $[\mathcal{B}_k]$ remains the same.

**2.4. Theorem.** *Let $Y$ be a topological space and $X$ an open subset of $\mathbb{R}^n$. Then $\mathcal{B}_0(X, Y) = Y^X$ and $\mathcal{S}_0(X, Y) = \mathcal{C}(X, Y)$, the class of continuous functions from $X$ into $Y$.*

**Proof.** We shall use the notations of the definition. It is trivial that $\mathcal{B}_0$ contains all functions from $X$ into $Y$.

Now let us prove that any continuous function $f : X \to Y$ is in $\mathcal{S}_0$. Since $U = \emptyset$ or $U = \{0\}$, clearly the function $p \mapsto f(\varphi(u, p))$ is continuous for each $u \in U$. This implies $f \in \mathcal{S}_0$.

The converse is proved by contradiction: if $f \in \mathcal{S}_0$, but not continuous, then there exists an $x_0 \in X$, a sequence $x_n \to x_0$, and a neighbourhood $W$ of $f(x_0)$ such that $f(x_n) \notin W$. Let $U = \{0\}$, $P = X$, $p_0 = x_0$, $\varphi(0, p) = p$ for $p \in P$. Choosing the sequence $p_m = x_m$ we have

$$f(\varphi(0, p_m)) = f(x_m) \to f(x_0) = f(\varphi(0, p_0))$$

hence we obtain a contradiction.

We shall prove that functions having the Baire property over an open subset $X$ of $\mathbb{R}^n$ are in $\mathcal{S}_n$. To make the connection with earlier results in [8] clear, we do the main part of the proof in the following abstract setting:

**2.5. Theorem.** *Let $P$, $U$ and $X$ be topological spaces. Suppose that $\varphi$ : $U \times P \to X$ is a continuous function with the following property:*

*(1) If $p \in P$ and $A \subset U$ is of second category then $\varphi_p(A)$ is also of second category.*

*Suppose, moreover, that $p_0 \in P$ and $f$ has values in a topological space and the restriction of $f$ to the complement of some subset of first category of $X$ is continuous. Then for $U$, $P$, $p_0$, $\varphi$ and $f$ the conditions (S) and (B) are satisfied.*

**Proof.** Let us first prove that (B) is satisfied. Let $F$ be a set of first category for which $f|X\backslash F$ is continuous. We may suppose that $F$ is a Borel set. Let $V$ be any open subset of $Y$. Since the set $A = (f|X\backslash F)^{-1}(V)$ is relatively open in $X\backslash F$, it is a Borel subset of $X$. The set $F$ is of first category hence by (1) the set $N = (f|F)^{-1}(V)$ is also of first category. Now let us observe that

$$(f \circ \varphi_p)^{-1}(V) = \varphi_p^{-1}(A) \cup \varphi_p^{-1}(N).$$

On the left hand side, $\varphi_p^{-1}(A)$ is a Borel set and by condition (1), the set $\varphi_p^{-1}(N)$ is of first category. This means that (B) is satisfied.

Now we will show that (S) is satisfied. With the set $F$ above we have that $\varphi_{p_m}^{-1}(F)$ is of first category for $m = 0, 1, 2, \ldots$. Let $E$ be the union of all these sets. If $u \in U\backslash E$, then $\varphi(u, p_m)$ and $\varphi(u, p_0)$ are in $X\backslash F$ and $\varphi(u, p_m) \to \varphi(u, p_0)$. Hence we have $f(\varphi(u, p_m)) \to f(\varphi(u, p_0))$. This proves (S).

**2.6. Theorem.** *Let $X$ be an open subset of $\mathbb{R}^n$. If $Y$ is a topological space having countable base then every function $f : X \to Y$ having the Baire property is contained in $\mathcal{S}_n(X, Y)$ and $\mathcal{B}_n(X, Y)$.*

**Proof.** By the analogue of Luzin's theorem from 2.1, there is a subset $F$ of first category of $X$ such that $f|X\backslash F$ is continuous. Let $U \subset \mathbb{R}^n$ be open, $P$ an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ a $\mathcal{C}^1$ function for which each $\varphi_p$, $p \in P$ is an embedding. We shall apply the previous theorem for $\varphi$ locally. Let $u_0 \in U$. Choosing a neighbourhood $U_0$ of $u_0$ and $P_0$ of $p_0$ such that $\varphi_p$ is a homeomorphism of $U_0$ onto an open subset of $X$ for each $p \in P_0$, we obtain that for any subset $A$ of $U_0$ which is of second category, the image $\varphi_p(A)$ is also of second category.

Now, the previous theorem can be applied to $\varphi|U_0 \times P_0$. As it was mentioned at the definition this is enough to prove that (S) and (B) are satisfied for $f$, $U$, $P$, $p_0$, $\varphi$.

## 3. The main results

**3.1. Theorem.** *Let $Z$, $Z_i$ $(i = 1, 2, \ldots, n)$ be topological spaces. Let $X_i$ $(i = 1, 2, \ldots, n)$ and $X$ be open subsets of Euclidean spaces and let $Y \subset \mathbb{R}^l$ be open. Let $D$ be an open subset of $X \times Y$. Consider the functions $f : X \to Z$, $f_i : X_i \to Z_i$, $h : D \times Z_1 \times \ldots \times Z_n \to Z$, $g_i : D \to X_i$ $(i = 1, 2, \ldots, n)$. Let $U \subset \mathbb{R}^k$ be open, $P$ be an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ a $\mathcal{C}^1$-function, for which $\varphi_p$ is an immersion of $U$ into $X$ for all $p \in P$, and suppose that the following conditions hold:*

*(1) For each $(x, y) \in D$*

$$f(x) = h\left(x, y, f_1\left(g_1(x, y)\right), \ldots, f_n\left(g_n(x, y)\right)\right);$$

*(2) for each fixed $y \in Y$, $h$ is continuous in the other variables;*

*(3) the function $f_i$ is in $\mathcal{S}_{k+l}$ on $X_i$ $(i = 1, 2, \ldots, n)$;*

*(4)* $g_i$ *is* $\mathcal{C}^1$ *on* $D$ $(i = 1, 2, \ldots, n)$;

*(5) for each* $u_0 \in U$ *there exists a* $y_0$ *such that* $(\varphi(u_0, p_0), y_0) \in D$ *and the rank of the derivative of*

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

*at* $(u_0, y_0)$ *is* $k + l$ *for each* $1 \leq i \leq n$.

