Kalkulus II.

Járai Antal

Ezek a programok csak szemléltetésre szolgálnak

1. Lineáris algebra

2. Többváltozós függvények

2.1. Határérték és folytonosság

```
> restart;
```

2.1.1. Távolság, környezetek, korlátosság.

```
> verify([3,3,3],[Pi,Pi,Pi],'neighborhood(1)');
```

true (2.1.1.1)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1)');
```

false (2.1.1.2)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1,closed)');
```

true (2.1.1.3)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1,p=1)');
```

false (2.1.1.4)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1,p=3)');
```

true (2.1.1.5)

2.1.2. Belső, külső, izolált, torlódási és határpontok.

2.1.3. Nyílt és zárt halmazok.

2.1.4. Állítás.

2.1.5. Állítás.

2.1.6. Tétel.

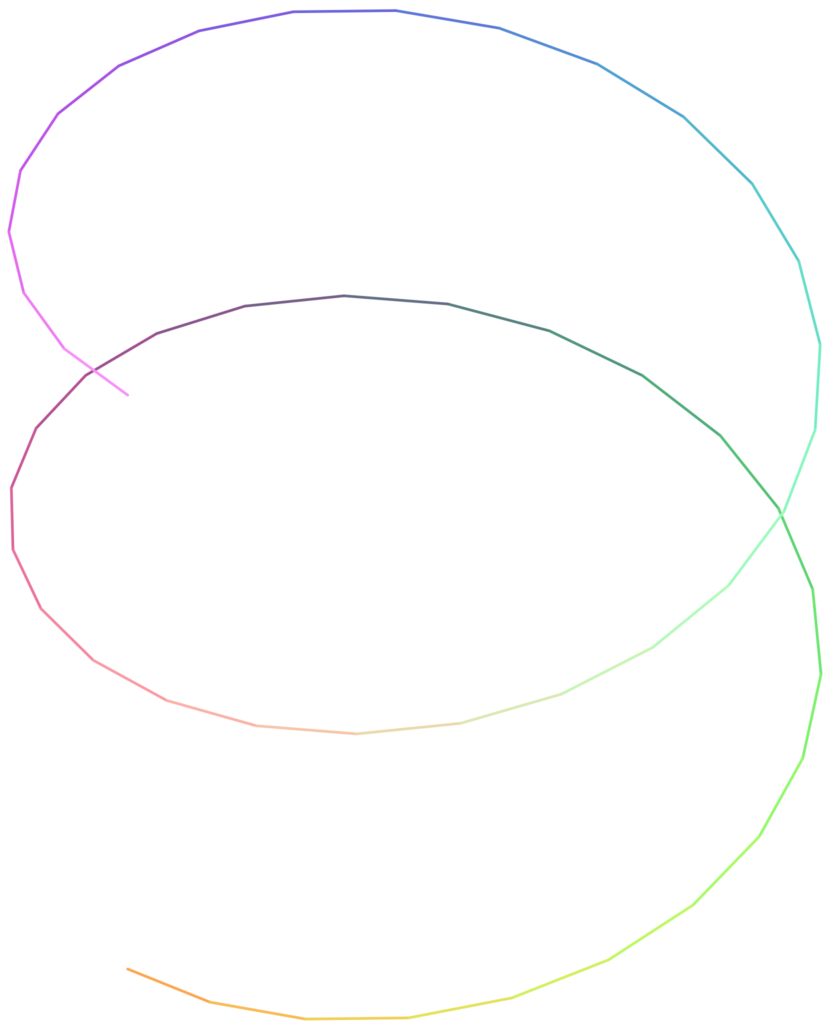
2.1.7. Tétel.

2.1.8. Következmény.
2.1.9. Következmény.
2.1.10. Következmény.
2.1.11. Tartomány.
2.1.12. Konvex halmaz.
2.1.13. Sűrű halmazok.
2.1.14. Kompakt halmazok.
2.1.15. Tétel.
2.1.16. Következmény: Weierstrass tétele.
2.1.17.Vektor-skalár, skalár-vektor és vektor-vektor függvények.

```
> with(plots);
```

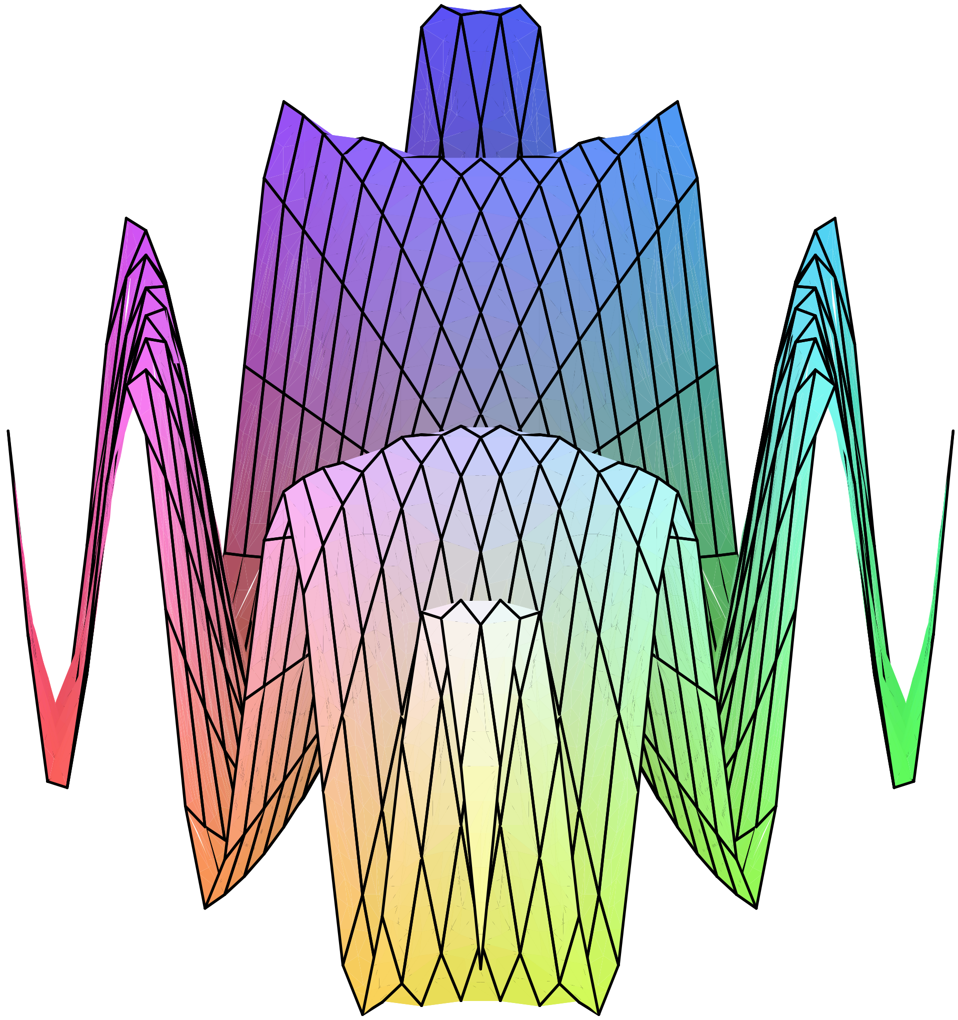
[Interactive, animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, graphplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot] (2.1.17.1)

```
> spacecurve([cos(t),sin(t),t],t=0..4*Pi);
```

