Bevezetés a matematikába

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Ezek a programok csak szemléltetésre szolgálnak.

1. Halmazok

2. Természetes számok

3. A számfogalom bővítése

```
> restart;
```

3.1. Egész számok

3.1.1. Osztályozás kompatibilitása művelettel.

```
> `type/ordpair`:=proc(x) type(x,list) and nops(x)=2 end;
```

$type/ordpair := \textbf{proc}(x)\ type(x, list)\ \textbf{and}\ nops(x) = 2\ \textbf{end proc}$ (3.1.1.1)

```
> iscompbinop:=proc(X::set,E::set(ordpair),f::procedure)
local x,xx,y,yy;
for x in X do for y in X do
  if not f(x,y) in X then return false fi;
  for xx in X do for yy in X do
    if [x,xx] in E and [y,yy] in E and not [f(x,y),f(xx,yy)
] in E
      then return false fi;
od; od; od; od; true; end;

X:={0,1,2,3,4,5}; E:={[0,0],[0,3],[3,0],[3,3],[1,1],[1,4],
[4,1],[4,4],[2,2],[2,5],[5,2],[5,5]}; f:=(x,y)->irem(x+y,6)
; iscompbinop(X,E,f);
```

iscompbinop := **proc**(*X*::*set*, *E*::(*set*(*ordpair*)), *f*::*procedure*)
    local *x*, *xx*, *y*, *yy*;
    for *x* **in** *X* **do**
        for *y* **in** *X* **do**
            if not *in*(*f*(*x*, *y*), *X*) **then**
                return *false*
            end if;

```
            for xx in X do
                for yy in X do
                    if in([x, xx], E) and in([y, yy],
                    E) and not in([f(x, y), f(xx, yy)], E) then
                        return false
                    end if
                end do
            end do
        end do
    end do;
    true
end proc
```

$$X := \{0, 1, 2, 3, 4, 5\}$$

$$E := \{[0, 0], [0, 3], [3, 0], [3, 3], [1, 1], [1, 4], [4, 1], [4, 4], [2, 2], [2, 5], [5, 2], [5, 5]\}$$

$$f := (x, y) \rightarrow irem(x + y, 6)$$

$$true \quad (3.1.1.2)$$

3.1.2. Példa.

3.1.3. Feladat.

3.1.4. Osztályozás kompatibilitása relációval.

```
> iscomprel:=proc(X::set,E::set(ordpair),R::set(ordpair))
  local x,xx,y,yy;
  for x in X do for y in X do
    for xx in X do for yy in X do
      if [x,xx] in E and [y,yy] in E and [x,y] in R and not
  [xx,yy] in R
        then return false fi;
  od; od; od; od; true; end;

  X:={0,1,2,3}; E:={[0,0],[0,2],[2,0],[2,2],[1,1],[1,3],[3,1]
  ,[3,3]};
  R:={[0,1],[0,3],[2,1],[2,3]}; iscomprel(X,E,R);
```

```
iscomprel := proc(X::set, E::(set(ordpair)), R::(set(ordpair)))
    local x, xx, y, yy;
    for x in X do
        for y in X do
            for xx in X do
                for yy in X do
```

if $in([x, xx], E)$ **and** $in([y, yy],$
$E)$ **and** $in([x, y], R)$ **and** **not** $in([xx, yy], R)$ **then**
return *false*
end if
end do
end do
end do
end do;
true
end proc

$$X := \{0, 1, 2, 3\}$$

$$E := \{[0, 0], [3, 3], [1, 1], [2, 2], [0, 2], [1, 3], [2, 0], [3, 1]\}$$

$$R := \{[0, 3], [0, 1], [2, 1], [2, 3]\}$$

$$true \qquad (3.1.4.1)$$

3.1.5. Példa.

3.1.6. Tétel.

```
> type(5,integer); type(-3,integer); type(0,integer);
```

$$true$$

$$true$$

$$true \qquad (3.1.6.1)$$

```
> `&~`:=(x,y)->x[1]+y[2]=x[2]+y[1];

`&+`:=(x,y)->[x[1]+y[1],x[2]+y[2]];

`&*`:=(x,y)->[x[1]*y[1]+x[2]*y[2],x[1]*y[2]+y[1]*x[2]];

`&le`:=(x,y)->x[1]+y[2]<=x[2]+y[1];

[7,4]&~[3,0]; [7,4]&+[2,6]; [2,1]&*[2,4]; [3,5]&le[2,3];
```

$$\&\sim := (x, y) \rightarrow x_1 + y_2 = x_2 + y_1$$

$$\&+ := (x, y) \rightarrow [x_1 + y_1, x_2 + y_2]$$

$$\&* := (x, y) \rightarrow [x_1 y_1 + x_2 y_2, x_1 y_2 + y_1 x_2]$$

$$\&le := (x, y) \rightarrow x_1 + y_2 \leq x_2 + y_1$$

$$7 = 7$$

$$[9, 10]$$

$$[8, 10]$$

$$(3.1.6.2)$$

$$6 \leq 7 \quad (3.1.6.2)$$

3.1.7. Tétel: ℕ beágyazása ℤ-be.

3.1.8. Az egész számok rendezése.

3.1.9. Az egész számok szorzása.

3.1.10. Az egész számok számítógépes ábrázolása.

3.1.11. Hatványozás egész kitevővel.

```
> (a^(-1))^5; a^(m+n); expand(%); (a*b)^5;
  (a^n)^m; combine(%,power) assuming integer;
```

$$\frac{1}{a^5}$$

$$a^{m+n}$$

$$a^m a^n$$

$$a^5 b^5$$

$$\left(a^n\right)^m$$

$$a^{n m} \quad (3.1.11.1)$$

3.1.12. Példa.

3.1.13. Gyűrűk.

```
> isgrupoid:=proc(G::set,f::procedure) local x,y;
  for x in G do for y in G do if not f(x,y) in G then return
  false fi;
  od; od; true; end;
```

isgrupoid := **proc**(*G*::*set*, *f*::*procedure*) (3.1.13.1)

```
    local x, y;
    for x in G do
        for y in G do
            if not in(f(x, y), G) then
                return false
            end if
        end do
    end do;
    true
end proc
```

```
> neutral:=proc(G::set,f::procedure) local x,y,s,S;
  if not isgrupoid(G,f) then return NULL fi;
```

```
for x in G do s:=true; for y in G do
  if f(x,y)<>y or f(y,x)<>y then s:=false; break; fi;
od; if s then return x fi; od; NULL end;

G:={a,b,c};neutral(G,(x,y)->y);neutral(G,(x,y)->y);

neutral({0,1,2},(x,y)->irem(x+y,3));
```

```
neutral := proc(G::set, f::procedure)
    local x, y, s, S;
    if not isgrupoid(G, f) then
        return NULL
    end if;
    for x in G do
        s := true;
        for y in G do
            if f(x, y) <> y or f(y, x) <> y then
                s := false;
                break
            end if
        end do;
        if s then
            return x
        end if
    end do;
    NULL
end proc
```