*Then condition (S) is satisfied for* $f$, $U$, $P$, $p_0$, $\varphi$.

**Proof.** Suppose that $p_m \to p_0$. Let us choose open neighbourhoods $U_0$, $P_0$, $Y_0$ of $u_0$, $p_0$, $y_0$ such that $(\varphi(u, p), y)$ is in $D$ whenever $u \in U_0$, $p \in P_0$, $y \in Y_0$, moreover, the rank of the derivative of the mapping $(u, y) \mapsto g_i(\varphi(u, p), y)$ is equal to $k + l$ for all $u \in U_0$, $p \in P_0$, $y \in Y_0$ and for $1 \leq i \leq n$. This is possible, because $D$ is open, $g_i$ and $\varphi$ are $\mathcal{C}^1$-functions, the rank is lower semicontinuous and $U \times Y$ has dimension $k + l$, hence the rank cannot increase above $k + l$.

Since the function $f_1$ is in $\mathcal{S}_{k+l}$, we have that, except for pairs $(u, y) \in U_0 \times Y_0$ from a set $E_1$ of first category,

$$f_1(g_1(\varphi(u, p_m), y)) \to f_1(g_1(\varphi(u, p_0), y)).$$

Now using that $f_2$ is in $\mathcal{S}_{k+l}$ we obtain that, except for pairs $(u, y) \in U_0 \times Y_0$ from a set $E_2$ of first category

$$f_2(g_2(\varphi(u, p_m), y)) \to f_2(g_2(\varphi(u, p_0), y)),$$

etc. Finally, we obtain that except for a set $E = \cup_{i=1}^n E_i$ of pairs $(u, y) \in U_0 \times Y_0$ of first category we have

$$f_i(g_i(\varphi(u, p_m), y)) \to f_i(g_i(\varphi(u, p_0), y))$$

for $i = 1, 2, \ldots, n$. By the theorem of Kuratowski and Ulam, except for a set of first category of $y$'s from $Y_0$ we have that the set of all $u \in U_0$ for which $(u, y) \in E$ is of first category. Fixing any such $y$, from the functional equation and from the continuity of $h$ for fixed $y$ we obtain that

$$f(\varphi(u, p_m)) \to f(\varphi(u, p_0)),$$

except for a set of $u$'s which is of first category. This is condition (S) with the function $\varphi | U_0 \times P_0$.

By the remark in the definition we obtain that (S) is satisfied.

The following example is from [14].

**3.2. Example.** Let us consider the following example:

$$\sum_{i=0}^{n} a_i(x, y) f(x + g_i(y)) = 0$$

whenever $x \in \mathbb{R}^m$, $y \in \mathbb{R}$. Suppose that the functions $a_i : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous and the functions $g_i : \mathbb{R} \to \mathbb{R}^m$ are in $\mathcal{C}^1$. Introducing the variable $x_j = x + g_j(y)$ instead of $x$, we obtain

$$f(x_j) = -\sum_{i \neq j} \frac{a_i(x_j - g_j(y), y)}{a_j(x_j - g_j(y), y)} f(x_j - g_j(y) + g_i(y)). \tag{1}$$

To see that condition (5) is satisfied we have to check the rank of the matrix

$$\begin{pmatrix} \frac{\partial \varphi_{p_0}^{(1)}}{\partial u_1}(u) & \dots & \frac{\partial \varphi_{p_0}^{(1)}}{\partial u_k}(u) & \frac{d\, g_i^{(1)}}{d\, y}(y) - \frac{d\, g_j^{(1)}}{d\, y}(y) \\ \vdots & & \vdots & \vdots \\ \frac{\partial \varphi_{p_0}^{(m)}}{\partial u_1}(u) & \dots & \frac{\partial \varphi_{p_0}^{(m)}}{\partial u_k}(u) & \frac{d\, g_i^{(m)}}{d\, y}(y) - \frac{d\, g_i^{(m)}}{d\, y}(y) \end{pmatrix},$$

where $\varphi_p^{(i)}$ are the coordinate functions of $\varphi_p$. If this is $k+1$, then we may apply our theorem with $l = 1$. This means, geometrically, that the vector $g_i'(y) - g_j'(y)$ is not contained in the range of the linear operator $\varphi_{p_0}'(u)$ (which is known to be $k$-dimensional). This range can be any $k$-dimensional linear subspace in $\mathbb{R}^m$. It may happen that for each $k$-dimensional linear subspace, there exists a $y \in \mathbb{R}$ such that none of the vectors $g_i'(y) - g_j'(y)$, $i \neq j$ is contained in the linear subspace. Then our theorem can be applied directly and proves that $f \in \mathcal{S}_{k+1}$ implies $f \in \mathcal{S}_k$. If this is the case for $k = m-1, m-2, \dots, 0$ then we obtain that every solution having the Baire property is continuous. But there are situations when this is not the case. If, for example, the derivative of the functions $g_i$ is constant, i. e. if $g_i(y) = b_i + yc_i$, then for any fixed $j$, equation (1) cannot be applied to get $f \in \mathcal{S}_k$ from $f \in \mathcal{S}_{k+1}$, because for some $\varphi$'s the range of $\varphi_{p_0}'(u)$ will contain some of the vectors $g_i'(y) - g_j'(y) = c_i - c_j$. But we have the possibility to use *any* of the equations (1). Using the locality mentioned in the definition, it is enough to prove that for any $k$-dimensional linear subspace of $\mathbb{R}^n$ there exists a $j$ such that none of the vectors $c_i - c_j$, $i \neq j$ is contained in the given subspace. For example this is the situation if $n \geq m$ and the vectors $c_0, \dots, c_n$ are in general position. If this condition is not satisfied, then it is still possible that our theorem can be applied. A similar (but somewhat simpler) situation was studied in the paper [15], in the proof of Theorem 2.3.

