Komputeralgebrai algoritmusok

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Ezek a programok csak szemléltetésre szolgálnak.

1. Történet

2. Algebrai alapok

3. Normál formák, reprezentáció

4. Aritmetika

5. Kínai maradékolás

6. Newton-iteráció, Hensel-felemelés

7. Legnagyobb közös osztó

```
> restart;
```

E 7.1. Példa.

```
> A:=x^8+x^6-3*x^4-3*x^3+8*x^2+2*x-5; B:=3*x^6+5*x^4-4*x^2-9*x+21;
```

$$A := x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$

$$B := 3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x + 21 \tag{7.1.1}$$

```
> p:=23; `mod`:=mods; A mod p; B mod p; Rem(%%,%,x) mod p; Rem(%%,%,x) mod p; Rem(%%,%,x) mod p; Rem(%%,%,x) mod p;
```

$$p := 23$$

$$mod := mods$$

$$x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$

$$3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x - 2$$

$$2\,x^4 - 5\,x^2 - 8$$

$$-x^2 - 9\,x + 2$$

$$10\,x - 8$$

(7.1.2)

$$-4 \tag{7.1.2}$$

E 7.2. Példa.

```
> A; B; rem(%%,%,x); rem(%%,%,x); rem(%%,%,x); rem(%%,%,x);
```

$$x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$

$$3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x + 21$$

$$-\frac{1}{3} - \frac{5}{9}\,x^4 + \frac{1}{9}\,x^2$$

$$\frac{441}{25} - \frac{117}{25}\,x^2 - 9\,x$$

$$-\frac{102500}{6591} + \frac{233150}{19773}\,x$$

$$-\frac{1288744821}{543589225} \tag{7.2.1}$$

E 7.3. Példa.

```
> A; B; rem(3^3*%%,%,x); rem((-15)^3*%%,%,x); rem(15795^3*%%,%,
  x);
  rem(1254542875143750^2*%%,%,x);
```

$$x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$

$$3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x + 21$$

$$-9 - 15\,x^4 + 3\,x^2$$

$$-59535 + 15795\,x^2 + 30375\,x$$

$$-1654608338437500 + 1254542875143750\,x$$

$$12593338795500743100931141992187500 \tag{7.3.1}$$

E 7.4. Példa.

```
> A; B; rem(3^3*%%,%,x); %/igcd(coeffs(%)); rem(5^3*%%%,%,x);
  %/igcd(coeffs(%)); rem((-13)^3*%%%,%,x); %/igcd(coeffs(%));
  rem((-4663)^2*%%%,%,x); %/igcd(coeffs(%));
```

$$x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$

$$3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x + 21$$

$$-9 - 15\,x^4 + 3\,x^2$$

$$-3 - 5\,x^4 + x^2$$

$$2205 - 585\,x^2 - 1125\,x$$

$$49 - 13\,x^2 - 25\,x$$

$$307500 - 233150\,x$$

$$6150 - 4663\,x$$

$$-143193869$$

$$-1 \tag{7.4.1}$$

E 7.5. Példa.

```
> R[0]:=A; R[1]:=B;
  r[1]:=lcoeff(R[1]); delta[1]:=degree(R[0])-degree(R[1]);
  alpha[1]:=r[1]^(delta[1]+1); beta[1]:=1;
  R[2]:=rem(alpha[1]*R[0],R[1],x)/beta[1];
```

$$R_0 := x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$

$$R_1 := 3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x + 21$$

$$r_1 := 3$$

$$\delta_1 := 2$$

$$\alpha_1 := 27$$

$$\beta_1 := 1$$

$$R_2 := -9 - 15\,x^4 + 3\,x^2 \tag{7.5.1}$$

```
> r[2]:=lcoeff(R[2]); delta[2]:=degree(R[1])-degree(R[2]);
  alpha[2]:=r[2]^(delta[2]+1); beta[2]:=alpha[1];
  R[3]:=rem(alpha[2]*R[1],R[2],x)/beta[2];
```