$$G := \{a, b, c\}$$

$$0 \tag{3.1.13.2}$$

```
> issemigroup:=proc(G::set,f::procedure) local x,y,z;
if not isgrupoid(G,f) then return false fi;
for x in G do for y in G do for z in G do
  if f(x,f(y,z))<>f(f(x,y),z) then return false fi;
od; od; od; true end;

issemigroup({a,b,c},(x,y)->x);

issemigroup({true,false},(x,y)-> x implies y);
```

```
issemigroup := proc(G::set, f::procedure)
    local x, y, z;
    if not isgrupoid(G, f) then
```

```
        return false
    end if;
    for x in G do
        for y in G do
            for z in G do
                if f(x, f(y, z)) <> f(f(x, y), z) then
                    return false
                end if
            end do
        end do
    end do;
    true
end proc
```

true

false (3.1.13.3)

```
> isgroup:=proc(G::set,f::procedure) local x,y,n,i;
  if not isgrupoid(G,f) then return false fi;
  if not issemigroup(G,f) then return false fi;
  n:=neutral(G,f); if n=NULL then return false fi;
  for x in G do i:=false; for y in G do
    if f(x,y)=n and f(y,x)=n then i:=true; break fi;
  od; if i=false then return false fi; od; true; end;

  isgroup({0,1,2},(x,y)->irem(x+y,3));
```

```
isgroup := proc(G::set, f::procedure)
    local x, y, n, i;
    if not isgrupoid(G, f) then
        return false
    end if;
    if not issemigroup(G, f) then
        return false
    end if;
    n := neutral(G, f);
    if n = NULL then
        return false
    end if;
    for x in G do
        i := false;
        for y in G do
```

```
            if f(x, y) = n and f(y, x) = n then
                i := true;
                break
            end if
        end do;
        if i = false then
            return false
        end if
    end do;
    true
end proc
```

true (3.1.13.4)

```
> iscommutative:=proc(G::set,f::procedure) local x,y;
  if not isgrupoid(G,f) then return false fi;
  for x in G do for y in G do
    if f(x,y)<>f(y,x) then return false fi;
  od; od; true; end;

  iscommutative({0,1,2},(x,y)->irem(x+y,3));
```

```
iscommutative := proc(G::set, f::procedure)
    local x, y;
    if not isgrupoid(G, f) then
        return false
    end if;
    for x in G do
        for y in G do
            if f(x, y) <> f(y, x) then
                return false
            end if
        end do
    end do;
    true
end proc
```

true (3.1.13.5)

```
> isabeliangroup:=proc(G::set,f::procedure)
  isgroup(G,f) and iscommutative(G,f) end;

  iscommutative({0,1,2},(x,y)->irem(x+y,3));
```

isabeliangroup := **proc**(*G*::*set*, *f*::*procedure*)
    isgroup(*G*, *f*) **and** *iscommutative*(*G*, *f*)
end proc

true (3.1.13.6)

```
> isleftdistributive:=proc(R::set,f::procedure,g::procedure)
  local x,y,z;
  if not isgrupoid(R,f) then return false fi;
  if not isgrupoid(R,g) then return false fi;
  for x in R do for y in R do for z in R do
    if g(x,f(y,z))<>f(g(x,y),g(x,z)) then return false fi;
  od; od; od; true end;
```

isleftdistributive := **proc**(*R*::*set*, *f*::*procedure*, *g*::*procedure*) (3.1.13.7)

```
    local x, y, z;
    if not isgrupoid(R, f) then
        return false
    end if;
    if not isgrupoid(R, g) then
        return false
    end if;
    for x in R do
        for y in R do
            for z in R do
                if g(x, f(y, z)) <> f(g(x, y), g(x, z)) then
                    return false
                end if
            end do
        end do
    end do;
    true
end proc
```

```
> isrightdistributive:=proc(R::set,f::procedure,g::procedure)
  local x,y,z;
  if not isgrupoid(R,f) then return false fi;
  if not isgrupoid(R,g) then return false fi;
  for x in R do for y in R do for z in R do
    if g(f(y,z),x)<>f(g(y,x),g(z,x)) then return false fi;
  od; od; od; true end;
```

isrightdistributive := **proc**(*R*::*set*, *f*::*procedure*, *g*::*procedure*) (3.1.13.8)

```
    local x, y, z;
    if not isgrupoid(R, f) then
        return false
```

```
    end if;
    if not isgrupoid(R, g) then
        return false
    end if;
    for x in R do
        for y in R do
            for z in R do
                if g(f(y, z), x) <> f(g(y, x), g(z, x)) then
                    return false
                end if
            end do
        end do
    end do;
    true
end proc
```

```
> isring:=proc(R::set,f::procedure,g::procedure)
  isabeliangroup(R,f) and issemigroup(R,g)
  and isleftdistributive(R,f,g) and isrightdistributive(R,f,
  g) end;
```

isring := proc(R::set, f::procedure, g::procedure) (3.1.13.9)
    isabeliangroup(R, f) and issemigroup(R,
    g) and isleftdistributive(R, f, g) and isrightdistributive(R, f, g)
end proc

```
> iscommutativering:=proc(R::set,f::procedure,g::procedure)
  isring(R,f,g) and iscommutative(R,g) end;
```

iscommutativering := proc(R::set, f::procedure, g::procedure) (3.1.13.10)
    isring(R, f, g) and iscommutative(R, g)
end proc

```
> isringwithunity:=proc(R::set,f::procedure,g::procedure)
  isring(R,f,g) and neutral(R,g)<>NULL end;
```

isringwithunity := proc(R::set, f::procedure, g::procedure) (3.1.13.11)
    isring(R, f, g) and neutral(R, g) <> NULL
end proc

```
> isringwithunity({0},(x,y)->0,(x,y)->0);
```

true (3.1.13.12)

```
> X:={a,b,c}; P:=combinat[powerset](X);
  isring(P,(x,y)->symmdiff(x,y),(x,y)->{});
```

$X := \{a, b, c\}$

$P := \{\{\}, \{a, b, c\}, \{b, c\}, \{c\}, \{a, c\}, \{a\}, \{b\}, \{a, b\}\}$

true (3.1.13.13)

3.1.14. Példa.

```
> iscommutativering(P,(x,y)->symmdiff(x,y),(x,y)->x intersect y);
  isringwithunity(P,(x,y)->symmdiff(x,y),(x,y)->x intersect y);
```

$$true$$

$$true \qquad (3.1.14.1)$$

->3.1.15. Feladat.

3.1.16. Példák.

->3.1.17. Feladat.

->3.1.18. Feladat.

->3.1.19. Feladat.

3.1.20. Az általános disztributivitás tétele.

```
> A:=sum(a[i],i=1..4); B:=sum(b[j],j=1..5); A*B; expand(%);
```

$$A := a_1 + a_2 + a_3 + a_4$$

$$B := b_1 + b_2 + b_3 + b_4 + b_5$$

$$(a_1 + a_2 + a_3 + a_4)(b_1 + b_2 + b_3 + b_4 + b_5)$$

$$a_1 b_1 + a_1 b_2 + a_1 b_3 + a_1 b_4 + a_1 b_5 + a_2 b_1 + a_2 b_2 + a_2 b_3 + a_2 b_4 + a_2 b_5 + a_3 b_1 + a_3 b_2 + a_3 b_3 + a_3 b_4 + a_3 b_5 + a_4 b_1 + a_4 b_2 + a_4 b_3 + a_4 b_4 + a_4 b_5 \qquad (3.1.20.1)$$

3.1.21. Nullosztók, integritási tartomány, rendezett integritási tartomány.

3.1.22. Tétel.

3.1.23. Tétel.

->3.1.24. Feladat.

->3.1.25. Feladat.