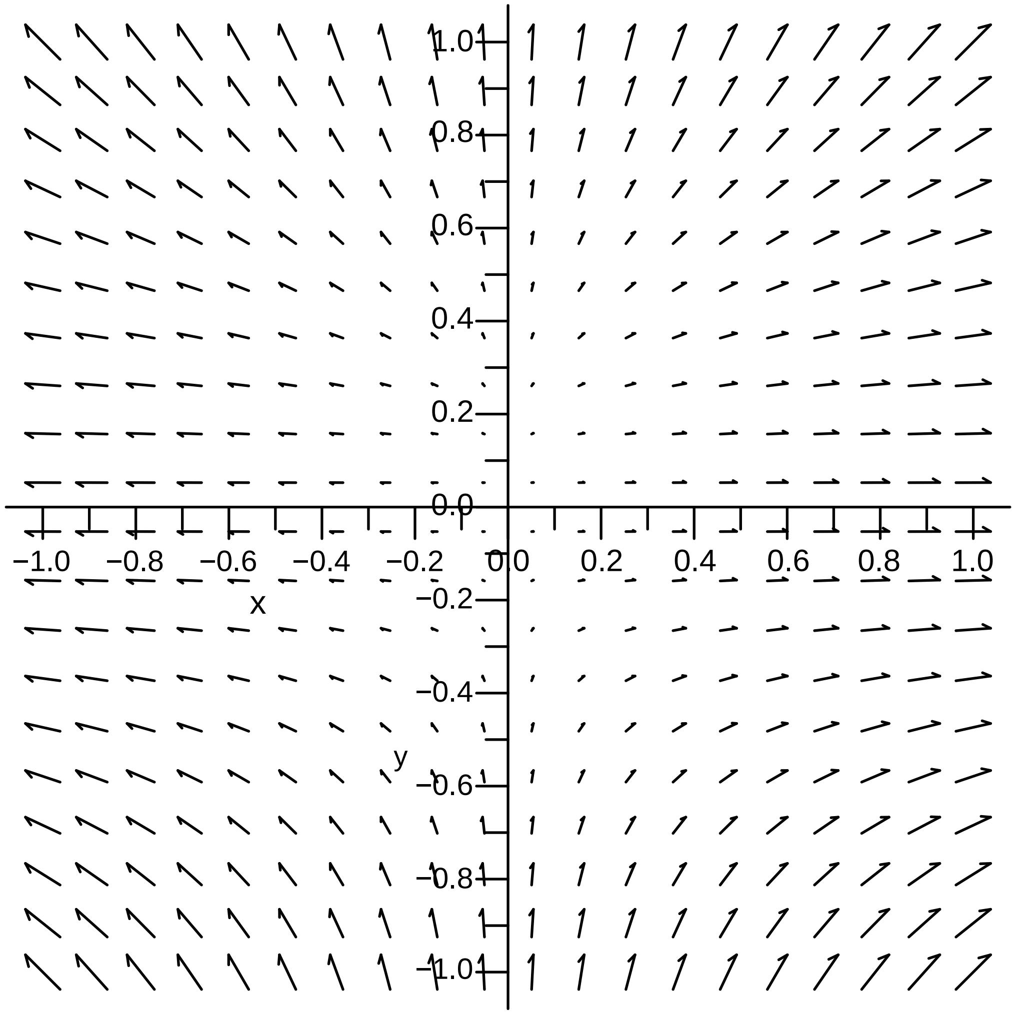


```
> p:=proc(x,y) sin(x*y) end; plot3d(p,-Pi..Pi,-Pi..Pi);
```

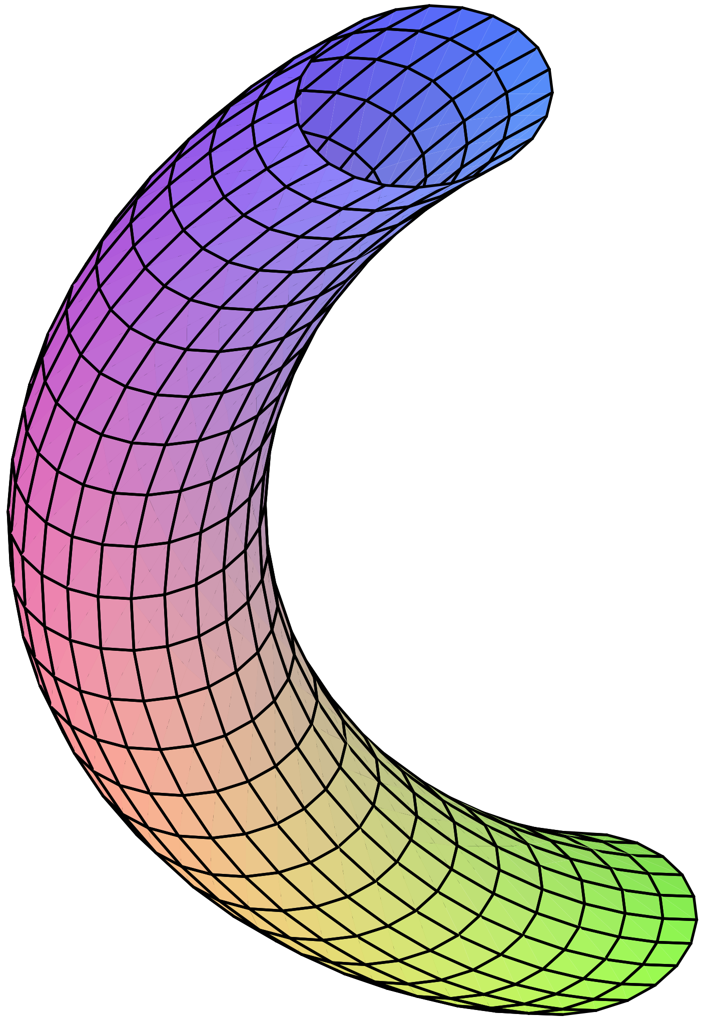
$$p := \mathbf{proc}(x, y)\ \sin(y * x)\ \mathbf{end\ proc}$$



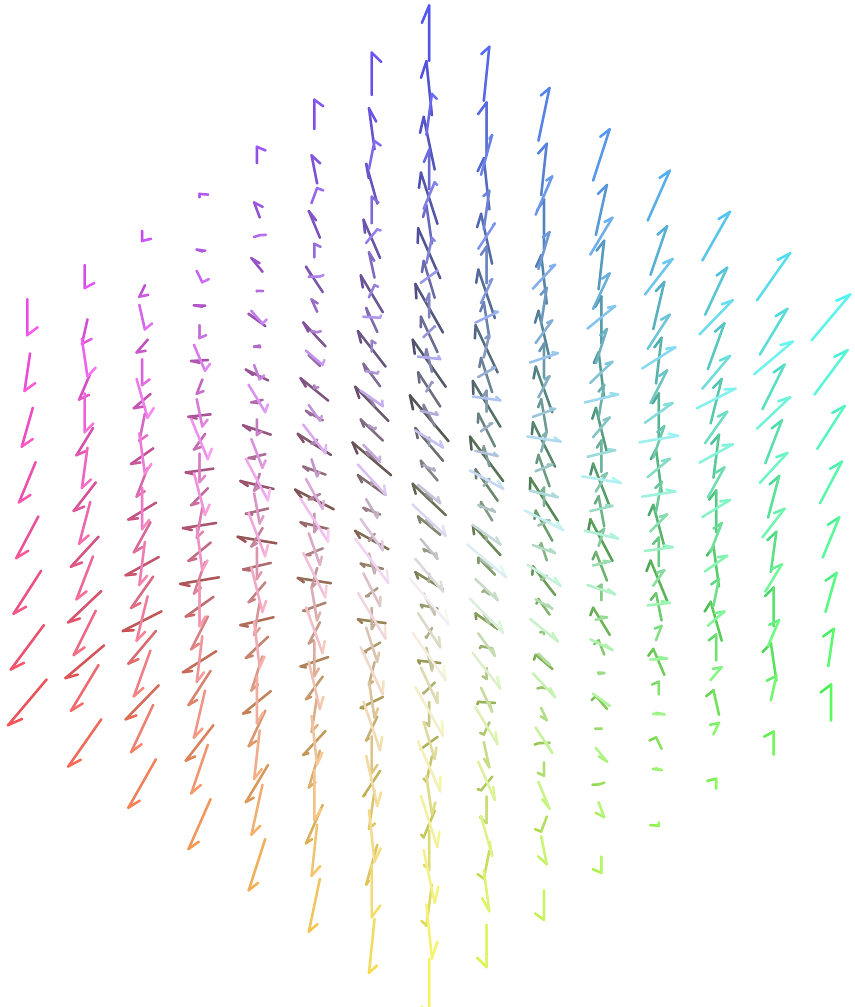
```
> fieldplot([x,y^2],x=-1..1,y=-1..1);
```




```
> plot3d([sin(x)+2*sin(0.4*y),cos(x)+2*cos(0.4*y),y],x=0..2*
  Pi,y=0..10);
```