**3.3. Remark.** Although, as the example above shows, Theorem 3.1 can be applied in several cases, it is not satisfying because condition (5) is too strong. If we want to apply theorem 3.1 to prove that $f \in \mathcal{S}_k$ then $\varphi$ can be arbitrary. Hence condition (5) implicitly means that the rank of $\frac{\partial g_i}{\partial x}$ has to be large, even if $\frac{\partial g_i}{\partial y}$ has a large rank. This in practice means that the $g_i$ have to depend on all coordinates of $x$, which is not comfortable. We want to relax this condition. Instead of supposing that

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

has maximal possible rank $k + l$ at $(u_0, y_0)$ we shall only suppose that it has a constant rank $k_i$ (depending on $i$) on a neighbourhood of $(u_0, p_0, y_0)$. But in this case we have to work with functions from $\mathcal{S}_k \cap \mathcal{B}_k$, and, roughly speaking, our theorem says that solutions in $\mathcal{S}_{k+1} \cap \mathcal{B}_{k+1}$ are also in $\mathcal{S}_k \cap \mathcal{B}_k$.

First we only deal with the Baire-type condition (B). We shall use the following lemma to prove that condition (B) for the unknown functions $f_i$ implies condition (B) for $f$.

**3.4. Lemma.** *Let $X$ be an open subset of $\mathbb{R}^n$, $Y$ a topological space, $0 \le k \le n$ and $f \in \mathcal{B}_k(X, Y)$. If $\psi$ is a subimmersion of the open subset $U$ of $\mathbb{R}^m$ into $X$ with rank $k$ of the derivative everywhere, then $f \circ \psi$ has the Baire property.*

**Proof.** The lemma directly follows from the rank theorem. Indeed, the rank theorem implies, that for each $u_0 \in U$ there exists an open neighbourhood $U_0$ such that $\psi|U_0$ can be written as $\alpha \circ p \circ \beta$. Here, with the notation $I = ]-1, 1[$, the mapping $\beta$ is a diffeomorphism of $U_0$ onto $I^m$ such that $\beta(u_0) = 0$, the projection $p$ of $I^m$ into $I^n$ has the form $p(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$, and $\alpha$ is a diffeomorphism of $I^n$ onto an open set $X_0$ mapping $0$ into $x_0 = \psi(u_0)$. Identifying the set $I^k \times \{0\} \subset I^n$ with $I^k$ we have that $\alpha|I^k$ is an immersion, hence $\left(f \circ (\alpha|I^k)\right)^{-1}(V)$ has the Baire property for each open subset $V$ of $Y$. Since $p^{-1}(A)$ has the Baire property for each subset $A$ of $I^k$ which has the Baire property, and $\beta^{-1}(B)$ has the Baire property for each subset $B$ of $I^m$ which has the Baire property, we obtain that $f \circ (\psi|U_0)$ has the Baire property. Now using locality mentioned at the definition of (B), we get the general case.

**3.5. Theorem.** *Let $Z$ be a topological space and let $Z_i$ $(i = 1, 2, \ldots, n)$ be topological spaces having countable bases. Let $X_i$ $(i = 1, 2, \ldots, n)$ and $X$ be open subsets of Euclidean spaces and let $Y \subset \mathbb{R}^l$ be open. Let $D$ be an open subset of $X \times Y$. Consider the functions $f : X \to Z$, $f_i : X_i \to Z_i$, $h : D \times Z_1 \times \ldots \times Z_n \to Z$, $g_i : D \to X_i$ $(i = 1, 2, \ldots, n)$. Let $U \subset \mathbb{R}^k$ be open, $\psi : U \to X$ be a $\mathcal{C}^1$ immersion of $U$ into $X$, and suppose that the following conditions hold:*

*(1) For each $(x, y) \in D$*

$$f(x) = h\left(x, y, f_1\left(g_1(x, y)\right), \ldots, f_n\left(g_n(x, y)\right)\right);$$

*(2) for each fixed $y \in Y$, $h$ is continuous in the other variables;*

*(3) the function $f_i$ is in $\mathcal{B}_{k_i}$ on $X_i$ $(i = 1, 2, \ldots, n)$;*