$$r_2 := -15$$

$$\delta_2 := 2$$

$$\alpha_2 := -3375$$

$$\beta_2 := 27$$

$$R_3 := -2205 + 585\,x^2 + 1125\,x \tag{7.5.2}$$

```
> r[3]:=lcoeff(R[3]); delta[3]:=degree(R[2])-degree(R[3]);
  alpha[3]:=r[3]^(delta[3]+1); beta[3]:=alpha[2];
  R[4]:=rem(alpha[3]*R[2],R[3],x)/beta[3];
```

$$r_3 := 585$$

$$\delta_3 := 2$$

$$\alpha_3 := 200201625$$

$$\beta_3 := -3375$$

$$R_4 := 24907500 - 18885150\,x \tag{7.5.3}$$

```
> r[4]:=lcoeff(R[4]); delta[4]:=degree(R[3])-degree(R[4]);
  alpha[4]:=r[4]^(delta[4]+1); beta[4]:=alpha[3];
  R[5]:=rem(alpha[4]*R[3],R[4],x)/beta[4];
```

$$r_4 := -18885150$$
$$\delta_4 := 1$$
$$\alpha_4 := 356648890522500$$
$$\beta_4 := 200201625$$
$$R_5 := 527933700 \tag{7.5.4}$$

E 7.6. Példa.

```
> R[0]:=A; R[1]:=B;
  r[1]:=lcoeff(R[1]); delta[1]:=degree(R[0])-degree(R[1]);
  alpha[1]:=r[1]^(delta[1]+1); psi[1]:=-1; beta[1]:=(-1)^(delta
  [1]+1);
  R[2]:=rem(alpha[1]*R[0],R[1],x)/beta[1];
```
$$R_0 := x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$
$$R_1 := 3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x + 21$$
$$r_1 := 3$$
$$\delta_1 := 2$$
$$\alpha_1 := 27$$
$$\psi_1 := -1$$
$$\beta_1 := -1$$
$$R_2 := 9 + 15\,x^4 - 3\,x^2 \tag{7.6.1}$$

```
> r[2]:=lcoeff(R[2]); delta[2]:=degree(R[1])-degree(R[2]);
  alpha[2]:=r[2]^(delta[2]+1);
  psi[2]:=(-r[1])^delta[1]*psi[1]^(1-delta[1]);
  beta[2]:=-r[1]*psi[2]^delta[2];
  R[3]:=rem(alpha[2]*R[1],R[2],x)/beta[2];
```
$$r_2 := 15$$
$$\delta_2 := 2$$
$$\alpha_2 := 3375$$
$$\psi_2 := -9$$
$$\beta_2 := -243$$
$$R_3 := -245 + 65\,x^2 + 125\,x \tag{7.6.2}$$

```
> r[3]:=lcoeff(R[3]); delta[3]:=degree(R[2])-degree(R[3]);
  alpha[3]:=r[3]^(delta[3]+1);
  psi[3]:=(-r[2])^delta[2]*psi[2]^(1-delta[2]);
  beta[3]:=-r[2]*psi[3]^delta[3];
  R[4]:=rem(alpha[3]*R[2],R[3],x)/beta[3];
```

$$r_3 := 65$$

$$\delta_3 := 2$$

$$\alpha_3 := 274625$$

$$\psi_3 := -25$$

$$\beta_3 := -9375$$

$$R_4 := -12300 + 9326\,x \tag{7.6.3}$$

```
> r[4]:=lcoeff(R[4]); delta[4]:=degree(R[3])-degree(R[4]);
  alpha[4]:=r[4]^(delta[4]+1);
  psi[4]:=(-r[3])^delta[3]*psi[3]^(1-delta[3]);
  beta[4]:=-r[3]*psi[4]^delta[4];
  R[5]:=rem(alpha[4]*R[3],R[4],x)/beta[4];
```

$$r_4 := 9326$$

$$\delta_4 := 1$$

$$\alpha_4 := 86974276$$

$$\psi_4 := -169$$

$$\beta_4 := 10985$$

$$R_5 := 260708 \tag{7.6.4}$$

E 7.7. Példa.

```
> with(LinearAlgebra);
```

[*&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main, LUDecomposition, LeastSquares, LinearSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse,* (7.7.1)