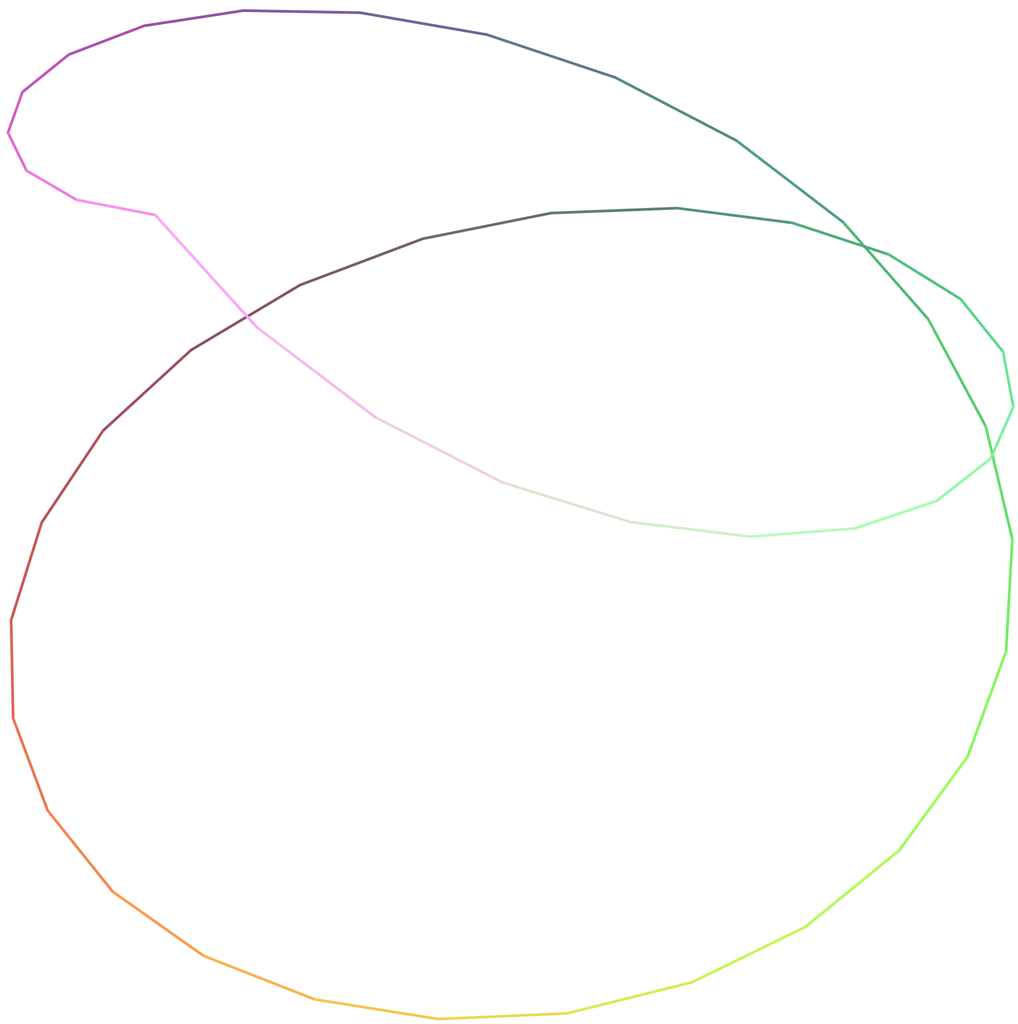


```
> fieldplot3d([x-y,x+z,z-y-x],x=-1..1,y=-1..1,z=-1..1);
```

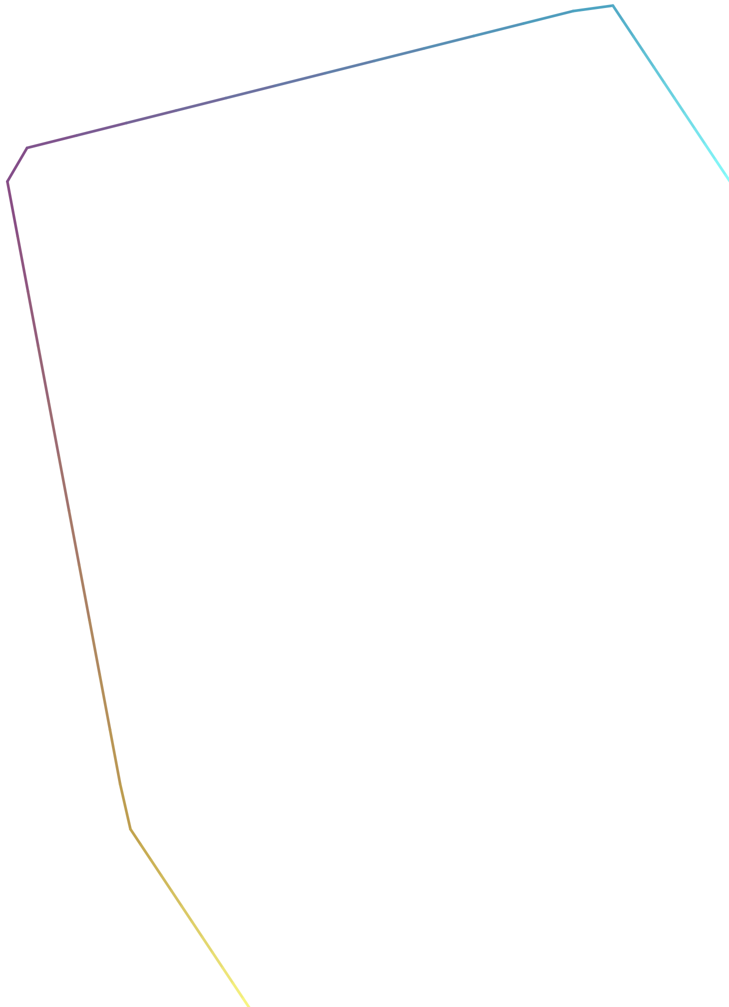


- 2.1.18. Folytonosság.
- 2.1.19. Példák.
- 2.1.20. Tétel.
- 2.1.21. Segédtétel.
- 2.1.22. Tétel.
- 2.1.23. Tétel.
- 2.1.24. Példák.
- 2.1.25. Görbék.

```
> spacecurve([cos(t),sin(t),t*(t-4*Pi)],t=0..4*Pi);
```



```
> spacecurve([abs(t-1),3+abs(t-2),4+abs(t-3)],t=0..4);
```



2.1.26. Tétel.

2.1.27. Következmény.

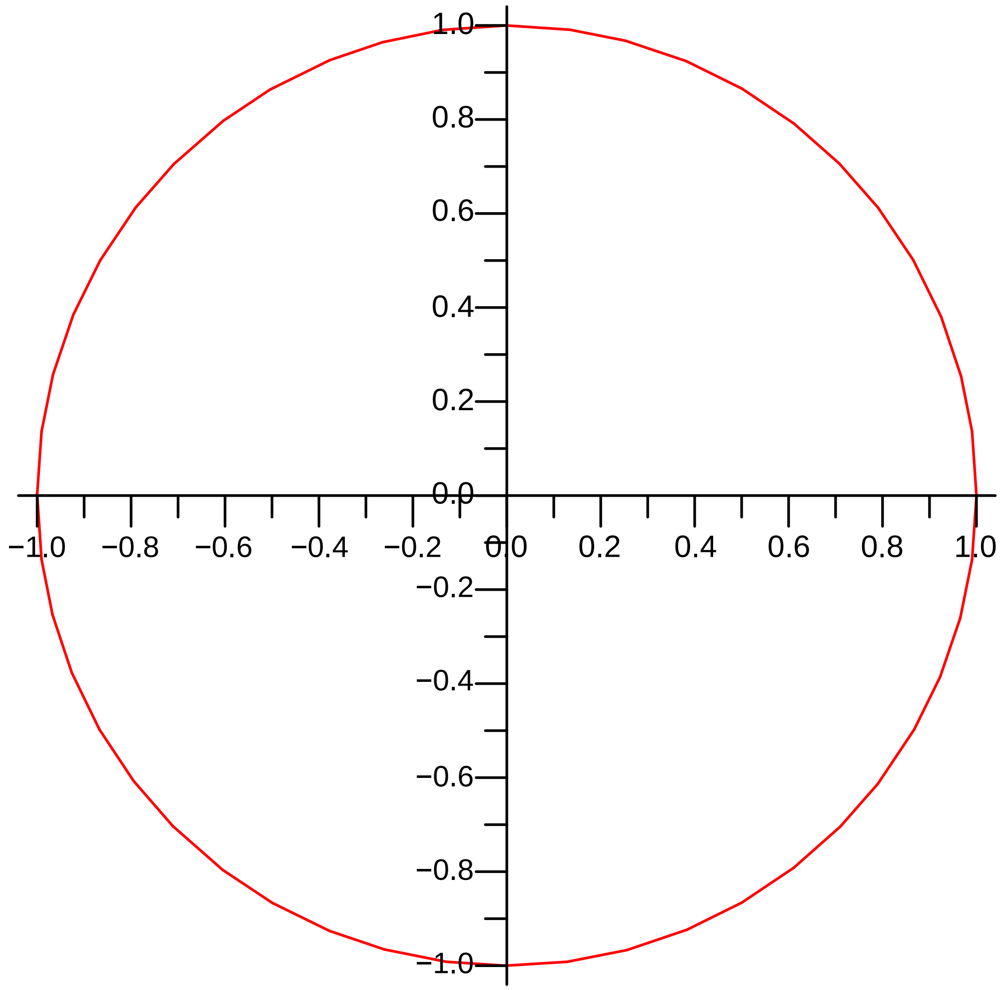
2.1.28. Tétel.