*(4) $g_i$ is $\mathcal{C}^1$ on $D$ $(i = 1, 2, \ldots, n)$;*

*(5) for each $u_0 \in U$ there exists a $y_0$ such that $(\psi(u_0), y_0) \in D$ and the rank of the derivative of*

$$(u, y) \mapsto g_i(\psi(u), y)$$

*is $k_i$ on a neighbourhood of $(u_0, y_0)$ for each $1 \le i \le n$.*

*Then $u \mapsto f(\psi(u))$ has the Baire property.*

**Proof.** Let us choose an open neighbourhood $U_0$ of $u_0$ and $Y_0$ of $y_0$ such that $(\psi(u), y)$ is in $D$ whenever $u \in U_0$, $y \in Y_0$, moreover, the rank of the derivative of the mapping $(u,y) \mapsto g_i(\psi(u),y)$ is equal to $k_i$ for all $u \in U_0$, $y \in Y_0$, $1 \le i \le n$. This is possible by condition (5). By the previous lemma we obtain that the mapping $(u,y) \mapsto f_i(g_i(\psi(u),y))$ has the Baire property. By the analogue of Fubini's theorem stated in section 2.1, except for a set $E_i$ of $y$'s from $Y_0$ which is of first category, the mapping $u \mapsto f_i(g_i(\psi(u),y))$ has the Baire property on $U_0$. Hence, except for the set $E = \cup_{i=1}^n E_i$, for all $y \in Y_0$ the mapping

$$u \mapsto (\psi(u), f_1(g_1(\psi(u),y)), \dots, f_n(g_n(\psi(u),y)))$$

of $U_0$ into $D_y \times Z_1 \times \cdots \times Z_n$ has the Baire property. Since for any fixed $y$ the function $h$ is continuous in other variables, we obtain that for any fixed $y \in Y_0 \setminus E$ the mapping

$$u \mapsto h(\psi(u), y, f_1(g_1(\psi(u),y)), \dots, f_n(g_n(\psi(u),y)))$$

has the Baire property. This means that $u \mapsto f(\psi(u))$ has the Baire property on $U_0$.

Now by the locality principle mentioned at the definition of (B) the statement follows.

The following theorem is the key to the generalization 3.7 of theorem 3.1.

**3.6. Theorem.** *Let $U \subset \mathbb{R}^m$, $X$ and $P$ be open subsets of Euclidean spaces, $p_0 \in P$, $Y$ a metric space, $\varphi : U \times P \to X$ a $\mathcal{C}^1$ function, for which* $\operatorname{rank} \varphi_p'(u) = k$ *for each $u \in U$, $p \in P$. If $f \in \mathcal{B}_k(X,Y) \cap \mathcal{S}_k(X,Y)$ then the condition (S) is satisfied for $f$, $U$, $P$, $p_0$ and $\varphi$.*

**Proof.** Let $u_0 \in U$, and let $p_j \to p_0$ be a sequence. Since the rank of $\varphi_{p_0}'(u_0)$ is equal to $k$, we may write $u$ as $u = (u_1,u_2) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ such that the determinant of

$$\frac{\partial \varphi}{\partial u_1}(u_0,p_0)$$

is not equal to 0. Hence there exists a neighbourhood $U_1 \times U_2$ of $u_0$ and a neighbourhood $P_0$ of $p_0$ such that the mapping

$$u_1 \mapsto \varphi(u_1,u_2,p)$$

is an immersion of $U_1$ for each $u_2 \in U_2$, $p \in P_0$. Since $f \in \mathcal{S}_k$, for each $u_2 \in U_2$ there exists a subset $F_{u_2}$ of $U_1$ of first category such that if $u_1 \in U_1 \setminus F_{u_2}$ then we have $f(\varphi(u_1,u_2,p_j)) \to f(\varphi(u_1,u_2,p_0))$. By the previous lemma, $u \mapsto f(\varphi(u,p))$ has the Baire property. Hence the set

$$\{(u_1,u_2) \in U_1 \times U_2 : f(\varphi(u_1,u_2,p_j)) \to f(\varphi(u_1,u_2,p_0))\}$$

has the Baire property (see 2.1). By the Kuratowski-Ulam theorem we obtain that its complement is of first category.

**3.7. Theorem.** *Let $Z$ be a topological space and let $Z_i$ $(i = 1, 2, \dots, n)$ be separable metric spaces. Let $X_i$ $(i = 1, 2, \dots, n)$ and $X$ be open subsets of Euclidean spaces and $Y \subset \mathbb{R}^l$ be open. Let $D$ be an open subset of $X \times Y$. Consider the functions $f : X \to Z$, $f_i : X_i \to Z_i$, $h : D \times Z_1 \times \ldots \times Z_n \to Z$, $g_i : D \to X_i$ $(i = 1, 2, \dots, n)$. Let $U \subset \mathbb{R}^k$ be open, $P$ an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ a $\mathcal{C}^1$-function for which each $\varphi_p$, $p \in P$ is an immersion of $U$ into $X$, and suppose that the following conditions hold:*

*(1) For each $(x, y) \in D$*

$$f(x) = h\left(x, y, f_1\left(g_1(x, y)\right), \dots, f_n\left(g_n(x, y)\right)\right);$$

*(2) for each fixed $y \in Y$, $h$ is continuous in the other variables;*
*(3) the function $f_i$ is in $\mathcal{S}_{k_i} \cap \mathcal{B}_{k_i}$ $(i = 1, 2, \dots, n)$;*
*(4) $g_i$ is $\mathcal{C}^1$ on $D$ $(i = 1, 2, \dots, n)$;*
*(5) for each $u_0 \in U$ there exists a $y_0$ such that $(\varphi(u_0, p_0), y_0) \in D$ and the rank of the derivative of*

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

*is $k_i$ on a neighbourhood of the point $(u_0, p_0, y_0)$ for each $1 \le i \le n$.*

*Then the conditions (S) and (B) are satisfied for $f$, $U$, $P$, $p_0$, $\varphi$.*

**Proof.** From Theorem 3.5 it follows that condition (B) is satisfied by $f$, $U$, $P$, $p_0$, $\varphi$. Let us fix an $u_0 \in U$ and let us choose a $y_0$ for $u_0$ by (5). Let us choose open neighbourhoods $U_0$, $P_0$ and $Y_0$ of $u_0$, $p_0$ and $y_0$ such that $(\varphi(u, p), y) \in D$ whenever $u \in U_0$, $p \in P_0$ and $y \in Y_0$, moreover the rank of the derivative of

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

is $k_i$ on $U_0 \times P_0 \times Y_0$ for each $1 \le i \le n$. Now the proof that condition (S) is also satisfied is exactly the same as in Theorem 3.1, but we have to use the previous theorem instead of the definition.