MatrixMatrixMultiply, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*]

```
> A:=3*x^4+3*x^3+x^2-x-2; B:=x^3-3*x^2+x+5;
```

$$A := 3\,x^4 + 3\,x^3 + x^2 - x - 2$$

$$B := x^3 - 3\,x^2 + x + 5 \qquad (7.7.2)$$

```
> S0:=SylvesterMatrix(A,B,x);
```

$$S0 := \begin{bmatrix} 3 & 3 & 1 & -1 & -2 & 0 & 0 \\ 0 & 3 & 3 & 1 & -1 & -2 & 0 \\ 0 & 0 & 3 & 3 & 1 & -1 & -2 \\ 1 & -3 & 1 & 5 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 5 \end{bmatrix} \qquad (7.7.3)$$

E 7.8. Példa.

```
> Determinant(S0);
```

$$0 \qquad (7.8.1)$$

```
> S1:=SubMatrix(S0,[1..2,4..6],[1..5]):
  S1[1,5]:=x*A: S1[2,5]:=A: S1[3,5]:=x^2*B: S1[4,5]:=x*B: S1[5,
  5]:=B:
  S1; Determinant(S1);
```

$$\begin{bmatrix} 3 & 3 & 1 & -1 & x\left(3x^4+3x^3+x^2-x-2\right) \\ 0 & 3 & 3 & 1 & 3x^4+3x^3+x^2-x-2 \\ 1 & -3 & 1 & 5 & x^2\left(x^3-3x^2+x+5\right) \\ 0 & 1 & -3 & 1 & x\left(x^3-3x^2+x+5\right) \\ 0 & 0 & 1 & -3 & x^3-3x^2+x+5 \end{bmatrix}$$

$$1192\,x+1192 \tag{7.8.2}$$

```
> S2:=SubMatrix(S0,[1..1,4..5],[1..3]):
  S2[1,3]:=A: S2[2,3]:=x*B: S2[3,3]:=B:
  S2; Determinant(S2);
```

$$\begin{bmatrix} 3 & 3 & 3x^4+3x^3+x^2-x-2 \\ 1 & -3 & x\left(x^3-3x^2+x+5\right) \\ 0 & 1 & x^3-3x^2+x+5 \end{bmatrix}$$

$$34\,x^2-28\,x-62 \tag{7.8.3}$$

```
> S3:=SubMatrix(S0,[4],[1]):
  S3[1,1]:=B:
  S3; Determinant(S3);
```

$$\begin{bmatrix} x^3-3x^2+x+5 \end{bmatrix}$$

$$x^3-3\,x^2+x+5 \tag{7.8.4}$$

E 7.9. Példa.

```
> A:=x^8+x^6-3*x^4-3*x^3+8*x^2+2*x-5; B:=3*x^6+5*x^4-4*x^2-9*
  x+21;
```

$$A:=x^8+x^6-3\,x^4-3\,x^3+8\,x^2+2\,x-5$$

$$B:=3\,x^6+5\,x^4-4\,x^2-9\,x+21 \tag{7.9.1}$$

```
> S0:=SylvesterMatrix(A,B,x);
```

$$S0:=\begin{bmatrix} \textit{14 x 14 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{7.9.2}$$

```
> Determinant(S0);
```

$$260708 \tag{7.9.3}$$

```
> S1:=SubMatrix(S0,[1..5,7..13],[1..12]):
  S1[1,12]:=x^4*A: S1[2,12]:=x^3*A:  S1[3,12]:=x^2*A:  S1[4,12]
  :=x*A:  S1[5,12]:=A:  S1[6,12]:=x^6*B:  S1[7,12]:=x^5*B: S1
  [8,12]:=x^4*B: S1[9,12]:=x^3*B: S1[10,12]:=x^2*B: S1[11,12]:=
```

```
x*B: S1[12,12]:=B:
S1; Determinant(S1);
```