2.1.29. Következmény: Weierstrass tétele.

2.1.30. Hausdorff tétele.

o 2.1.31. Példa.

```
> plot([cos(t),sin(t),t=0..2*Pi]);
```



▶ ***2.1.32. Lebesgue-szám.***

▶ ***2.1.33. Tétel.***

▶ ***2.1.34. Egyenletes folytonosság.***

▶ ***2.1.35. Heine tétele.***

▶ ***2.1.36. Jobb és bal oldali folytonosság.***

▼ ***2.1.37. Határérték.***

```
> limit((x^4-y^4)/(x^2+y^2),{x=0,y=0});
```

$$0 \tag{2.1.37.1}$$

```
> limit((x^2-y^2)/(x^2+y^2),{x=0,y=0});
```

$$\mathit{undefined} \tag{2.1.37.2}$$

```
> limit((sin(x^2)-sin(y^2))/(x-y), {x=0,y=0});
```

$$\mathit{limit}\left(\frac{\sin\left(x^2\right)-\sin\left(y^2\right)}{x-y}, \{x=0, y=0\}\right) \tag{2.1.37.3}$$

▶ *2.1.38. Tétel.*
▶ *2.1.39. Tétel.*
▶ *2.1.40. Jobb és bal oldali határérték.*
▶ *2.1.41. Szakadások.*
▶ *2.1.42. Végtelen, mint határérték, határérték a végtelenben.*
▶ *2.1.43. Tétel.*
▶ *2.1.44. Tétel.*
▶ *2.1.45. Tétel: rendőr-elv.*
▶ *2.1.46. Sorozatok.*
▶ *2.1.47. Tétel.*
▶ *2.1.48. Tétel.*
▶ *2.1.49. Tétel.*
▶ *2.1.50. Következmény: Bolzano-Weierstrass-féle kiválasztási tétel.*
▶ *2.1.51. Cauchy-sorozatok.*
▶ *2.1.52. Cauchy-féle konvergenciakritérium.*
▶ *2.1.53. Banach-terek és Hilbert-terek.*
▼ *2.1.54. Sorok.*

```
> sum('1/k!','k'=0..infinity);
```

$$e \tag{2.1.54.1}$$

```
> sum('1/k^2','k'=1..infinity);
```

$$\frac{1}{6}\pi^2 \tag{2.1.54.2}$$

```
> sum('1/k^3','k'=1..infinity);
```

$$\zeta(3) \tag{2.1.54.3}$$

```
> ?Zeta
```

▶ *2.1.55. Tétel.*
▶ *2.1.56. Cauchy-féle konvergenciakritérium.*
▶ *2.1.57. Következmény.*
▶ *2.1.58. Öszehasonlítókritérium.*
▶ *2.1.59. Következmény.*
▶ *2.1.60. Cauchy-féle gyökkritérium.*
▶ *2.1.61. d'Alembert-féle hányadoskritérium.*

▶ ***2.1.62. Kettős sor tétel.***
▶ ***2.1.63. Következmény: sorok átrendezése.***
▶ ***2.1.64. Következmény.***
▶ ***2.1.65. Ekvivalens normák.***
▶ ***2.1.66. Tétel.***
▶ ****2.1.67. Fixpont és kontrakció.***
▶ ****2.1.68 Banach-féle fixponttétel.***
▶ ****2.1.69. Metrikus terek.***

2.2. Differenciálszámítás

```
> restart; with(LinearAlgebra); with(VectorCalculus);
```

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main, LUDecomposition, LeastSquares, LinearSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*]

[*&x*, ***, *+*, *-*, *.*, *<,>*, *<|>*, *AddCoordinates*, *ArcLength*, *BasisFormat*, *Binormal*, *CrossProd*, *CrossProduct*, *Curl*, *Curvature*, D, *Del*, *DirectionalDiff*, *Divergence*, *DotProd*, *DotProduct*, *Flux*, *GetCoordinateParameters*, *GetCoordinates*, *Gradient*, *Hessian*, *Jacobian*, *Laplacian*, *LineInt*, *MapToBasis*, *Nabla*, *Norm*, *Normalize*, *PathInt*, *PrincipalNormal*, *RadiusOfCurvature*, *ScalarPotential*, *SetCoordinateParameters*, *SetCoordinates*, *SurfaceInt*, *TNBFrame*, *Tangent*, *TangentLine*, *TangentPlane*, *TangentVector*, *Torsion*, *Vector*, *VectorField*, *VectorPotential*, *Wronskian*, *diff*, *evalVF*, *int*, *limit*, *series*] (2.2.1)

2.2.1. Lineáris operátorok normája.

```
> a:=Matrix([[1,0],[2,3]]);
```

$$a := \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \tag{2.2.1.1}$$

```
> MatrixNorm(a); MatrixNorm(a,'infinity');
  MatrixNorm(a,1); MatrixNorm(a,2); evalf(%); MatrixNorm(a,
  Frobenius);
```

$$5$$

$$5$$

$$3$$

$$\sqrt{\mathrm{RootOf}\left(\_Z^2 - 14\,\_Z + 9,\ index = 2\right)}$$

$$3.650281540$$

$$\sqrt{14} \tag{2.2.1.2}$$

2.2.2. Feladat.

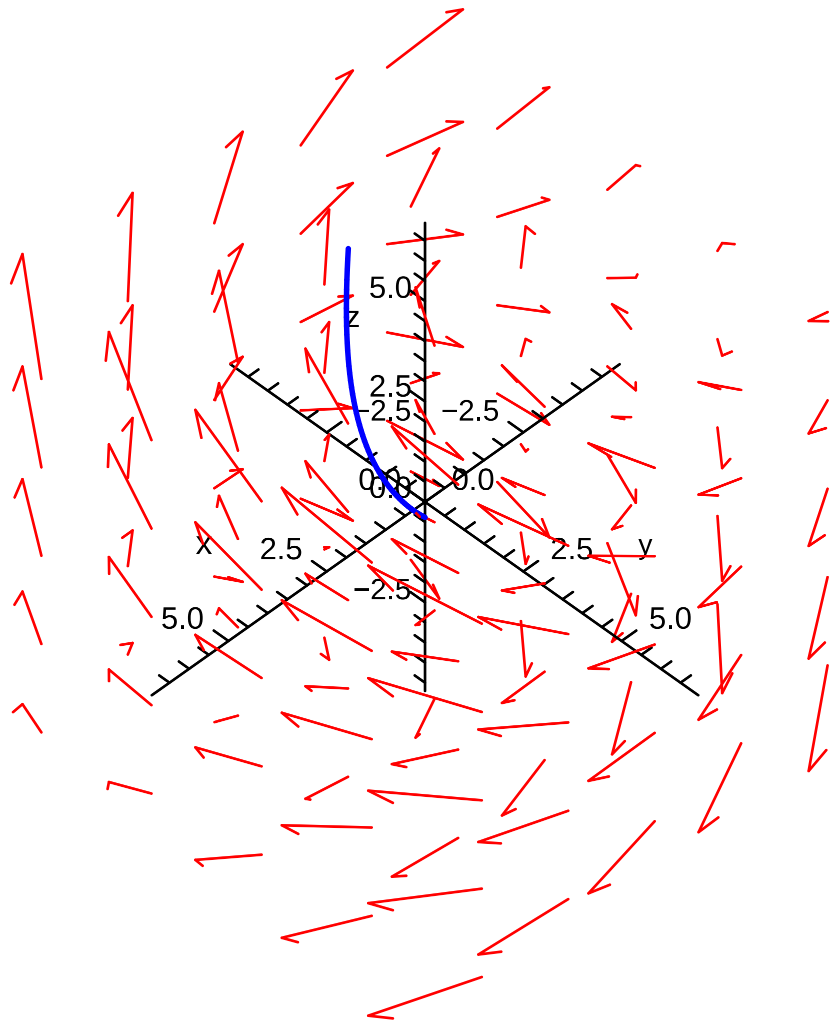
2.2.3. Feladat.