## 4. Further investigation of the new notions

**4.1. Conditions.** In what follows we shall only investigate the situation, where $X$ is a nonvoid open subset of $\mathbb{R}^n$ and $f$ maps $X$ into a separable metric space $Y$, because we want to avoid any difficulties arising only from the poor topology of the range $Y$.

**4.2. Remark.** There is another kind of locality than the one treated after Definition 2.2. We have $f \in \mathcal{S}_k(X, Y)$ if and only if each $x_0 \in X$ has an open neighbourhood $X_0 \subset X$ such that $f|X_0 \in \mathcal{S}_k(X_0, Y)$. The "only if" part is trivial. To prove the "if" part we shall use the notation of Definition 2.1. Let us note that for each point $u_0 \in U$ there exist open neighbourhoods $U_0$ and $P_0$ of $u_0$ and $p_0$, respectively, such that for $x_0 = \varphi(u_0, p_0)$ the set $\varphi(U_0, P_0)$ is contained in $X_0$. This means that (S) is satisfied for $\varphi|U_0 \times P_0$. Now from the locality principle in the definition we have that $f \in \mathcal{S}_k(X, Y)$. The same locality is true (and the same proof works) for $\mathcal{B}_k$.

**4.3. The class $\mathcal{B}_k$.** Let

$$\mathcal{A}_k = \{A \subset X : \xi_A \in \mathcal{B}_k(X, \{0,1\})\}$$

where $\{0,1\}$ is taken as a discrete space. It is easy to see that $\mathcal{A}_k$ is a $\sigma$-algebra, and a function $f : X \to Y$ is in $\mathcal{B}_k(X,Y)$ if and only if $f^{-1}(V)$ is in $\mathcal{A}_k$ for each open subset $V$ of $Y$. Hence the investigation of $\mathcal{B}_k(X,Y)$ is reduced to the investigation of the $\sigma$-algebra $\mathcal{A}_k$. It is easy to see that $\mathcal{A}_n$ is the class of all subsets of $X$ having the Baire property and $\mathcal{A}_0$ is the class of all subsets of $X$. Each $\mathcal{A}_k$ contains the Borel subsets of $X$.

We shall prove that $A \in \mathcal{A}_k$ *if and only if* $A \cap \operatorname{rng}\psi$ *has the Baire property in the subspace* $\operatorname{rng}\psi$ *for any embedding* $\psi$ *of some open subset of* $\mathbb{R}^k$ *into* $X$. Indeed, if $A \in \mathcal{A}_k$ then $\psi^{-1}(A)$ has the Baire property in $\operatorname{dmn}\psi$ and since $\psi$ is a homeomorphism of $\operatorname{dmn}\psi$ onto $\operatorname{rng}\psi$, the set $\psi(\psi^{-1}(A)) = A \cap \operatorname{rng}\psi$ has the Baire property in $\operatorname{rng}\psi$. Similarly, if $A \cap \operatorname{rng}\psi$ has the Baire property in $\operatorname{rng}\psi$, then $\psi^{-1}(A)$ has the Baire property in $\operatorname{dmn}\psi$. By 2.3 (4) if $\psi^{-1}(A)$ has Baire property for any embedding $\psi$ of some open subset of $\mathbb{R}^k$ into $X$ then $A \in \mathcal{A}_k$.

Similarly, $A \in \mathcal{A}_k$ *if and only if* $A \cap \operatorname{rng}\psi$ *has the Baire property in the subspace* $\operatorname{rng}\psi$ *for any immersion* $\psi$ *of some open subset of* $\mathbb{R}^k$ *into* $X$. Let us represent $U = \operatorname{dmn}\psi$ as a countable union of open subsets $U_i$ of $U$ such that $\psi|U_i$ is an embedding of $U_i$ into $X$. If $A \in \mathcal{A}_k$ then $\psi^{-1}(A)$ has the Baire property in $U$, hence $U_i \cap \psi^{-1}(A)$ also has the Baire property in $U_i$. From this $A_i = (\psi|U_i)(\psi^{-1}(A))$ has the Baire property in $\psi(U_i)$, i. e. $A_i \triangle V_i \subset F_i$ for some relatively open subset $V_i$ of $\psi(U_i)$ and some $F_i$ which is of first category in $\psi(U_i)$. Since $F_i$ is of first category in $\operatorname{rng}\psi$ too, $\cup_i F_i$ is of first category in $\operatorname{rng}\psi$. Since

$$(\cup_i A_i) \triangle (\cup_i V_i) \subset \cup_i F_i,$$

we have that the symmetric difference of $A \cap \operatorname{rng}\psi = \cup_i A_i$ and the $\sigma$-compact set $\cup_i V_i$ is of first category. This proves that $A \cap \operatorname{rng}\psi$ has the Baire property in $\operatorname{rng}\psi$. In the other direction, if $A \cap \psi(U_i)$ has the Baire property, then $(\psi|U_i)^{-1}(A)$ has the Baire property in $U_i$, hence in $U$, too. Since this is true for any immersion $\psi$ of some open subset of $\mathbb{R}^k$ into $X$, we obtain that $A \in \mathcal{A}_k$.