$$\left[\begin{array}{c} \textit{12 x 12 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right]$$

$$-12300 + 9326\,x \tag{7.9.4}$$

```
> S2:=SubMatrix(S0,[1..4,7..12],[1..10]):
S2[1,10]:=x^3*A: S2[2,10]:=x^2*A:  S2[3,10]:=x*A:  S2[4,10]:=
A:  S2[5,10]:=x^5*B:  S2[6,10]:=x^4*B:  S2[7,10]:=x^3*B: S2
[8,10]:=x^2*B: S2[9,10]:=x*B: S2[10,10]:=B:
S2; Determinant(S2);
```

$[1, 0, 1, 0, -3, -3, 8, 2, -5, x^3\left(x^8+x^6-3\,x^4-3\,x^3+8\,x^2+2\,x-5\right)], [0, 1, 0, 1,$
$0, -3, -3, 8, 2, x^2\left(x^8+x^6-3\,x^4-3\,x^3+8\,x^2+2\,x-5\right)], [0, 0, 1, 0, 1, 0, -3,$
$-3, 8, x\left(x^8+x^6-3\,x^4-3\,x^3+8\,x^2+2\,x-5\right)], [0, 0, 0, 1, 0, 1, 0, -3, -3,$
$x^8+x^6-3\,x^4-3\,x^3+8\,x^2+2\,x-5], [3, 0, 5, 0, -4, -9, 21, 0, 0,$
$x^5\left(3\,x^6+5\,x^4-4\,x^2-9\,x+21\right)], [0, 3, 0, 5, 0, -4, -9, 21, 0,$
$x^4\left(3\,x^6+5\,x^4-4\,x^2-9\,x+21\right)], [0, 0, 3, 0, 5, 0, -4, -9, 21,$
$x^3\left(3\,x^6+5\,x^4-4\,x^2-9\,x+21\right)], [0, 0, 0, 3, 0, 5, 0, -4, -9,$
$x^2\left(3\,x^6+5\,x^4-4\,x^2-9\,x+21\right)], [0, 0, 0, 0, 3, 0, 5, 0, -4,$
$x\left(3\,x^6+5\,x^4-4\,x^2-9\,x+21\right)], [0, 0, 0, 0, 0, 3, 0, 5, 0,$

$$3x^6+5x^4-4x^2-9x+21]$$

$$-637+325x+169x^2 \tag{7.9.5}$$

```
> S3:=SubMatrix(S0,[1..3,7..11],[1..8]):
  S3[1,8]:=x^2*A: S3[2,8]:=x*A:  S3[3,8]:=A:  S3[4,8]:=x^4*B:
  S3[5,8]:=x^3*B:  S3[6,8]:=x^2*B:  S3[7,8]:=x*B: S3[8,8]:=B:
  S3; Determinant(S3);
```

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -3 & -3 & 8 & x^2\left(x^8+x^6-3x^4-3x^3+8x^2+2x-5\right) \\ 0 & 1 & 0 & 1 & 0 & -3 & -3 & x\left(x^8+x^6-3x^4-3x^3+8x^2+2x-5\right) \\ 0 & 0 & 1 & 0 & 1 & 0 & -3 & x^8+x^6-3x^4-3x^3+8x^2+2x-5 \\ 3 & 0 & 5 & 0 & -4 & -9 & 21 & x^4\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 3 & 0 & 5 & 0 & -4 & -9 & x^3\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 0 & 3 & 0 & 5 & 0 & -4 & x^2\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 0 & 0 & 3 & 0 & 5 & 0 & x\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 0 & 0 & 0 & 3 & 0 & 5 & 3x^6+5x^4-4x^2-9x+21 \end{bmatrix}$$

$$-245+65x^2+125x \tag{7.9.6}$$

```
> S4:=SubMatrix(S0,[1..2,7..10],[1..6]):
  S4[1,6]:=x*A: S4[2,6]:=A:  S4[3,6]:=x^3*B:  S4[4,6]:=x^2*B:
  S4[5,6]:=x*B:  S4[6,6]:=B:
  S4; Determinant(S4);
```