▶ *2.2.4. Feladat.*

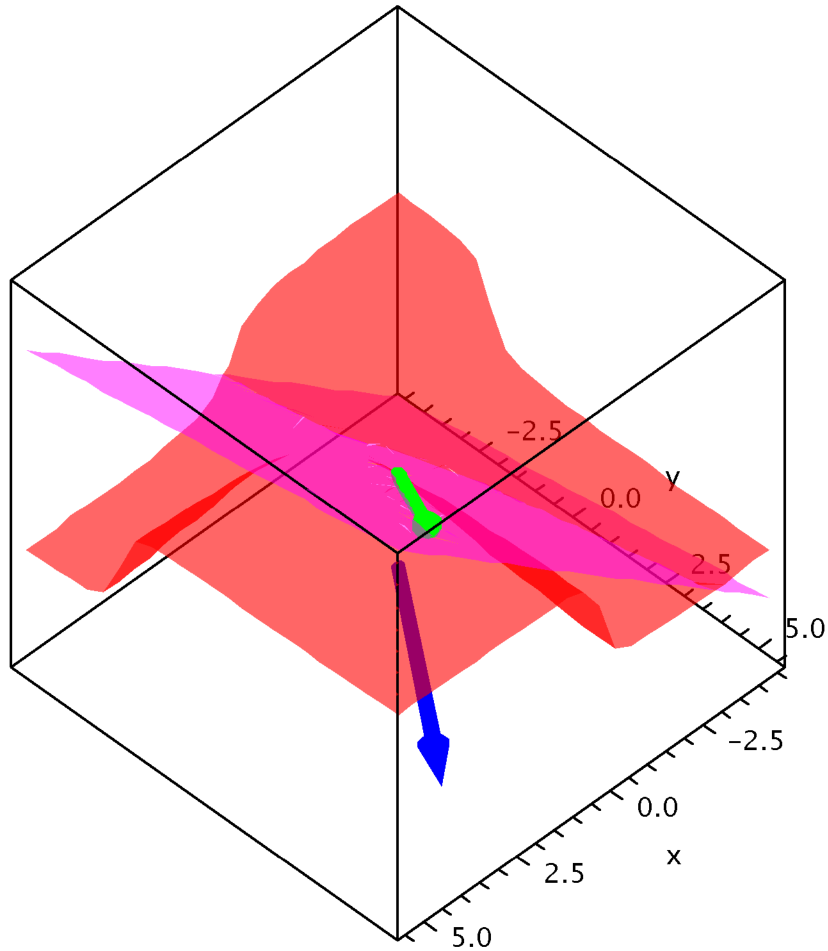
▶ *2.2.5. Feladat.*

▼ *2.2.6. Definíció.*

```
> Student[VectorCalculus][VectorFieldTutor]();
```




```
> Student[MultivariateCalculus][DirectionalDerivativeTutor]()
;
```




```
> f:=<x^2,x*y,x*z>;
  Jacobian(f,[x,y,z]);
  diff(f[1],x); diff(f[1],y); diff(f[1],z);
  diff(f[2],x); diff(f[2],y); diff(f[2],z);
  diff(f[3],x); diff(f[3],y); diff(f[3],z);
```

$$f := \left(x^2\right)e_x + \left(x\,y\right)e_y + \left(x\,z\right)e_z$$

$$\begin{bmatrix} 2\,x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \end{bmatrix}$$

$$2\,x$$

$$0$$

$$0$$

$$y$$

$$x$$

$$
\begin{matrix} 0 \\ z \\ 0 \\ x \end{matrix} \tag{2.2.6.1}
$$

```
> f:=<x^2,x*y,x*z,z*y>; Jacobian(f,[x,y,z]);
```

$$f := (x^2)e_{x1} + (x\,y)e_{x2} + (x\,z)e_{x3} + (z\,y)e_{x4}$$

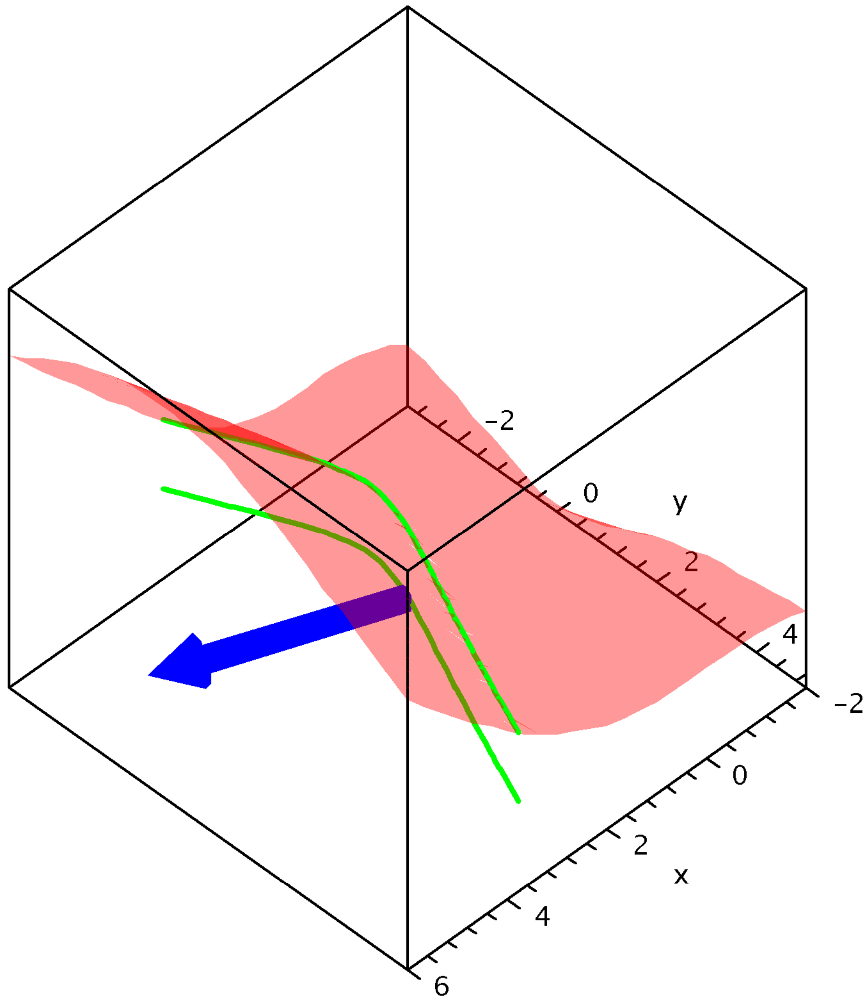
$$
\begin{bmatrix} 2\,x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix} \tag{2.2.6.2}
$$

```
> f:=3*x^2+2*y*z; Gradient(f,[x,y,z]);
```

$$f := 3\,x^2 + 2\,z\,y$$

$$6\,x\bar{e}_x + 2\,z\bar{e}_y + 2\,y\bar{e}_z \tag{2.2.6.3}$$

```
> Student[MultivariateCalculus][GradientTutor]();
```