3.2. Racionális számok

3.2.1. Tétel.

```
> type(5/7,rational); type(0,rational);
```

$$true$$
$$true \qquad (3.2.1.1)$$

```
> `&~`:=(x,y)->x[1]*y[2]=y[1]*x[2];

  `&+`:=(x,y)->[x[1]*y[2]+x[2]*y[1],x[2]*y[2]];

  `&*`:=(x,y)->[x[1]*y[1],x[2]*y[2]];

  `&le`:=(x,y)->(y[1]*x[2]-y[2]*x[1])*x[2]*y[2]>=0;

  [1,5]&~[2,10]; [1,2]&+[2,3]; [1,2]&*[2,3]; [1,2]&le[2,3];
```

$$\&\sim := (x, y) \rightarrow x_1\, y_2 = y_1\, x_2$$
$$\&+ := (x, y) \rightarrow \left[x_1\, y_2 + y_1\, x_2,\ x_2\, y_2\right]$$
$$\&* := (x, y) \rightarrow \left[x_1\, y_1,\ x_2\, y_2\right]$$
$$\&le := (x, y) \rightarrow 0 \le \left(y_1\, x_2 - x_1\, y_2\right) x_2\, y_2$$
$$10 = 10$$
$$[7, 6]$$
$$[2, 6]$$
$$0 \le 6 \qquad (3.2.1.2)$$

3.2.2. Tétel: ℤ beágyazása ℚ-ba.

3.2.3. Ferdetest, test, rendezett test.

```
> isskewfield:=proc(R::set,f::procedure,g::procedure) local
  n;
  n:=neutral(R,f); if n=NULL then return false fi;
  isring(R,f,g) and isgroup(R minus {n},g) end;
```

isskewfield := **proc**(*R*::*set*, *f*::*procedure*, *g*::*procedure*) (3.2.3.1)

local *n*;
n := *neutral*(*R*, *f*);
if *n* = *NULL* **then**
return *false*
end if;
isring(*R*, *f*, *g*) **and** *isgroup*(*minus*(*R*, {*n*}), *g*)
end proc

```
> isfield:=proc(R::set,f::procedure,g::procedure) local n;
  n:=neutral(R,f); if n=NULL then return false fi;
```

```
isring(R,f,g) and isabeliangroup(R minus {n},g) end;
```

$isfield := \mathbf{proc}(R::set, f::procedure, g::procedure)$ (3.2.3.2)

local n;

$n := neutral(R, f)$;

if $n = NULL$ **then**

return $false$

end if;

$isring(R, f, g)$ **and** $isabeliangroup(minus(R, \{n\}), g)$

end proc

```
> &+(0,0):=0; &+(0,1):=1; &+(1,0):=1; &+(1,1):=0;

  &*(0,0):=0; &*(0,1):=0; &*(1,0):=0; &*(1,1):=1;
```

$$0 \;\&+\; 0 := 0$$
$$0 \;\&+\; 1 := 1$$
$$1 \;\&+\; 0 := 1$$
$$1 \;\&+\; 1 := 0$$
$$0 \;\&*\; 0 := 0$$
$$0 \;\&*\; 1 := 0$$
$$1 \;\&*\; 0 := 0$$
$$1 \;\&*\; 1 := 1 \qquad (3.2.3.3)$$

```
> isfield({0,1},(x,y)->x&+y,(x,y)->x&*y);
```

$$true \qquad (3.2.3.4)$$

3.2.4. *Példák.*

```
> `&+`:=(x,y)->irem(x+y,5); `&*`:=(x,y)->irem(x*y,5); 3&+4;
  3&*4;
```

$$\&+ := (x, y) \rightarrow irem(x + y, 5)$$
$$\&* := (x, y) \rightarrow irem(x\, y, 5)$$
$$2$$
$$2 \qquad (3.2.4.1)$$

3.2.5 *Tétel: $\mathbb{Q}$ beágyazása rendezett testbe.*

->3.2.6. Feladat.

->3.2.7. Feladat.

->3.2.8. Feladat.

3.2.9. Feladat.

3.3. Valós számok

```
> restart;
```

3.3.1. Állítás.

```
> i:='i': x:=0:
  for i from 0 do while (x+1)^2<2*10^(2*i) do x:=x+1: od; x;
  x:=x*10: od;
```

$$1$$
$$x := 10$$
$$14$$
$$x := 140$$
$$141$$
$$x := 1410$$
$$1414$$
$$x := 14140$$
$$14142$$
$$x := 141420$$
$$141421$$
$$x := 1414210$$
$$1414213$$
$$x := 14142130$$
$$14142135$$
$$x := 141421350$$
$$141421356$$
$$x := 1414213560$$
$$1414213562$$
$$x := 14142135620$$
$$14142135623$$
$$x := 141421356230$$
$$141421356237$$
$$x := 1414213562370$$
$$1414213562373$$
$$x := 14142135623730$$
$$14142135623730$$
$$x := 141421356237300$$
$$141421356237309$$
$$x := 1414213562373090$$

$$1414213562373095$$
$$x := 14142135623730950$$
$$14142135623730950$$
$$x := 141421356237309500$$
$$141421356237309504$$
$$x := 1414213562373095040$$
$$1414213562373095048$$
$$x := 14142135623730950480$$
$$14142135623730950488$$
$$x := 141421356237309504880$$
$$141421356237309504880$$
$$x := 1414213562373095048800$$
$$1414213562373095048801$$
$$x := 14142135623730950488010$$
$$14142135623730950488016$$
$$x := 141421356237309504880160$$
$$141421356237309504880168$$
$$x := 1414213562373095048801680$$
$$1414213562373095048801688$$
$$x := 14142135623730950488016880$$
$$14142135623730950488016887$$
$$x := 141421356237309504880168870$$
$$141421356237309504880168872$$
$$x := 1414213562373095048801688720$$
$$1414213562373095048801688724$$
$$x := 14142135623730950488016887240$$
$$14142135623730950488016887242$$
$$x := 141421356237309504880168872420$$
$$141421356237309504880168872420$$
$$x := 1414213562373095048801688724200$$
$$1414213562373095048801688724209$$
$$x := 14142135623730950488016887242090$$
$$14142135623730950488016887242096$$
$$x := 141421356237309504880168872420960$$
$$141421356237309504880168872420969$$
$$x := 1414213562373095048801688724209690$$
$$1414213562373095048801688724209698$$
$$x := 14142135623730950488016887242096980$$