Finally, $A \in \mathcal{A}_k$ *if and only if* $A \cap M$ *has the Baire property in the subspace* $M$ *for each pure* $k$ *dimensional submanifold* $M$ *of* $X$. Indeed, if this is true, then in particular $A \cap \operatorname{rng}\psi$ has the Baire property in $\operatorname{rng}\psi$ for each immersion $\psi$ of some open subset of $\mathbb{R}^k$ into $X$, hence $A \in \mathcal{A}_k$. On the other hand, each pure $k$ dimensional submanifold $M$ of $X$ can be represented as the range of some immersion $\psi$ of some open subset of $\mathbb{R}^k$. Hence $A \cap M = A \cap \operatorname{rng}\psi$ has the Baire property in $M = \operatorname{rng}\psi$.

**4.4. Connections between $\mathcal{B}_k$ and $\mathcal{S}_k$.** One of the simplest questions is, whether $f \in \mathcal{B}_k$ implies $f \in \mathcal{S}_k$. We know that this is true for $k = n$. If $k < n$ then the characteristic function of the intersection of $X$ and an appropriate $k$-dimensional plane is in $\mathcal{B}_k$ but not contained in $\mathcal{S}_k$.

In the other direction, suppose, that $f \in \mathcal{S}_k$. The question is, whether $f \in \mathcal{B}_k$ is satisfied. This is trivial for $k = 0$. We shall show that this cannot be proved in ZFC for $0 < k \leq n$. Namely, we shall give an example $f$ under the continuum hypothesis for which $f \in \mathcal{S}_k$ but $f \notin \mathcal{B}_k$. By the famous results of Gödel and Cohen, the continuum hypothesis is independent from the axioms of ZFC. This means that $\mathcal{B}_k \subset \mathcal{S}_k$ cannot be proved in ZFC.

**4.5. Hierarchy between function classes belonging to different dimensions.** Let us fix dimensions $0 \leq k < l \leq n$ and let us investigate the connection between the classes $\mathcal{B}_k$ and $\mathcal{S}_k$ and classes $\mathcal{B}_l$ and $\mathcal{S}_l$.

We may hope that decreasing the dimension condition (S) becomes stronger. One of the only two positive results in this direction is that this is true for condition (S) under condition (B):

$$\mathcal{B}_k \cap \mathcal{B}_l \cap \mathcal{S}_k \subset \mathcal{S}_l.$$

The proof of this statement is very similar to the proof of Theorem 3.6, therefore we do not repeat the argument.

We shall show by a counterexample under the continuum hypothesis that for $k > 0$

$$\text{ZFC} \nvDash \mathcal{B}_k \cap \mathcal{S}_k \subset \mathcal{B}_l \cup \mathcal{S}_l.$$

Similarly we shall show by a counterexample under the continuum hypothesis that

$$\text{ZFC} \nvDash \mathcal{B}_k \cap \mathcal{S}_k \cap \mathcal{S}_l \subset \mathcal{B}_l$$

except for the trivial case $k = 0$.

It is much easier to see that inclusions in the other direction do not hold in general. Although

$$\mathcal{B}_l \subset \mathcal{B}_0$$

is satisfied trivially, in general

$$\mathcal{B}_l \not\subset \mathcal{B}_k \qquad \text{if} \qquad k > 0.$$

This is shown by the characteristic function of a subset in the the intersection of $X$ and an appropriate $k$ dimensional plane which does not have the Baire property in the given plane. The same example shows that

$$\mathcal{B}_l \cap \mathcal{S}_l \not\subset \mathcal{B}_k \cup \mathcal{S}_k.$$

If we take the characteristic function of the intersection of $X$ and an appropriate $k$ dimensional plane, then we see that

$$\mathcal{B}_l \cap \mathcal{S}_l \cap \mathcal{B}_k \not\subset \mathcal{S}_k.$$

We shall show that

$$\mathcal{B}_l \cap \mathcal{S}_k \subset \mathcal{B}_k.$$

Let us see the proofs.

**4.6.Theorem.** *Under the conditions of 4.1 and 4.5 we have $\mathcal{B}_l \cap \mathcal{S}_k \subset \mathcal{B}_k$.*

**Proof.** This is trivial for $k = 0$. Otherwise, let $\psi$ be an immersion of an open subset $U \subset \mathbb{R}^k$. Let $u_0 \in U$ and let $V$ be an $l - k$ dimensional subspace of $\mathbb{R}^n$ orthogonal to $\operatorname{rng} \psi'(u_0)$. Let $\pi : \mathbb{R}^{l-k} \to V$ be a linear isometry, and let us define $\varphi$ by $\varphi(u,p) = \psi(u) + \pi(p)$. Then for $p_0 = 0$ we have $\varphi_{p_0} = \psi$. Let us choose open neighbourhoods $U_0$ and $P_0$ of $u_0$ and $p_0$, respectively, such that $\varphi(U_0, P_0) \subset X$ and $\varphi$ is an immersion of $U_0 \times P_0$ into $X$. Since $f \in \mathcal{B}_l$, the mapping $(u,p) \mapsto f(\varphi(u,p))$ has the Baire property. Hence, by the analogue of Fubini's theorem (see 2.1), except for a set of first category, for all $p \in P_0$ the mapping $u \mapsto f(\varphi(u,p))$ has the Baire property. Let us choose a sequence $p_m \to p_0$ such that each $u \mapsto f(\varphi(u,p_m))$ has the Baire property. By $f \in \mathcal{B}_k$ we have that

$$f(\varphi(u,p_m)) \to f(\varphi(u,p_0))$$

for all $u \in U_0$ except for a set of first category. Hence $u \mapsto f(\psi(u))$ has the Baire property on $U_0$, i.ẽ. locally. This implies that $f \in \mathcal{B}_k$.