▶ *2.2.7. Sima görbék.*

▶ *2.2.8. Összegszabály.*

▶ *2.2.9. Láncszabály.*

▶ *2.2.10. Koordinátafüggvények.*

▶ *2.2.11. Parciális<deriváltak.*

▶ *2.2.12. Középérték egyenlőtlenség.*

▶ *2.2.13. Következmény.*

▶ *2.2.14. Következmény.*

▶ *2.2.15. Tétel.*

▼ *2.2.16. Magasabbrendű deriváltak.*

```
> f:=(x,y,z)->x^2*y*z^3;
  D[1](D[1](f)); D[1](D[2](f)); D[2](D[1](f));
```

$$f := (x, y, z) \rightarrow VectorCalculus{:-}{*}\left(VectorCalculus{:-}{*}\left(x^2, y\right), z^3\right)$$

$$(x, y, z) \rightarrow 2\, y\, z^3$$

$$(x, y, z) \rightarrow 2\, x\, z^3$$

$$(x, y, z) \rightarrow 2\, x\, z^3 \tag{2.2.16.1}$$

```
> f:=<3*x^3+2*y*z*x+x^2*y^2*z^2>; Jacobian(f,[x,y,z]);
  convert(%,list); Jacobian(%,[x,y,z]);
  convert(%,listlist); map(u->Jacobian(u,[x,y,z]),%);
```

$$f := \left(3\, x^3 + 2\, z\, y\, x + x^2\, y^2\, z^2\right) e_x$$

$$\begin{bmatrix} 9\, x^2 + 2\, z\, y + 2\, x\, y^2\, z^2 & 2\, x\, z + 2\, x^2\, y\, z^2 & 2\, x\, y + 2\, x^2\, y^2\, z \end{bmatrix}$$

$$\left[9\, x^2 + 2\, z\, y + 2\, x\, y^2\, z^2, 2\, x\, z + 2\, x^2\, y\, z^2, 2\, x\, y + 2\, x^2\, y^2\, z\right]$$

$$\begin{bmatrix} 18\, x + 2\, y^2\, z^2 & 2\, z + 4\, x\, y\, z^2 & 2\, y + 4\, x\, y^2\, z \\ 2\, z + 4\, x\, y\, z^2 & 2\, x^2\, z^2 & 2\, x + 4\, x^2\, y\, z \\ 2\, y + 4\, x\, y^2\, z & 2\, x + 4\, x^2\, y\, z & 2\, x^2\, y^2 \end{bmatrix}$$

$$\left[\left[18\, x + 2\, y^2\, z^2, 2\, z + 4\, x\, y\, z^2, 2\, y + 4\, x\, y^2\, z\right], \left[2\, z + 4\, x\, y\, z^2, 2\, x^2\, z^2, 2\, x + 4\, x^2\, y\, z\right], \left[2\, y + 4\, x\, y^2\, z, 2\, x + 4\, x^2\, y\, z, 2\, x^2\, y^2\right]\right]$$

$$\left[\begin{bmatrix} 18 & 4\, y\, z^2 & 4\, y^2\, z \\ 4\, y\, z^2 & 4\, x\, z^2 & 2 + 8\, z\, y\, x \\ 4\, y^2\, z & 2 + 8\, z\, y\, x & 4\, x\, y^2 \end{bmatrix}, \begin{bmatrix} 4\, y\, z^2 & 4\, x\, z^2 & 2 + 8\, z\, y\, x \\ 4\, x\, z^2 & 0 & 4\, x^2\, z \\ 2 + 8\, z\, y\, x & 4\, x^2\, z & 4\, x^2\, y \end{bmatrix}, \begin{bmatrix} 4\, y^2\, z & 2 + 8\, z\, y\, x & 4\, x\, y^2 \\ 2 + 8\, z\, y\, x & 4\, x^2\, z & 4\, x^2\, y \\ 4\, x\, y^2 & 4\, x^2\, y & 0 \end{bmatrix}\right] \tag{2.2.16.2}$$

```
> f:=<x^3,x*y*z,x*z^2,z*y^2>; Jacobian(f,[x,y,z]);
  convert(%,listlist); map(u->Jacobian(u,[x,y,z]),%);
```

$$f := \left(x^3\right) e_{x1} + (z\, y\, x)\, e_{x2} + \left(x\, z^2\right) e_{x3} + \left(y^2\, z\right) e_{x4}$$

$$\begin{bmatrix} 3\, x^2 & 0 & 0 \\ z\, y & x\, z & x\, y \\ z^2 & 0 & 2\, x\, z \\ 0 & 2\, z\, y & y^2 \end{bmatrix}$$

$$\left[\left[3\, x^2, 0, 0\right], [z\, y, x\, z, x\, y], \left[z^2, 0, 2\, x\, z\right], \left[0, 2\, z\, y, y^2\right]\right]$$

$$\left[\begin{bmatrix}6x & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & z & y\\ z & 0 & x\\ y & x & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 2z\\ 0 & 0 & 0\\ 2z & 0 & 2x\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 2z & 2y\\ 0 & 2y & 0\end{bmatrix}\right] \tag{2.2.16.3}$$

2.2.17. Magasabbrendű deriváltak és parciális deriváltak kapcsolata.

2.2.18. A második derivált mint bilineáris leképezés.

```
> Hessian(x^2*y*z^3,[x,y,z]);
```

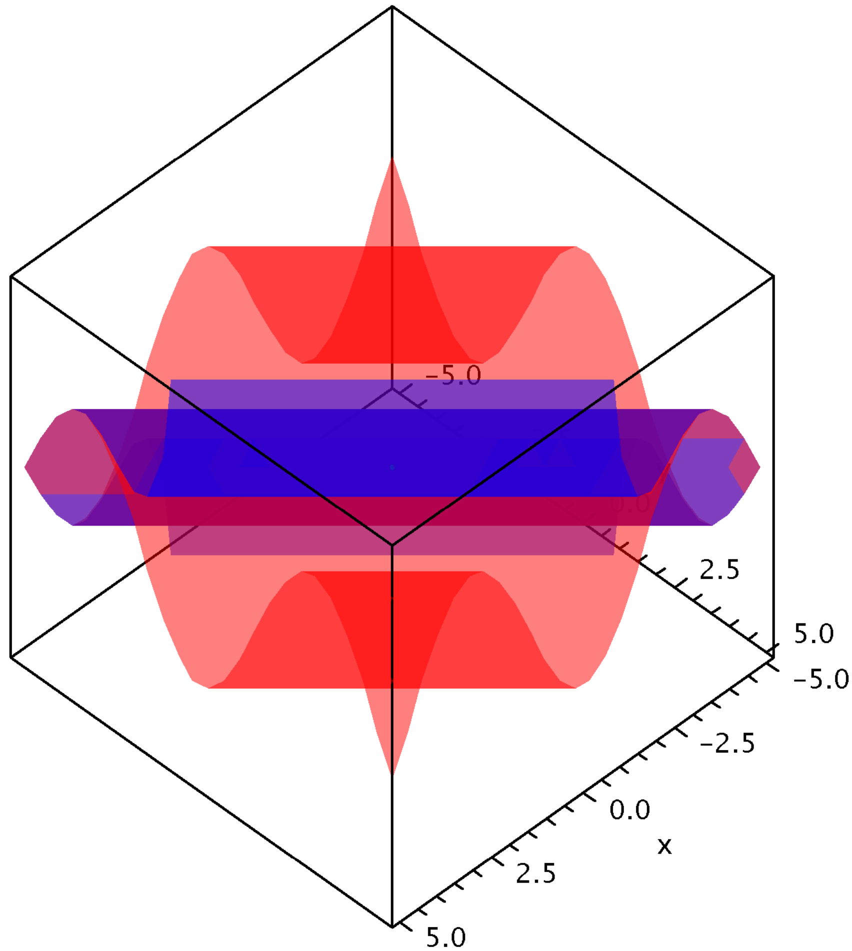
$$\begin{bmatrix}2yz^3 & 2xz^3 & 6xyz^2\\ 2xz^3 & 0 & 3x^2z^2\\ 6xyz^2 & 3x^2z^2 & 6x^2yz\end{bmatrix} \tag{2.2.18.1}$$

**2.2.19. Magasabbrendű deriváltak mint multilineáris leképezések.*

2.2.20. Young tétele.

2.2.21. Taylor-formula valós értékű függvényekre.

```
> Student[MultivariateCalculus][TaylorApproximationTutor]();
```




```
> mtaylor(sin(x^2+y^2),[x,y]);
  mtaylor(sin(x^2+y^2),[x=0,y=0]);
  mtaylor(sin(x^2+y^2),[x,y],8);
  mtaylor(sin(x^2+y^2),[x=1,y=2],3);
  mtaylor(g(x,y),[x,y]);
```

$$x^2 + y^2$$

$$x^2 + y^2$$

$$x^2 + y^2 - \frac{1}{6} x^6 - \frac{1}{2} y^2 x^4 - \frac{1}{2} y^4 x^2 - \frac{1}{6} y^6$$

$$\sin(5) + 2\cos(5)(x-1) + 4\cos(5)(y-2) + (-2\sin(5) + \cos(5))(x-1)^2 - 8\sin(5)(y-2)(x-1) + (-8\sin(5) + \cos(5))(y-2)^2$$

$$g(0, 0) + \mathrm{D}_1(g)(0, 0)\, x + \mathrm{D}_2(g)(0, 0)\, y + \frac{1}{2} \mathrm{D}_{1,\,1}(g)(0, \qquad (2.2.21.1)$$