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -3 & x\left(x^8+x^6-3x^4-3x^3+8x^2+2x-5\right) \\ 0 & 1 & 0 & 1 & 0 & x^8+x^6-3x^4-3x^3+8x^2+2x-5 \\ 3 & 0 & 5 & 0 & -4 & x^3\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 3 & 0 & 5 & 0 & x^2\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 0 & 3 & 0 & 5 & x\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 0 & 0 & 3 & 0 & 3x^6+5x^4-4x^2-9x+21 \end{bmatrix}$$

$$15-5x^2+25x^4 \tag{7.9.7}$$

```
> S5:=SubMatrix(S0,[1..1,7..9],[1..4]):
  S5[1,4]:=A: S5[2,4]:=x^2*B:  S5[3,4]:=x*B:  S5[4,4]:=B:
  S5; Determinant(S5);
```

$$\begin{bmatrix} 1 & 0 & 1 & x^8+x^6-3x^4-3x^3+8x^2+2x-5 \\ 3 & 0 & 5 & x^2\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 3 & 0 & x\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 0 & 3 & 3x^6+5x^4-4x^2-9x+21 \end{bmatrix}$$

$$9+15x^4-3x^2 \tag{7.9.8}$$

```
> S6:=SubMatrix(S0,[7..8],[1..2]):
  S6[1,2]:=x*B: S6[2,2]:=B:
  S6; Determinant(S6);
```

$$\begin{bmatrix} 3 & x\left(3x^6+5x^4-4x^2-9x+21\right) \\ 0 & 3x^6+5x^4-4x^2-9x+21 \end{bmatrix}$$

$$9x^6+15x^4-12x^2-27x+63 \tag{7.9.9}$$

E 7.10. Példa.

```
> A:=3*x^4+3*x^3+x^2-x-2; B:=x^3-3*x^2+x+5;
  A=(3*x+12)*B+34*x^2-28*x-62; expand(%);
```

$$A := 3x^4+3x^3+x^2-x-2$$

$$B := x^3-3x^2+x+5$$

$$3x^4+3x^3+x^2-x-2 = (3x+12)\left(x^3-3x^2+x+5\right)+34x^2-28x-62$$

$$3x^4+3x^3+x^2-x-2 = 3x^4+3x^3+x^2-x-2 \tag{7.10.1}$$

```
> 0; 1192*x-1192; 34*x^2-288*x-62;
```

$$0$$

$$1192x-1192$$

$$34x^2-288x-62 \tag{7.10.2}$$

E 7.11. Példa.

E 7.12. Példa.

E 7.13. Példa.

```
> A:=x^8+x^6-3*x^4-3*x^3+8*x^2+2*x-5; B:=3*x^6+5*x^4-4*x^2-9*x+21;
```

$$A := x^8 + x^6 - 3\,x^4 - 3\,x^3 + 8\,x^2 + 2\,x - 5$$

$$B := 3\,x^6 + 5\,x^4 - 4\,x^2 - 9\,x + 21 \tag{7.13.1}$$

```
> Gcd(A,B) mod 23;
```

$$1 \tag{7.13.2}$$

E 7.14. Példa.

```
> Gcd(A,B) mod 2;
```

$$x^2 + x + 1 \tag{7.14.1}$$

E 7.15. Példa.

```
> A:=x^4+25*x^3+145*x^2-171*x-360; B:=x^5+14*x^4+15*x^3-x^2-14*x-15;
```

$$A := x^4 + 25\,x^3 + 145\,x^2 - 171\,x - 360$$

$$B := x^5 + 14\,x^4 + 15\,x^3 - x^2 - 14\,x - 15 \tag{7.15.1}$$

```
> A mod 5; B mod 5; Gcd(A,B) mod 5;
```