14142135623730950488016887242096980

$x :=$ 141421356237309504880168872420969800

141421356237309504880168872420969807

$x :=$ 1414213562373095048801688724209698070

1414213562373095048801688724209698078

$x :=$ 14142135623730950488016887242096980780

14142135623730950488016887242096980785

$x :=$ 141421356237309504880168872420969807850

141421356237309504880168872420969807856

$x :=$ 1414213562373095048801688724209698078560

1414213562373095048801688724209698078569

$x :=$ 14142135623730950488016887242096980785690

14142135623730950488016887242096980785696

$x :=$ 141421356237309504880168872420969807856960

141421356237309504880168872420969807856967

$x :=$ 1414213562373095048801688724209698078569670

1414213562373095048801688724209698078569671

$x :=$ 14142135623730950488016887242096980785696710

14142135623730950488016887242096980785696718

$x :=$ 141421356237309504880168872420969807856967180

141421356237309504880168872420969807856967187

$x :=$ 1414213562373095048801688724209698078569671870

1414213562373095048801688724209698078569671875

$x :=$ 14142135623730950488016887242096980785696718750

14142135623730950488016887242096980785696718753

$x :=$ 141421356237309504880168872420969807856967187530

141421356237309504880168872420969807856967187537

$x :=$ 1414213562373095048801688724209698078569671875370

1414213562373095048801688724209698078569671875376

$x :=$ 14142135623730950488016887242096980785696718753760

14142135623730950488016887242096980785696718753769

$x :=$ 141421356237309504880168872420969807856967187537690

141421356237309504880168872420969807856967187537694

$x :=$ 1414213562373095048801688724209698078569671875376940

1414213562373095048801688724209698078569671875376948

$x :=$ 14142135623730950488016887242096980785696718753769480

14142135623730950488016887242096980785696718753769480

$x :=$ 141421356237309504880168872420969807856967187537694800

1414213562373095048801688724209698078569671875376948 07

$x :=$ 14142135623730950488016887242096980785696718753769480 70

1414213562373095048801688724209698078569671875376948073

$x :=$ 14142135623730950488016887242096980785696718753769480730

14142135623730950488016887242096980785696718753769480731

$x :=$ 141421356237309504880168872420969807856967187537694807310

141421356237309504880168872420969807856967187537694807317

$x :=$ 1414213562373095048801688724209698078569671875376948073170

1414213562373095048801688724209698078569671875376948073176

$x :=$
1414213562373095048801688724209698078569671875376948073\
1760

14142135623730950488016887242096980785696718753769480731766

$x :=$
1414213562373095048801688724209698078569671875376948073\
17660

141421356237309504880168872420969807856967187537694807317667

$x :=$
1414213562373095048801688724209698078569671875376948073\
176670

1414213562373095048801688724209698078569671875376948073176679

$x :=$
1414213562373095048801688724209698078569671875376948073\
1766790

1414213562373095048801688724209698078569671875376948073176679\
7

$x :=$
1414213562373095048801688724209698078569671875376948073\
17667970

1414213562373095048801688724209698078569671875376948073176679\
73

$x :=$
1414213562373095048801688724209698078569671875376948073\
176679730

1414213562373095048801688724209698078569671875376948073176679\
737

$x :=$
1414213562373095048801688724209698078569671875376948073\

1766797370

14142135623730950488016887242096980785696718753769480731766 79\
7379

x:=
1414213562373095048801688724209698078569671875376948073\
17667973790

14142135623730950488016887242096980785696718753769480731766 79\
73799

x:=
1414213562373095048801688724209698078569671875376948073\
176679737990

14142135623730950488016887242096980785696718753769480731766 79\
737990

x:=
1414213562373095048801688724209698078569671875376948073\
1766797379900

14142135623730950488016887242096980785696718753769480731766 79\
7379907

x:=
1414213562373095048801688724209698078569671875376948073\
17667973799070

14142135623730950488016887242096980785696718753769480731766 79\
73799073

x:=
1414213562373095048801688724209698078569671875376948073\
176679737990730

14142135623730950488016887242096980785696718753769480731766 79\
737990732

x:=
1414213562373095048801688724209698078569671875376948073\
1766797379907320

14142135623730950488016887242096980785696718753769480731766 79\
7379907324

x:=
1414213562373095048801688724209698078569671875376948073\
17667973799073240

14142135623730950488016887242096980785696718753769480731766 79\
73799073247

x :=
1414213562373095048801688724209698078569671875376948073\
176679737990732470
1414213562373095048801688724209698078569671875376948073176679\
737990732478
x :=
1414213562373095048801688724209698078569671875376948073\
1766797379907324780
1414213562373095048801688724209698078569671875376948073176679\
7379907324784
x :=
1414213562373095048801688724209698078569671875376948073\
17667973799073247840
1414213562373095048801688724209698078569671875376948073176679\
73799073247846
x :=
1414213562373095048801688724209698078569671875376948073\
176679737990732478460
1414213562373095048801688724209698078569671875376948073176679\
737990732478462
x :=
1414213562373095048801688724209698078569671875376948073\
1766797379907324784620
1414213562373095048801688724209698078569671875376948073176679\
7379907324784621
x :=
1414213562373095048801688724209698078569671875376948073\
17667973799073247846210
1414213562373095048801688724209698078569671875376948073176679\
73799073247846210
x :=
1414213562373095048801688724209698078569671875376948073\
176679737990732478462100
1414213562373095048801688724209698078569671875376948073176679\
737990732478462107
x :=
1414213562373095048801688724209698078569671875376948073\
1766797379907324784621070

141421356237309504880168872420969807856967187537694807317667\
737990732478462107O

$x :=$

14142135623730950488016887242096980785696718753769480731\
76679737990732478462107OO

141421356237309504880168872420969807856967187537694807317667\
7379907324784621O7O3

$x :=$

14142135623730950488016887242096980785696718753769480731\
766797379907324784621O7O3O

141421356237309504880168872420969807856967187537694807317667\
737990732478462107O38

$x :=$

14142135623730950488016887242096980785696718753769480731\
766797379907324784621O7O38O

141421356237309504880168872420969807856967187537694807317667\
737990732478462107O388

$x :=$

14142135623730950488016887242096980785696718753769480731\
7667973799073247846210703880

141421356237309504880168872420969807856967187537694807317667\
7379907324784621070388 5

$x :=$

14142135623730950488016887242096980785696718753769480731\
766797379907324784621070388 50

141421356237309504880168872420969807856967187537694807317667\
73799073247846210703885O

$x :=$

14142135623730950488016887242096980785696718753769480731\
766797379907324784621070388500

141421356237309504880168872420969807856967187537694807317667\
7379907324784621070388503

$x :=$

14142135623730950488016887242096980785696718753769480731\
7667973799073247846210703885030

141421356237309504880168872420969807856967187537694807317667\
73799073247846210703885038

$x :=$

```
        1414213562373095048801688724209698078569671875376948073\
        176679737990732478462107038850380
1414213562373095048801688724209698078569671875376948073176679\
        737990732478462107038850387
x:=
        1414213562373095048801688724209698078569671875376948073\
        1766797379907324784621070388503870
1414213562373095048801688724209698078569671875376948073176679\
        7379907324784621070388503875
x:=
        1414213562373095048801688724209698078569671875376948073\
        17667973799073247846210703885038750
1414213562373095048801688724209698078569671875376948073176679\
        73799073247846210703885038753
x:=
        1414213562373095048801688724209698078569671875376948073\
        176679737990732478462107038850387530
1414213562373095048801688724209698078569671875376948073176679\
        737990732478462107038850387534
x:=
        1414213562373095048801688724209698078569671875376948073\
        1766797379907324784621070388503875340
1414213562373095048801688724209698078569671875376948073176679\
        7379907324784621070388503875343
Warning,  computation interrupted
```

▶ ***3.3.2. Arkhimédészi tulajdonság.***

▶ ***3.3.3. Állítás.***

▶ ***3.3.4. Állítás.***

▶ ****3.3.5. Tétel.***

▶ ***3.3.6. Tétel.***

▼ ***3.3.7. Valós számok.***

```
> abs(7.4); abs(-3); abs(0); signum(7.4); signum(-3); signum
  (0);
```

7.4

3

0

$$1$$
$$-1$$
$$0 \qquad (3.3.7.1)$$

```
> floor(3.14); ceil(3.14); ceil(-3.14);
```

$$3$$
$$4$$
$$-3 \qquad (3.3.7.2)$$

```
> Rmod:=proc(x::realcons,y::realcons) if y=0 then x else x-
  floor(x/y)*y fi; end;

  Rmod(5,0); Rmod(3.1415,2.78);
```