For the following counterexamples we need a lemma. The counterexamples are related to the existence of the so-called almost invariant sets. These sets were used by Kakutani and Oxtoby to prove that the Lebesgue measure on the complex unit circle can be extended to an invariant measure such that the Hilbert space dimension of the corresponding $\mathbf{L}^2$ space becomes $2^{\mathbf{c}}$, where $\mathbf{c}$ is the cardinal number continuum. The proof of the lemma below is a refinement of the construction from the paper [7] of the author, where the result of Kakutani and Oxtoby was extended — among others — to arbitrary locally compact groups. The ideas there are combined with the well-known ideas of Sierpinski to construct under the continuum hypothesis a subset of the unit square with outer measure 1 and containing at most two points on each line. To better understand the following abstract lemma, we can think of the case when $X$ is the plane, $T$ is the class of all diffeomorphisms mapping some open subset of the plane onto some other open subset of the plane, $\mathcal{F}$ is the class of all Borel subsets of the plane of second category, $\mathcal{G}$ is the class of all one dimensional $\mathcal{C}^1$ submanifolds of the plane and $\mathbf{n} = \mathbf{c} = \aleph_1$.

**4.7. Lemma.** *Let $X$ be a set and $T$ a class of one-to-one transformations each mapping a subset of $X$ into $X$ and let $\mathcal{F}$, $\mathcal{G}$ be classes of subsets of $X$. Suppose that there exists an infinite cardinal number $\mathbf{n} > \aleph_0$ with the following properties:*

*(1)* $\operatorname{card}(X) = \mathbf{n}$;
*(2)* $\operatorname{card}(T) \le \mathbf{n}$;
*(3)* $\operatorname{card}(\mathcal{F}) \le \mathbf{n}$ *and for every $F \in \mathcal{F}$ we have* $\operatorname{card}(F) = \mathbf{n}$;
*(4)* $\operatorname{card}(\mathcal{G}) \le \mathbf{n}$ *and for every $F \in \mathcal{F}$ and $\mathcal{G}_0 \subset \mathcal{G}$ for which* $\operatorname{card}(\mathcal{G}_0) < \mathbf{n}$ *we have* $\operatorname{card}(F \setminus \cup \mathcal{G}_0) = \mathbf{n}$;
*(5) The class $\mathcal{G}$ is $T$ invariant, i.e. if $G \in \mathcal{G}$, $\tau \in T$ then $\tau(G) \in \mathcal{G}$ and $\tau^{-1}(G) \in \mathcal{G}$.*

*Then there exists a family* $\{X_\gamma\}_{\gamma\in\Gamma}$ *of subsets* $X_\gamma$ *of* $X$ *with the following properties:*

*(6)* $\operatorname{card}(\Gamma) = \mathbf{n}$*;*

*(7) the sets* $X_\gamma$*,* $\gamma \in \Gamma$ *are pairwise disjoint;*

*(8) for each* $\gamma \in \Gamma$ *and* $G \in \mathcal{G}$ *we have* $\operatorname{card}(X_\gamma \cap G) < \mathbf{n}$*;*

*(9)* $\operatorname{card}(F \cap X_\gamma) = \mathbf{n}$ *whenever* $\gamma \in \Gamma$ *and* $F \in \mathcal{F}$*;*

*(10) for every subset* $\Gamma_0$ *of* $\Gamma$ *and for every* $\tau \in T$

$$\operatorname{card}\left(\tau(\cup_{\gamma\in\Gamma_0} X_\gamma) \triangle \left(\tau(X) \cap (\cup_{\gamma\in\Gamma_0} X_\gamma)\right)\right) < \mathbf{n}.$$

The proof was given in Járai [14], Lemma 4.8.

**4.8. Counterexample.** *Using the conditions of 4.1, under the continuum hypothesis for* $0 < k \le n$ *we have* $\mathcal{S}_k \not\subset \mathcal{B}_k$*.*

**Proof.** We shall give a function $f \in \mathcal{S}_k$ for which $f \notin \mathcal{B}_k$. We want to apply the previous lemma. We shall use only that the functions $\varphi$ in the definition of $\mathcal{S}_k$ are continuous and that by Remark 2.3.(4) we may suppose that the functions $\varphi_p$ are one-to-one. Let $T$ denote the class of all one-to-one functions $\tau$ which can be represented in the form $\varphi_p \circ \varphi_{p'}^{-1}$, where $U$ is an open subset of $\mathbb{R}^k$, $P$ is an open subset of some Euclidean space and $\varphi : U \times P \to X$ is a continuous function for which all $\varphi_p$, $p \in P$ is one-to-one. Since the cardinality of all pairs $U$, $P$ is continuum and any continuous function $\varphi$ is uniquely determined by the values on a countable dense subset, the cardinality of the class $T$ is continuum.

Let $\mathcal{F}$ denote the class of all subsets of $X$ representable in the form $\psi(G)$ where $\emptyset \ne U \subset \mathbb{R}^k$ is open, $\psi : U \to X$ is an embedding and $G \subset U$ is a Borel subset of $U$ of second category in $U$. Each element of $\mathcal{F}$ is a Borel subset of $X$, hence the cardinality of $\mathcal{F}$ is at most $\mathbf{c}$ (continuum). Moreover, by a theorem of Piccard, $G - G$ contains a neighbourhood of the origin, hence each elements of $\mathcal{F}$ has cardinality $\mathbf{c}$.

Applying the previous lemma with $\mathcal{G} = \emptyset$ we obtain a class of subsets $X_\gamma$, $\gamma \in \mathbb{R}$ of $X$. Our counterexample will be the characteristic function $f$ of $X_0$.

If $f$ were in $\mathcal{B}_k$ then for any embedding $\psi$ of some nonvoid open subset $U$ of $\mathbb{R}^k$ the set $A_0 = \psi^{-1}(X_0)$ would be a Baire set. $A_0$ cannot be of first category, because then for a Borel set $G \subset U\backslash A_0$ of second category $\psi(G)$ would not intersect with $X_0$. Similarly, if $A_0$ is of second category, then choosing a Borel set $B \subset A_0$ of second category we obtain $\psi(G) \subset X_0$.