$$x^4 - x$$

$$x^5 - x^4 - x^2 + x$$

$$x^4 - x \tag{7.15.2}$$

```
> A mod 7; B mod 7; Gcd(A,B) mod 7;
```

$$x^4 - 3\,x^3 - 2\,x^2 - 3\,x - 3$$

$$x^5 + x^3 - x^2 - 1$$

$$x^2 + 1 \tag{7.15.3}$$

```
> A mod 11; B mod 11; Gcd(A,B) mod 11;
```

$$x^4 + 3\,x^3 + 2\,x^2 + 5\,x + 3$$

$$x^5 + 3\,x^4 + 4\,x^3 - x^2 - 3\,x - 4$$

$$x^2 + 3\,x + 4 \tag{7.15.4}$$

```
> C:=x^2+14*x+15; C mod 7; C mod 11;
```

$$C := x^2 + 14\,x + 15$$

$$x^2+1$$
$$x^2+3x+4 \quad (7.15.5)$$

```
> rem(A,C,x); rem(B,C,x);
```

$$0$$
$$0 \quad (7.15.6)$$

E 7.16. Példa.

```
> `mod`:=mods;
```

$$mod := mods \quad (7.16.1)$$

```
> A:=9*x^5+2*x^4*y*z-189*x^3*y^3*z+117*x^3*y*z^2+3*x^3-42*x^2*
  y^4*z^2+26*x^2*y^2*z^3+18*x^2-63*x*y^3*z+39*x*y*z^2+4*x*y*
  z+6;
  B:=6*x^6-126*x^4*y^3*z+78*x^4*y*z^2+x^4*y+x^4*z+13*x^3-21*
  x^2*y^4*z-21*x^2*y^3*z^2+13*x^2*y^2*z^2+13*x^2*y*z^3-21*x*
  y^3*z+13*x*y*z^2+2*x*y+2*x*z+2;
```

$$A := 9x^5+2x^4yz-189x^3y^3z+117x^3yz^2+3x^3-42x^2y^4z^2 \\ +26x^2y^2z^3+18x^2-63xy^3z+39xyz^2+4xyz+6$$
$$B := 6x^6-126x^4y^3z+78x^4yz^2+x^4y+x^4z \\ +13x^3-21x^2y^4z-21x^2y^3z^2+13x^2y^2z^2+13x^2yz^3-21xy^3z \\ +13xyz^2+2xy+2xz+2 \quad (7.16.2)$$

```
> A11:=A mod 11; B11:=B mod 11;
```

$$A11 := -2x^5+2x^4yz-2x^3y^3z-4x^3yz^2+3x^3+2x^2y^4z^2 \\ +4x^2y^2z^3-4x^2+3xy^3z-5xyz^2+4xyz-5$$
$$B11 := -5x^6-5x^4y^3z+x^4yz^2+x^4y+x^4z+2x^3+x^2y^4z+x^2y^3z^2 \\ +2x^2y^2z^2+2x^2yz^3+xy^3z+2xyz^2+2xy+2xz+2 \quad (7.16.3)$$

```
> A11_2:=subs(z=2,A11) mod 11; B11_2:=subs(z=2,B11) mod 11;
```

$$A11\_2 := -2x^5+4x^4y-4x^3y^3-5x^3y \\ +3x^3-3x^2y^4-x^2y^2-4x^2-5xy^3-xy-5$$
$$B11\_2 := -5x^6+x^4y^3+5x^4y+2x^4+2x^3+2x^2y^4+4x^2y^3-3x^2y^2 \\ +5x^2y+2xy^3-xy+4x+2 \quad (7.16.4)$$

```
> A11_3_2:=subs(y=3,A11_2) mod 11; B11_3_2:=subs(y=3,B11_2) mod
  11;
```

$$A11\_3\_2 := -2x^5+x^4+x^3-3x^2+5x-5$$
$$B11\_3\_2 := -5x^6+2x^3+5x^2+2 \quad (7.16.5)$$

```
> Gcd(A11_3_2,B11_3_2) mod 11;
```

$$x^3+x+2 \quad (7.16.6)$$

```
> Gcd(subs(y=5,A11_2)mod 11,subs(y=5,B11_2)mod 11) mod 11;
```