Rmod := **proc**(*x*::*realcons*, *y*::*realcons*)
  if *y* = 0 **then**
    x
  else
    x − floor(*x* / *y*) * *y*
  end if
end proc

$$5$$
$$0.3615 \qquad (3.3.7.3)$$

▼ ***3.3.8. Bővített valós számok.***

▼ ***3.3.9. Valós számok kerekítése és fixpontos ábrázolása számítógépben.***

▼ ***3.3.10. Valós számok lebegőpontos ábrázolása számítógépben.***

▶ ***3.3.11. Tétel: a valós számok létezése.***

▶ ****3.3.12. A valós számok más bevezetései.***

▼ ***3.3.13. Tétel: gyökvonás.***

▶ ***3.3.14. Következmény.***

▼ ***3.3.15. A természetes, az egész és a racionális számok bevezetése a valós számok segítségével.***

▶ ***->3.2.16. Feladat.***

▶ ***->3.2.17. Feladat.***

▶ ***3.2.18. Feladat.***

▶ ***3.2.19. Feladat.***

▶ ***3.2.20. Feladat.***

- *3.2.21. Feladat.*
- *–>3.2.22. Feladat.*
- *3.2.23. Feladat.*
- *3.2.24. Feladat.*
- *3.2.25. Feladat.*
- *–>3.2.26. Feladat.*
- *–>3.2.27. Feladat.*
- *–>3.2.28. Feladat.*
- *–>3.2.29. Feladat.*
- *–>3.2.30. Feladat.*
- *3.2.31. Feladat.*
- *3.2.32. Feladat.*
- *3.2.33. Feladat.*
- *3.2.34. Feladat: öröknaptár.*
- *3.2.35. Feladat.*
- *–>3.2.36. Feladat.*
- *–>3.2.37. Feladat.*
- *3.2.38. Feladat.*
- *3.2.39. Feladat.*
- *3.2.40. Feladat.*
- *3.2.41. Feladat.*
- *3.2.42. Feladat.*
- *3.2.43. Feladat.*
- *3.2.44. Feladat.*
- *3.2.45. Feladat.*
- *3.2.46. Feladat.*
- *3.2.47. Feladat.*
- **3.2.48. Feladat.*
- *3.2.49. Feladat.*
- **3.2.50. Feladat.*
- *3.3.51. További feladatok.*

3.4. Komplex számok

```
> restart;
```

3.4.1. Komplex számok.

```
> `&+`:=proc(z,w) [z[1]+w[1],z[2]+w[2]] end;
  `&*`:=proc(z,w) [z[1]*w[1]-z[2]*w[2],z[1]*w[2]+z[2]*w[1]]
  end;

  [x,y]&+[0,0]; [x,y]&+[-x,-y]; [x,y]&*[1,0];

  [x,y]&*[x/(x^2+y^2),-y/(x^2+y^2)]; simplify(%);

  [0,1]&*[0,1];
```

$\&+ := \textbf{proc}(z, w)\ [z[1]+w[1], z[2]+w[2]]\ \textbf{end proc}$

$\&* := \textbf{proc}(z, w)$
$\quad [z[1]*w[1]-z[2]*w[2], z[1]*w[2]+z[2]*w[1]]$
$\textbf{end proc}$

$$[x, y]$$
$$[0, 0]$$
$$[x, y]$$
$$\left[\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}, 0\right]$$
$$[1, 0]$$
$$[-1, 0] \qquad (3.4.1.1)$$

```
> Complex(3,5); z:=3+5*I; w:=-2-6*I; z*w; Re(z); Im(z);
  conjugate(z);
```

$$3+5\,I$$
$$z := 3+5\,I$$
$$w := -2-6\,I$$
$$24-28\,I$$
$$3$$
$$5$$
$$3-5\,I \qquad (3.4.1.2)$$

```
> z:='z';w:='w'; conjugate(z);

  conjugate(conjugate(z));conjugate(z+w);expand(%);conjugate
  (1/z);
```

$$z := z$$
$$w := w$$

$$\bar{z}$$

$$\frac{z}{z+w}$$

$$\bar{z}+\bar{w}$$

$$\frac{1}{\bar{z}} \qquad (3.4.1.3)$$

3.4.2. Példa.

```
> 64/(3^(1/2)+I); evalc(%);
```

$$\frac{64}{\sqrt{3}+I}$$

$$16\sqrt{3}-16\,I \qquad (3.4.2.1)$$

3.4.3. Komplex szám abszolút értéke.

```
> z:=x+I*y;abs(z);evalc(%);evalc(1/(x+I*y));evalc(conjugate
  (z)/abs(z)^2);
```

$$z := x+I\,y$$

$$|x+I\,y|$$

$$\sqrt{x^2+y^2}$$

$$\frac{x}{x^2+y^2}-\frac{I\,y}{x^2+y^2}$$

$$\frac{x}{x^2+y^2}-\frac{I\,y}{x^2+y^2} \qquad (3.4.3.1)$$

```
> signum(3+4*I); signum(-5); signum(0);
```

$$\frac{3}{5}+\frac{4}{5}\,I$$

$$-1$$

$$0 \qquad (3.4.3.2)$$

->3.4.4. Feladat.

->3.4.5. Feladat.

->3.4.6. Feladat.

```
> evalc(2/(1-I)/(3+I)); evalc(1/(3+4*I)^2); evalc((2+I)/I/(
  -3+4*I));
  evalc((3^(1/2)+I)/(1-I)/(3^(1/2)-I)); simplify(%);
  evalc(1/I/(3-2*I)/(1+I)); evalc(1/(1-I)/(1-2*I)/(1+2*I));
```

$$\frac{2}{5}+\frac{1}{5}\,I$$

$$-\frac{7}{625}-\frac{24}{625}\,I$$

$$-\frac{11}{25}+\frac{2}{25}\,I$$

$$\frac{1}{4}\left(\frac{1}{2}\sqrt{3}-\frac{1}{2}\right)\sqrt{3}-\frac{1}{8}\sqrt{3}-\frac{1}{8}+I\left(\frac{1}{4}\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\right)\sqrt{3}+\frac{1}{8}\sqrt{3}-\frac{1}{8}\right)$$

$$-\frac{1}{4}\sqrt{3}+\frac{1}{4}+\frac{1}{4}\,I+\frac{1}{4}\,I\sqrt{3}$$

$$-\frac{1}{26}-\frac{5}{26}\,I$$

$$\frac{1}{10}+\frac{1}{10}\,I \qquad (3.4.6.1)$$

->3.4.7. Feladat.