$0)\, x^2 + x\, \mathrm{D}_{1,\,2}(g)(0,\,0)\, y + \frac{1}{2}\, \mathrm{D}_{2,\,2}(g)(0,$

$0)\, y^2 + \frac{1}{6}\, x^3\, \mathrm{D}_{1,\,1,\,1}(g)(0,\,0) + \frac{1}{2}\, x^2\, \mathrm{D}_{1,\,1,\,2}(g)(0,$

$0)\, y + \frac{1}{2}\, x\, y^2\, \mathrm{D}_{1,\,2,\,2}(g)(0,\,0) + \frac{1}{6}\, y^3\, \mathrm{D}_{2,\,2,\,2}(g)(0,$

$0) + \frac{1}{24}\, x^4\, \mathrm{D}_{1,\,1,\,1,\,1}(g)(0,\,0) + \frac{1}{6}\, x^3\, \mathrm{D}_{1,\,1,\,1,\,2}(g)(0,$

$0)\, y + \frac{1}{4}\, x^2\, y^2\, \mathrm{D}_{1,\,1,\,2,\,2}(g)(0,\,0) + \frac{1}{6}\, x\, y^3\, \mathrm{D}_{1,\,2,\,2,\,2}(g)(0,$

$0) + \frac{1}{24}\, y^4\, \mathrm{D}_{2,\,2,\,2,\,2}(g)(0,\,0) + \frac{1}{120}\, x^5\, \mathrm{D}_{1,\,1,\,1,\,1,\,1}(g)(0,$

$0) + \frac{1}{24}\, x^4\, \mathrm{D}_{1,\,1,\,1,\,1,\,2}(g)(0,\,0)\, y + \frac{1}{12}\, x^3\, y^2\, \mathrm{D}_{1,\,1,\,1,\,2,\,2}(g)(0,$

$0) + \frac{1}{12}\, x^2\, y^3\, \mathrm{D}_{1,\,1,\,2,\,2,\,2}(g)(0,\,0) + \frac{1}{24}\, x\, y^4\, \mathrm{D}_{1,\,2,\,2,\,2,\,2}(g)(0,$

$0) + \frac{1}{120}\, y^5\, \mathrm{D}_{2,\,2,\,2,\,2,\,2}(g)(0,\,0)$

▶ *2.2.22. Megjegyzés.*

▼ *2.2.23. Implicit függvény tétel.*

```
> x:='x';y:='y';z:='z'; implicitdiff(x^2+y^2=1,y(x),x);
  implicitdiff(x^2+y^2=z^2,z(x,y),x);
  implicitdiff({x^2+y^2+z^2=1,x-y=z},{y(x),z(x)},y,x);
  solve({x^2+y^2+z^2=1,x-y=z},{y,z});
```

$$x := x$$

$$y := y$$

$$z := z$$

$$-\frac{x}{y}$$

$$\frac{x}{z}$$

$$-\frac{x+z}{y-z}$$

$$\{z = \mathrm{RootOf}(2\, x^2 - 2\, x\, \_Z + 2\, \_Z^2 - 1), \quad y = x - \mathrm{RootOf}(2\, x^2 - 2\, x\, \_Z + 2\, \_Z^2 - 1)\} \tag{2.2.23.1}$$

2.2.24. Inverz függvény tétel.

*2.2.25. Megjegyzés.

2.2.26. Szélsőérték szükséges feltétele.

```
> f:=cos(x+y)*cos(y+z)*cos(x+z); g:=Gradient(f,[x,y,z]);
```

$$f := \cos(x+y)\cos(y+z)\cos(x+z)$$

$$g := (-\sin(x+y)\cos(y+z)\cos(x+z) - \cos(x+y)\cos(y+z)\sin(x+z))\bar{e}_x + (-\sin(x+y)\cos(y+z)\cos(x+z) - \cos(x+y)\sin(y+z)\cos(x+z))\bar{e}_y + (-\cos(x+y)\sin(y+z)\cos(x+z) - \cos(x+y)\cos(y+z)\sin(x+z))\bar{e}_z \quad (2.2.26.1)$$

```
> subs([x=0,y=0,z=0],g); subs([x=Pi/2,y=Pi/2,z=Pi/2],g);
  subs([x=Pi/2,y=0,z=Pi/2],g);
```

$$0\bar{e}_x + 0\bar{e}_y + 0\bar{e}_z$$

$$0\bar{e}_x + 0\bar{e}_y + 0\bar{e}_z$$

$$0\bar{e}_x + 0\bar{e}_y + 0\bar{e}_z \quad (2.2.26.2)$$

2.2.27. Lokális szélsőérték elégséges feltétele.

```
> with(Student[MultivariateCalculus]);
```

$[Add2DCoordinateNames, Add3DCoordinateNames, ApproximateInt, ApproximateIntTutor, CenterOfMass, ChangeOfVariables, CrossSection, CrossSectionTutor, DirectionalDerivative, DirectionalDerivativeTutor, FunctionAverage, Gradient, GradientTutor, Jacobian, LagrangeMultipliers, MultiInt, Revert, SecondDerivativeTest, SurfaceArea, TaylorApproximation, TaylorApproximationTutor]$ (2.2.27.1)

```
> SecondDerivativeTest(f,[x,y,z]=[0,0,0]);
```

$$LocalMin = [\,], LocalMax = ([[0, 0, 0]]), Saddle = [\,] \quad (2.2.27.2)$$

```
> SecondDerivativeTest(f,[x,y,z]=[0,0,0],output=hessian);
```

(2.2.27.3)

$$\begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \tag{2.2.27.3}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,Pi/2,Pi/2]);
```

$$LocalMin = \left(\left[\left[\frac{1}{2}\,\pi, \frac{1}{2}\,\pi, \frac{1}{2}\,\pi\right]\right]\right), LocalMax = [\,], Saddle = [\,] \tag{2.2.27.4}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,Pi/2,Pi/2],output=
  hessian);
```

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{2.2.27.5}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,0,Pi/2]);
```

$$LocalMin = [\,], LocalMax = [\,], Saddle = \left(\left[\left[\frac{1}{2}\,\pi, 0, \frac{1}{2}\,\pi\right]\right]\right) \tag{2.2.27.6}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,0,Pi/2],output=
  hessian);
```

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix} \tag{2.2.27.7}$$

2.2.28. Lagrange-elv: feltételes szélsőérték keresése.

```
> LagrangeMultipliers(x*y,[x^2/8+y^2/2-1],[x,y]);
```

$$[2, 1], [-2, -1], [-2, 1], [2, -1] \tag{2.2.28.1}$$

->2.2.29. Feladat.

->2.2.30. Feladat.

->2.2.31. Feladat.

->2.2.32. Feladat.

->2.2.33. Feladat.

2.2.34. Felületek.

▶ 2.2.35. Felületek előállítási módjai.

▼ 2.2.36. Felületi normális, érintősík, felületi görbék.

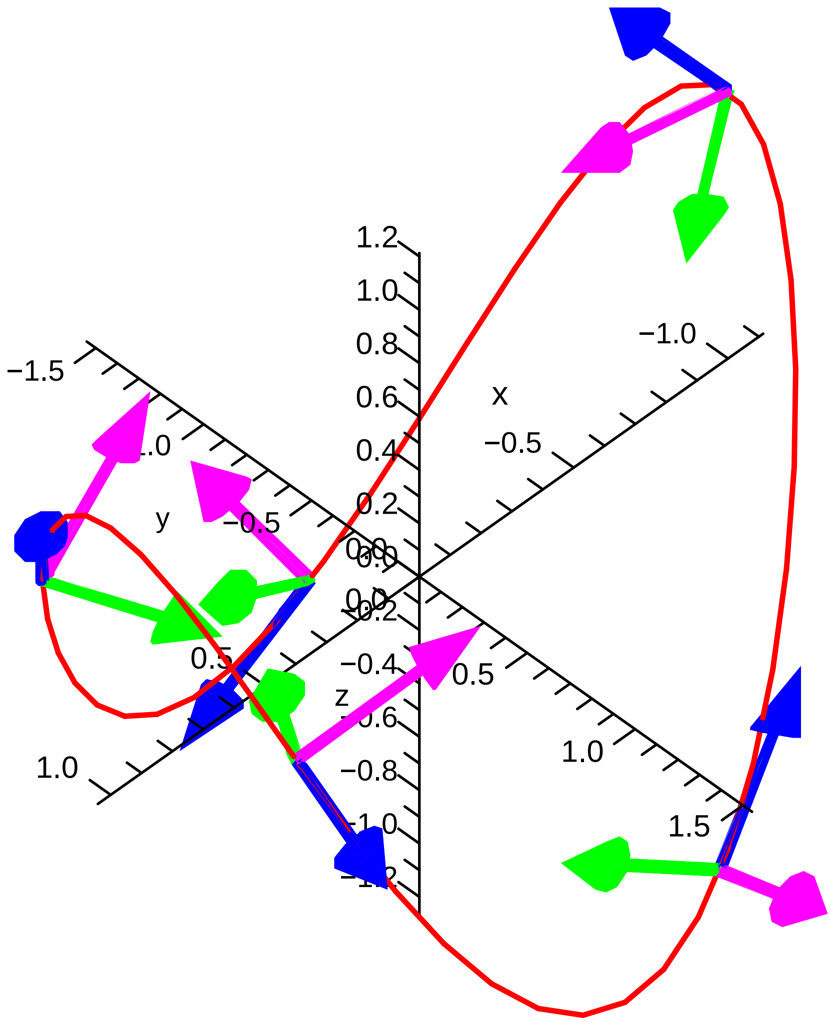
```
> TangentPlane(<s,t,s^2+t^2>,s=2,t=3);
```