$$x^3+4\,x+2 \tag{7.16.7}$$

```
> Gcd(subs(y=-4,A11_2)mod 11,subs(y=-4,B11_2)mod 11) mod 11;
```

$$x^3+5\,x+2 \tag{7.16.8}$$

```
> Gcd(subs(y=-2,A11_2)mod 11,subs(y=-2,B11_2)mod 11) mod 11;
```

$$x^3+x+2 \tag{7.16.9}$$

```
> Gcd(subs(y=2,A11_2)mod 11,subs(y=2,B11_2)mod 11) mod 11;
```

$$x^3-x+2 \tag{7.16.10}$$

```
> with(CurveFitting);
```

$$[\mathit{BSpline}, \mathit{BSplineCurve}, \mathit{Interactive}, \mathit{LeastSquares}, \mathit{PolynomialInterpolation}, \mathit{RationalInterpolation}, \mathit{Spline}, \mathit{ThieleInterpolation}] \tag{7.16.11}$$

```
> PolynomialInterpolation([3,5,-4,-2,2],[x^3+x+2,x^3+4*x+2,
  x^3+5*x+2,x^3+x+2,x^3-x+2],y,form=Lagrange) mod 11; expand(%)
  mod 11;
```

$$-3\left(x^3+x+2\right)(y-5)(y+4)(y+2)(y-2)+3\left(x^3+4\,x+2\right)(y-3)(y+4)(y+2)(y-2)-4\left(x^3+5\,x+2\right)(y-3)(y-5)(y+2)(y-2)+2\left(x^3+x+2\right)(y-3)(y-5)(y+4)(y-2)+2\left(x^3-x+2\right)(y-3)(y-5)(y+4)(y+2)$$

$$2+x^3-3\,x\,y+2\,x\,y^3 \tag{7.16.12}$$

```
> PolynomialInterpolation([2,-5,-3,5],[x^3+2*x*y^3-3*x*y+2,x^3
  -5*x*y^3-5*x*y+2,x^3-3*x*y^3-4*x*y+2,x^3+5*x*y^3-5*x*y+2],z,
  form=Lagrange) mod 11; expand(%) mod 11;
```

$$-2\left(2+x^3-3\,x\,y+2\,x\,y^3\right)(z+5)(z+3)(z-5)+4\left(x^3-5\,x\,y^3-5\,x\,y+2\right)(z-2)(z+3)(z-5)+4\left(x^3-3\,x\,y^3-4\,x\,y+2\right)(z-2)(z+5)(z-5)+5\left(x^3+5\,x\,y^3-5\,x\,y+2\right)(z-2)(z+5)(z+3)$$

$$2+x^3+2\,x\,y\,z^2+x\,y^3\,z \tag{7.16.13}$$

```
> chrem([3*x^3+3*x*y^3*z-5*x*y*z^2-5,3*x^3+2*x*y^3*z+6,3*x^3+5*
  x*y^3*z+5*x*y*z^2+6],[11,13,17]);
```

$$3\,x^3-63\,x\,y^3\,z+39\,x\,y\,z^2+6 \tag{7.16.14}$$

```
> C:=%/igcd(coeffs(%));
```

$$C := x^3-21\,x\,y^3\,z+13\,x\,y\,z^2+2 \tag{7.16.15}$$

```
> simplify(A/C); simplify(B/C);
```

$$9\,x^2+2\,x\,y\,z+3$$

$$6\,x^3+x\,y+x\,z+1 \tag{7.16.16}$$

A 7.1. Algoritmus.

>

A 7.2. Algoritmus.

>

E 7.17. Példa.

A 7.3. Algoritmus.

E 7.18. Példa.

E 7.19. Példa.

E 7.20. Példa.

E 7.21. Példa.

E 7.22. Példa.

A 7.4. Algoritmus.

>

8. Faktorizálás

9. Egyenletrendszerek

10. Gröbner-bázisok

11. Racionális törtfüggvények integrálása

12. A Risch-algoritmus.