3.4.8. Feladat.

3.4.9. Feladat.

3.4.10. Komplex szám argumentuma és trigonometrikus alakja.

```
> polar(x+I*y); op(1,%); op(2,%%); polar(3+4*I); evalc(%);
  argument(3+I*4);
```

$$\mathrm{polar}(|x+I\,y|, \mathrm{argument}(x+I\,y))$$

$$|x+I\,y|$$

$$\mathrm{argument}(x+I\,y)$$

$$\mathrm{polar}\left(5, \arctan\left(\frac{4}{3}\right)\right)$$

$$3+4\,I$$

$$\arctan\left(\frac{4}{3}\right) \qquad (3.4.10.1)$$

3.4.11. Példa.

```
> z:=16*sqrt(3)-I*16; polar(z);
```

$$z := 16\sqrt{3}-16\,I$$

$$\mathrm{polar}\left(32, -\frac{1}{6}\,\pi\right) \qquad (3.4.11.1)$$

3.4.12. Gyökvonás komplex számból.

3.4.13. Példa.

```
> z:='z'; i:='i'; w:=16*sqrt(3)-I*16; solve(z^5=w,z); z1:=w^
(1/5);

r:=abs(w); phi:=argument(w);

r^(1/5)*(cos(phi/5+i*2*Pi/5)+I*sin(phi/5+i*2*Pi/5))$i=0..4;
evalf(%);

solve(z^5=1); map(z->evalf(z*z1),[%]);
```

$$z := z$$

$$i := i$$

$$w := 16\sqrt{3} - 16\,\mathrm{I}$$

Warning, solutions may have been lost

$$z1 := \left(16\sqrt{3} - 16\,\mathrm{I}\right)^{1/5}$$

$$r := 32$$

$$\phi := -\frac{1}{6}\,\pi$$

$$32^{1/5}\left(\sin\left(\frac{7}{15}\,\pi\right) - \mathrm{I}\cos\left(\frac{7}{15}\,\pi\right)\right),\ 32^{1/5}\left(\sin\left(\frac{2}{15}\,\pi\right) + \mathrm{I}\cos\left(\frac{2}{15}\,\pi\right)\right),\ 32^{1/5}\left(-\sin\left(\frac{4}{15}\,\pi\right) + \mathrm{I}\cos\left(\frac{4}{15}\,\pi\right)\right),\ 32^{1/5}\left(-\frac{1}{2}\sqrt{3} - \frac{1}{2}\,\mathrm{I}\right),\ 32^{1/5}\left(\sin\left(\frac{1}{15}\,\pi\right) - \mathrm{I}\cos\left(\frac{1}{15}\,\pi\right)\right)$$

$$1.989043791 - 0.2090569258\,\mathrm{I},\ 0.8134732860 + 1.827090915\,\mathrm{I},\ -1.486289651 + 1.338261212\,\mathrm{I},\ -1.732050808 - 1.000000000\,\mathrm{I},\ 0.4158233818 - 1.956295201\,\mathrm{I}$$

Warning, solutions may have been lost

$$1,\ -\frac{1}{4} + \frac{1}{4}\sqrt{5} + \frac{1}{4}\,\mathrm{I}\sqrt{2}\sqrt{5+\sqrt{5}},\ -\frac{1}{4} - \frac{1}{4}\sqrt{5} + \frac{1}{4}\,\mathrm{I}\sqrt{2}\sqrt{5-\sqrt{5}},\ -\frac{1}{4} - \frac{1}{4}\sqrt{5} - \frac{1}{4}\,\mathrm{I}\sqrt{2}\sqrt{5-\sqrt{5}},$$

$$-\frac{1}{4}+\frac{1}{4}\sqrt{5}-\frac{1}{4}\mathrm{I}\sqrt{2}\sqrt{5+\sqrt{5}}$$

$$[1.989043791-0.2090569265\,\mathrm{I},\ 0.8134732858+1.827090915\,\mathrm{I},\ -1.486289651+1.338261212\,\mathrm{I},\ -1.732050807-0.9999999996\,\mathrm{I},\ 0.4158233815-1.956295201\,\mathrm{I}] \quad (3.4.13.1)$$

3.4.14. Az algebra alaptétele.

```
> f:=(x-1)^2*(x-2); f:=expand(f); solve(f,x); solve(x^3=1,x);
  r:=[%];
```

$$f := (x-1)^2\,(x-2)$$

$$f := x^3-4\,x^2+5\,x-2$$

```
Warning, solutions may have been lost
```

$$2,\ 1,\ 1$$

```
Warning, solutions may have been lost
```

$$1,\ -\frac{1}{2}+\frac{1}{2}\mathrm{I}\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\mathrm{I}\sqrt{3}$$

$$r := \left[1,\ -\frac{1}{2}+\frac{1}{2}\mathrm{I}\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\mathrm{I}\sqrt{3}\right] \quad (3.4.14.1)$$

->3.4.15. Feladat.
->3.4.16. Feladat.
->3.4.17. Feladat.
3.4.18. Feladat.
3.4.19. Feladat.
3.4.20. Feladat.
3.4.21. Feladat.
->3.4.22. Feladat.
->3.4.23. Feladat.
->3.4.24. Feladat.
->3.4.25. Feladat.
3.4.26. Feladat.
3.4.27. Feladat.
3.4.28. Kvaterniók.

```
> `&+`:=(p,q)->[p[1]+q[1],p[2]+q[2]];
```

```
`&*`:=(p,q)->[p[1]*q[1]-conjugate(q[2])*p[2],q[2]*p[1]+p[2]*conjugate(q[1])];
```

$$\&+ := (p, q) \rightarrow [p_1 + q_1, p_2 + q_2]$$

$$\&* := (p, q) \rightarrow [p_1 q_1 - \overline{q_2} p_2, p_1 q_2 + p_2 \overline{q_1}] \tag{3.4.28.1}$$

```
> p:=[a+I*b,c+I*d]; p&+[0,0]; p&+[-a-I*b,-c-I*d];
```

$$p := [a + \mathrm{I} b, c + \mathrm{I} d]$$

$$[a + \mathrm{I} b, c + \mathrm{I} d]$$

$$[0, 0] \tag{3.4.28.2}$$

```
> p&*[1,0]; [1,0]&*p;

q:=[(a-I*b)/(a^2+b^2+c^2+d^2),(-c-I*d)/(a^2+b^2+c^2+d^2)];
p&*q;evalc(%);simplify(%); q&*p;evalc(%);simplify(%);
```

$$[a + \mathrm{I} b, c + \mathrm{I} d]$$

$$[a + \mathrm{I} b, c + \mathrm{I} d]$$

$$q := \left[\frac{a - \mathrm{I} b}{a^2 + b^2 + c^2 + d^2}, \frac{-c - \mathrm{I} d}{a^2 + b^2 + c^2 + d^2}\right]$$

$$\left[\frac{(a + \mathrm{I} b)(a - \mathrm{I} b)}{a^2 + b^2 + c^2 + d^2} - \overline{\frac{-c - \mathrm{I} d}{a^2 + b^2 + c^2 + d^2}}\,(c + \mathrm{I} d),\right.$$
$$\left.\frac{(a + \mathrm{I} b)(-c - \mathrm{I} d)}{a^2 + b^2 + c^2 + d^2} + (c + \mathrm{I} d)\,\overline{\frac{a - \mathrm{I} b}{a^2 + b^2 + c^2 + d^2}}\right]$$

$$\left[\frac{a^2 + b^2}{a^2 + b^2 + c^2 + d^2} + \frac{c^2}{a^2 + b^2 + c^2 + d^2} + \frac{d^2}{a^2 + b^2 + c^2 + d^2},\right.$$
$$\frac{-a c + b d}{a^2 + b^2 + c^2 + d^2} + \frac{c a}{a^2 + b^2 + c^2 + d^2} - \frac{d b}{a^2 + b^2 + c^2 + d^2}$$
$$\left. + \mathrm{I}\left(\frac{-b c - a d}{a^2 + b^2 + c^2 + d^2} + \frac{d a}{a^2 + b^2 + c^2 + d^2} + \frac{c b}{a^2 + b^2 + c^2 + d^2}\right)\right]$$