We shall prove that $f \in \mathcal{S}_k$. Let $U$ be an open subset of $\mathbb{R}^k$, $P$ be an open subset of some euclidean space, $p_0 \in P$ and $\varphi : U \times P \to X$ a $\mathcal{C}^1$ function for which all $\varphi_p$ is an embedding. The set

$$\{u \in U : f(\varphi_{p_0}(u)) \ne f(\varphi_p(u))\}$$

is equal to the set

$$\varphi_{p_0}^{-1}\left(\{x \in \varphi_{p_0}(U) : x \in X_0 \triangle (\varphi_{p_0} \circ \varphi_p^{-1})(X_0)\}\right).$$

For the mapping $\tau = \varphi_{p_0} \circ \varphi_p^{-1}$ this set is a subset of the set

$$\varphi_{p_0}^{-1}\left((\tau(X) \cap X_0) \triangle \tau(X_0)\right).$$

If we suppose the continuum hypothesis then this set is countable.

**4.9. Counterexample.** *Using the conditions of 4.1 and 4.5, under the continuum hypothesis for* $0 < k < l \leq n$ *we have* $\mathcal{B}_k \cap \mathcal{S}_k \cap \mathcal{S}_l \not\subset \mathcal{B}_l$.

**Proof.** We shall give an example of a function $f \in \mathcal{B}_k \cap \mathcal{S}_k \cap \mathcal{S}_l$ but $f \notin \mathcal{B}_l$. We want to apply Lemma 4.7. We shall use that by Remark 2.3.(4) we may suppose that the functions $\varphi_p$ in the definition of $\mathcal{S}_l$ are embeddings. Let $T$ denote the class of all one-to-one functions $\tau$ which can be represented in the form $\varphi_p \circ \varphi_{p'}^{-1}$, where $U$ is an open subset of $\mathbb{R}^l$, $P$ is an open subset of some Euclidean space and $\varphi : U \times P \to X$ is a $\mathcal{C}^1$ function for which all $\varphi_p$, $p \in P$ is an embedding. Let $\mathcal{F}$ be the same as in the previous counterexample. Let $\mathcal{G}$ be the class of all Borel subsets of $X$ which are contained in a union of countably many $k$ dimensional submanifolds of $X$. We have to prove that $\mathcal{G}$ is $T$ invariant. The domain of any $\tau \in T$ is an $l$ dimensional submanifold of $X$. If a set $G \in \mathcal{G}$ is contained in $\cup_{j=1}^{\infty} M_j$, where each $M_j$ is a $k$ dimensional submanifold of $X$, then for each $x \in G \cap M_j \cap \operatorname{dmn} \tau$ it is possible to find an $\varepsilon > 0$ and a $k$ dimensional submanifold $M_j'$ of $\operatorname{dmn} \tau$ such that for each $y$ for which $|y - x| < \varepsilon$ we have $y \in X$, moreover, that $y \in G \cap M_j \cap \operatorname{dmn} \tau$ if and only if $y \in G \cap M_j' \cap \operatorname{dmn} \tau$. This proves that the Borel set $G \cap \operatorname{dmn} \tau$ can be covered by countably many $k$ dimensional submanifolds of $\operatorname{dmn} \tau$. Hence the Borel set $\tau(G)$ can also be covered by countably many $k$ dimensional submanifolds.

Since the topological dimension dim = ind = Ind of any subset $G$ of a $k$ dimensional submanifold is $\leq k$, the intersection of $G$ with an $l$ dimensional submanifold $L$ of $X$ is of first category in $L$. The same is true for any $G \in \mathcal{G}$, and, moreover, for the union $G$ of any countable subfamily $\mathcal{G}_0 \subset \mathcal{G}$. This proves that for any $F \in \mathcal{F}$ the set $F \backslash G$ has cardinality $\mathbf{c}$. Other conditions of Lemma 4.7 have already been checked at 4.8 .

Applying Lemma 4.7 we obtain a class $X_\gamma$, $\gamma \in \mathbb{R}$ where each $X_\gamma$ contains only countably many points from each $G \in \mathcal{G}$, but $X_\gamma \cap F \neq \emptyset$ for each $F \in \mathcal{F}$.

Let $f$ be the characteristic function of $X_0$. Along the same lines as in 4.8 we get that $f \in \mathcal{S}_l$ but $f \notin \mathcal{B}_l$. Since for any $\mathcal{C}^1$ embedding $\psi$ of an open subset of $\mathbb{R}^k$ into $X$ the function $f \circ \psi$ is zero except for a countable set, we get that $f \in \mathcal{B}_k$ and $f \in \mathcal{S}_k$, too. Hence the statement is proved.

**4.10. Counterexample.** *Using the conditions of 4.1 and 4.5, under the continuum hypothesis for* $0 < k < l \leq n$ *we have* $\mathcal{B}_k \cap \mathcal{S}_k \not\subset \mathcal{B}_l \cup \mathcal{S}_l$.

**Proof.** Let us apply Lemma 4.7 for the same $T$, $\mathcal{F}$ and $\mathcal{G}$ as in the previous counterexample. We obtain a class $X_\gamma$, $\gamma \in \mathbb{R}$ where each $X_\gamma$ contains only countably many points from each $G \in \mathcal{G}$, but $X_\gamma \cap F \neq \emptyset$ for each $F \in \mathcal{F}$.

Let $Y$ be an $l$ dimensional plane which has a nonempty intersection with $X$ and let $f$ be the characteristic function of the set $Y \cap X_0$. Then $f \in \mathcal{B}_k \cap \mathcal{S}_k$, but $f \notin \mathcal{B}_l$ and $f \notin \mathcal{S}_l$.

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