$$[1, 0]$$

$$\left[\frac{(a + \mathrm{I} b)(a - \mathrm{I} b)}{a^2 + b^2 + c^2 + d^2} - \frac{\overline{c + \mathrm{I} d}\,(-c - \mathrm{I} d)}{a^2 + b^2 + c^2 + d^2},\right.$$
$$\left.\frac{(a - \mathrm{I} b)(c + \mathrm{I} d)}{a^2 + b^2 + c^2 + d^2} + \frac{(-c - \mathrm{I} d)\,\overline{a + \mathrm{I} b}}{a^2 + b^2 + c^2 + d^2}\right]$$

$$\left[\frac{a^2 + b^2}{a^2 + b^2 + c^2 + d^2} + \frac{c^2 + d^2}{a^2 + b^2 + c^2 + d^2}, 0\right]$$

$$[1, 0] \tag{3.4.28.3}$$

```
> z:='z';w:='w';z1:='z1';p:=[z,w];p1:=[z1,w1];p2:=[z2,w2];
```

```
p&*(p1&*p2);expand(%);(p&*p1)&*p2;expand(%);
```

$$z := z$$
$$w := w$$
$$z1 := z1$$
$$p := [z, w]$$
$$p1 := [z1, w1]$$
$$p2 := [z2, w2]$$
$$\left[z\left(z1\,z2-\overline{w2}\,w1\right)-\overline{z1\,w2+w1\,\overline{z2}}\,w,\; z\left(z1\,w2+w1\,\overline{z2}\right)+w\,\overline{z1\,z2-\overline{w2}\,w1}\right]$$
$$\left[z\,z1\,z2-z\,\overline{w2}\,w1-w\,\overline{z1\,w2}-w\,z2\,\overline{w1},\; z\,z1\,w2+z\,w1\,\overline{z2}+w\,\overline{z1\,z2}-w\,w2\,\overline{w1}\right]$$
$$\left[\left(z\,z1-\overline{w1}\,w\right)z2-\overline{w2}\left(z\,w1+w\,\overline{z1}\right),\; \left(z\,z1-\overline{w1}\,w\right)w2+\left(z\,w1+w\,\overline{z1}\right)\overline{z2}\right]$$
$$\left[z\,z1\,z2-w\,z2\,\overline{w1}-z\,\overline{w2}\,w1-\overline{w2}\,w\,\overline{z1},\; z\,z1\,w2-w\,w2\,\overline{w1}+z\,w1\,\overline{z2}+\overline{z2}\,w\,\overline{z1}\right] \quad (3.4.28.4)$$

```
> p&*(p1&+p2);expand(%);(p&*p1)&+(p&*p2);
  (p1&+p2)&*p;expand(%);(p1&*p)&+(p2&*p);
```

$$\left[z\left(z1+z2\right)-\overline{w1+w2}\,w,\; z\left(w1+w2\right)+w\,\overline{z1+z2}\right]$$
$$\left[z\,z1+z\,z2-\overline{w1}\,w-w\,\overline{w2},\; z\,w1+z\,w2+w\,\overline{z1}+w\,\overline{z2}\right]$$
$$\left[z\,z1+z\,z2-\overline{w1}\,w-w\,\overline{w2},\; z\,w1+z\,w2+w\,\overline{z1}+w\,\overline{z2}\right]$$
$$\left[z\left(z1+z2\right)-\overline{w}\left(w1+w2\right),\; \left(z1+z2\right)w+\left(w1+w2\right)\overline{z}\right]$$
$$\left[z\,z1+z\,z2-\overline{w}\,w1-\overline{w}\,w2,\; w\,z1+w\,z2+\overline{z}\,w1+\overline{z}\,w2\right]$$
$$\left[z\,z1+z\,z2-\overline{w}\,w1-\overline{w}\,w2,\; w\,z1+w\,z2+\overline{z}\,w1+\overline{z}\,w2\right] \quad (3.4.28.5)$$

```
> j:=[0,1]; j&*j; [z,0]&+([w,0]&*j);
```

$$j := [0, 1]$$
$$[-1, 0]$$
$$[z, w] \quad (3.4.28.6)$$

```
> k:=[0,I]; k&*k; i:=[I,0]; i&*i; [a,0]&+([b,0]&*i)&+([c,0]&*
  j)&+([d,0]&*k);
```

$$k := [0, \mathrm{I}]$$
$$[-1, 0]$$
$$i := [\mathrm{I}, 0]$$
$$[-1, 0]$$
$$[a+\mathrm{I}\,b,\; c+\mathrm{I}\,d] \quad (3.4.28.7)$$

```
> p:=[a+I*b,c+I*d]; evalc([x,0]&*p); evalc(p&*[x,0]);
```

$$p := [a + \mathrm{I}\, b,\ c + \mathrm{I}\, d]$$
$$[x\, a + \mathrm{I}\, x\, b,\ x\, c + \mathrm{I}\, x\, d]$$
$$[x\, a + \mathrm{I}\, x\, b,\ x\, c + \mathrm{I}\, x\, d] \quad (3.4.28.8)$$

```
> j&*[z,0]; [z,0]&*j;
```

$$[0,\ \bar{z}]$$
$$[0,\ z] \quad (3.4.28.9)$$

```
> i&*j; j&*k; k&*i; j&*i; k&*j; i&*k;
```

$$[0,\ \mathrm{I}]$$
$$[\mathrm{I},\ 0]$$
$$[0,\ 1]$$
$$[0,\ -\mathrm{I}]$$
$$[-\mathrm{I},\ 0]$$
$$[0,\ -1] \quad (3.4.28.10)$$

```
> i:='i'; j:='j'; k:='k';

  C2toR4:=q->evalc(Re(q[1])+Im(q[1])*i+Re(q[2])*j+Im(q[2])*k)
  ; q:=C2toR4(p);
```

$$i := i$$
$$j := j$$
$$k := k$$
$$C2toR4 := q \rightarrow evalc\left(\Re(q_1) + \Im(q_1)\, i + \Re(q_2)\, j + \Im(q_2)\, k\right)$$
$$q := a + b\, i + c\, j + d\, k \quad (3.4.28.11)$$

```
> R4toC2:=q->[q-coeff(q,i)*i-coeff(q,j)*j-coeff(q,k)*k+I*
  coeff(q,i),coeff(q,j)+I*coeff(q,k)]; R4toC2(q);
```

$$R4toC2 := q \rightarrow [q - coeff(q, i)\, i - coeff(q, j)\, j - coeff(q, k)\, k + \mathrm{I}\, coeff(q, i),\ coeff(q, j) + \mathrm{I}\, coeff(q, k)]$$
$$[a + \mathrm{I}\, b,\ c + \mathrm{I}\, d] \quad (3.4.28.12)$$

```
> qIm:=q->coeff(q,i)*i+coeff(q,j)*j+coeff(q,k)*k;qRe:=q->q-
  qIm(q); qRe(q); qIm(q);
```

$$qIm := q \rightarrow coeff(q, i)\, i + coeff(q, j)\, j + coeff(q, k)\, k$$
$$qRe := q \rightarrow q - qIm(q)$$
$$a$$
$$b\, i + c\, j + d\, k \quad (3.4.28.13)$$

```
> qconjugate:=q->qRe(q)-qIm(q); qconjugate(q);
```

$$qconjugate := q \rightarrow qRe(q) - qIm(q)$$

$$a - b\,i - c\,j - d\,k \qquad (3.4.28.14)$$

```
> q; qconjugate(q); qconjugate(%); q+qconjugate(q); q-
  qconjugate(q);
```

$$a + b\,i + c\,j + d\,k$$

$$a - b\,i - c\,j - d\,k$$

$$a + b\,i + c\,j + d\,k$$

$$2\,a$$

$$2\,b\,i + 2\,c\,j + 2\,d\,k \qquad (3.4.28.15)$$

```
> q1:=a1+b1*i+c1*j+d1*k; q2:=a2+b2*i+c2*j+d2*k;

  q1+q2; collect(%,[i,j,k]); `&+`:=(q1,q2)->collect(q1+q2,[i,
  j,k]); q1&+q2;

  `&*`:=proc(q1,q2) local a1,a2,b1,b2,c1,c2,d1,d2;
  a1:=qRe(q1);a2:=qRe(q2);b1:=coeff(q1,i);b2:=coeff(q2,i);
  c1:=coeff(q1,j);c2:=coeff(q2,j);d1:=coeff(q1,k);d2:=coeff
  (q2,k);
  (a1*a2-b1*b2-c1*c2-d1*d2)+(a1*b2+a2*b1+c1*d2-d1*c2)*i+
  (a1*c2+c1*a2+d1*b2-b1*d2)*j+(a1*d2+d1*a2+b1*c2-c1*b2)*k;
  end;

  q1&*q2;

  qconjugate(q1&+q2); qconjugate(q1)&+qconjugate(q2);

  qconjugate(q1&*q2);qconjugate(q2)&*qconjugate(q1); expand
  (%%-%);
```

$$q1 := a1 + b1\,i + c1\,j + d1\,k$$

$$q2 := a2 + b2\,i + c2\,j + d2\,k$$

$$a1 + b1\,i + c1\,j + d1\,k + a2 + b2\,i + c2\,j + d2\,k$$

$$(b1 + b2)\,i + (c2 + c1)\,j + (d2 + d1)\,k + a1 + a2$$

$$\&+ := (q1, q2) \rightarrow collect(q1 + q2, [i, j, k])$$

$$(b1 + b2)\,i + (c2 + c1)\,j + (d2 + d1)\,k + a1 + a2$$

$\&* :=$ **proc**$(q1, q2)$

local $a1, a2, b1, b2, c1, c2, d1, d2$;

$a1 := qRe(q1)$;

$a2 := qRe(q2)$;

$b1 := coeff(q1, i)$;

$b2 := coeff(q2, i)$;

$c1 := coeff(q1, j)$;

$c2 := coeff(q2, j);$

$d1 := coeff(q1, k);$

$d2 := coeff(q2, k);$

$a1*a2 - b1*b2 - c1*c2 - d1*d2 + (a1*b2 + a2*b1 + c1*d2 - d1*c2)*i + (a1*c2 + c1*a2 + d1*b2 - b1*d2)*j + (a1*d2 + d1*a2 + b1*c2 - c1*b2)*k$

end proc

$$a1\,a2 - b1\,b2 - c1\,c2 - d1\,d2 + (a1\,b2 + a2\,b1 + c1\,d2 - d1\,c2)\,i + (a1\,c2 + c1\,a2 + d1\,b2 - b1\,d2)\,j + (a1\,d2 + d1\,a2 + b1\,c2 - c1\,b2)\,k$$

$$a1 + a2 - (b1 + b2)\,i - (c2 + c1)\,j - (d2 + d1)\,k$$

$$(-b1 - b2)\,i + (-c2 - c1)\,j + (-d2 - d1)\,k + a1 + a2$$

$$a1\,a2 - b1\,b2 - c1\,c2 - d1\,d2 - (a1\,b2 + a2\,b1 + c1\,d2 - d1\,c2)\,i - (a1\,c2 + c1\,a2 + d1\,b2 - b1\,d2)\,j - (a1\,d2 + d1\,a2 + b1\,c2 - c1\,b2)\,k$$

$$a1\,a2 - b1\,b2 - c1\,c2 - d1\,d2 + (-a2\,b1 - a1\,b2 + d1\,c2 - c1\,d2)\,i + (-c1\,a2 - a1\,c2 + b1\,d2 - d1\,b2)\,j + (-d1\,a2 - a1\,d2 + c1\,b2 - b1\,c2)\,k$$

$$0 \qquad (3.4.28.16)$$

3.4.29. Kvaterniók abszolút értéke.

```
> qabs:=q->sqrt(qRe(q)^2+coeff(q,i)^2+coeff(q,j)^2+coeff(q,k)
  ^2);

  qabs(q); q&*qconjugate(q);
```

$$qabs := q \rightarrow \sqrt{qRe(q)^2 + coeff(q, i)^2 + coeff(q, j)^2 + coeff(q, k)^2}$$

$$\sqrt{a^2 + b^2 + c^2 + d^2}$$

$$a^2 + b^2 + c^2 + d^2 \qquad (3.4.29.1)$$

->3.4.30. Feladat.

**3.4.31. Vektoriális szorzás.*

**3.4.32. Kvaterniók és a háromdimenziós euklidészi tér.*

```
> q1:=b1*i+c1*j+d1*k; q2:=b2*i+c2*j+d2*k; q3:=b3*i+c3*j+d3*k;

  scalarprod:=(q1,q2)->-qRe(q1&*q2); scalarprod(q1,q2);
```

```
vectorprod:=(q1,q2)->qIm(q1&*q2); vectorprod(q1,q2);

mixedprod:=(q1,q2,q3)->scalarprod(q1,vectorprod(q2,q3));
mixedprod(q1,q2,q3);
```

$$q1 := b1\, i + c1\, j + d1\, k$$
$$q2 := b2\, i + c2\, j + d2\, k$$
$$q3 := b3\, i + c3\, j + d3\, k$$
$$scalarprod := (q1, q2) \rightarrow -qRe(q1 \,\&^*\, q2)$$
$$b1\, b2 + c1\, c2 + d1\, d2$$
$$vectorprod := (q1, q2) \rightarrow qIm(q1 \,\&^*\, q2)$$
$$(c1\, d2 - d1\, c2)\, i + (d1\, b2 - b1\, d2)\, j + (b1\, c2 - c1\, b2)\, k$$
$$mixedprod := (q1, q2, q3) \rightarrow scalarprod(q1, vectorprod(q2, q3))$$
$$b1\, (c2\, d3 - d2\, c3) + c1\, (d2\, b3 - b2\, d3) + d1\, (b2\, c3 - c2\, b3) \qquad (3.4.32.1)$$

▼ ****3.4.33. A szorzások geometriai jelentése.***

▼ ****3.4.34. Forgatások.***

▼ ****3.4.35. A skaláris es vektoriális szorzás geometriai alkalmazásai.***

▼ **** 3.4.36. Oktávok vagy Cayley-számok.***

▼ ***->3.4.37. Feladat.***

▼ ***->3.4.38. Feladat.***

▼ ****3.4.39. Feladat.***

▶ ***3.4.40. További feladatok megoldásokkal.***

▶ ***3.4.41. További feladatok.***

▶ 4. Véges halmazok

▶ 5. Végtelen halmazok

▶ 6. Számelmélet

▶ 7. Gráfelmélet

▶ 8. Algebra

▶ **9. Kódolás**

▶ **10. Algoritmusok**