$$(2+s)e_x + (3+t)e_y + (13+4\,s+6\,t)e_z \tag{2.2.36.1}$$

```
> TangentVector(<t,t^2,t^3>,t);
```

$$e_x + 2\,te_y + 3\,t^2 e_z \tag{2.2.36.2}$$

```
> Student[VectorCalculus][SpaceCurveTutor]();
```



▼ 2.2.37. Approximáció.

```
> a:='a'; b:='b'; c:='c'; with(CurveFitting);
```

$$a := a$$

$$b := b$$

$$c := c$$

(2.2.37.1)

[*BSpline*, *BSplineCurve*, *Interactive*, *LeastSquares*, *PolynomialInterpolation*, *RationalInterpolation*, *Spline*, *ThieleInterpolation*] (2.2.37.1)

```
> data:=[[0,13.3],[20,31.6],[50,85.5],[80,169.0],[100,246.0]]
;
```

$data := [[0, 13.3], [20, 31.6], [50, 85.5], [80, 169.0], [100, 246.0]]$ (2.2.37.2)

```
> PolynomialInterpolation(data,x);
```

$7.597222250\ 10^{-7}\ x^4 - 0.0001288194449\ x^3 + 0.4866944442\ x + 0.02368777780\ x^2 + 13.3$ (2.2.37.3)

```
> PolynomialInterpolation(data,x,form='Lagrange');
```

$0.000001662500000\ (x-20)\ (x-50)\ (x-80)\ (x-100) - 0.00001097222222\ x\ (x-50)\ (x-80)\ (x-100) + 0.00003800000000\ x\ (x-20)\ (x-80)\ (x-100) - 0.00005868055556\ x\ (x-20)\ (x-50)\ (x-100) + 0.00003075000000\ x\ (x-20)\ (x-50)\ (x-80)$ (2.2.37.4)

```
> LeastSquares(data,x,curve=a*x^2+b*x+c);
```

$13.58761229 + 0.530547425866898403\ x + 0.0178665845648604306\ x^2$ (2.2.37.5)

▶ ***2.2.38. Newton-módszer.***

▶ ***2.2.39. Kapcsolat minimumfeladatokkal.***

▶ ***2.2.40. Az iránymenti csökkentés módszere.***

▶ ***2.2.41. Algebrai egyenletek megoldása.***

2.3. Integrálszámítás

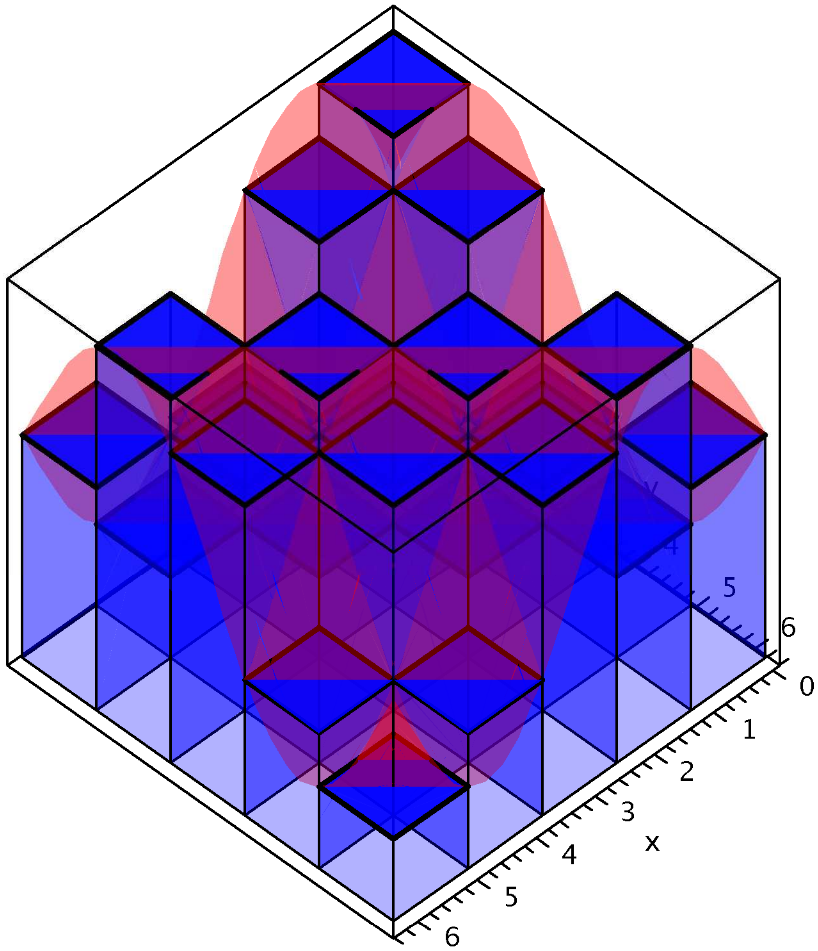
```
> restart;with(VectorCalculus);
```

[*&x*, ***, *+*, *-*, *.*, *<,>*, *<|>*, *AddCoordinates*, *ArcLength*, *BasisFormat*, *Binormal*, *CrossProd*, *CrossProduct*, *Curl*, *Curvature*, D, *Del*, *DirectionalDiff*, *Divergence*, *DotProd*, *DotProduct*, *Flux*, *GetCoordinateParameters*, *GetCoordinates*, *Gradient*, *Hessian*, *Jacobian*, *Laplacian*, *LineInt*, *MapToBasis*, *Nabla*, *Norm*, *Normalize*, *PathInt*, *PrincipalNormal*, *RadiusOfCurvature*, *ScalarPotential*, *SetCoordinateParameters*, *SetCoordinates*, *SurfaceInt*, *TNBFrame*, (2.3.1)

Tangent, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, Wronskian, diff, evalVF, int, limit, series]

2.3.1. Többváltozós függvények integrálja.

```
> Student[MultivariateCalculus][ApproximateIntTutor]();
```



2.3.2. Tétel.

2.3.3. Következmény.

2.3.4. Tétel: az integrál egyértelműsége.

2.3.5. Tétel: az integrál linearitása.

2.3.6. Tétel: integrál és koordinátafüggvények.

2.3.7. Tétel: az integrál nemnegativitása.

2.3.8. Következmény: az integrál monotonitása.

**2.3.9. Tétel: Cauchy-kritérium.*

▶ *2.3.10. Segédtétel.*

▶ *2.3.11. Tétel: az integrál mint halmazfüggvény additivitása.*

▶ *2.3.12. Nullahalmazok.*

▶ *2.3.13. Segédtétel.*

▶ *2.3.14. Lebesgue-feltétel.*

▶ *2.3.15. Következmény.*

▶ *2.3.16. Tétel.*

▶ *2.3.17. Megjegyzés.*

▼ *2.3.18. Ismételt integrálás.*

```
> int(x^2+y^2,[x,y]=Rectangle(-1..1,-2..2));
```

$$\frac{40}{3} \tag{2.3.18.1}$$

```
> int(x^2+y^2+z^4,[x,y,z]=Region(-1..1,-2..2,0..3));
```

$$\frac{2144}{5} \tag{2.3.18.2}$$

▶ *2.3.19. Abszolút integrálható függvények.*

▶ *2.3.20. Tétel.*

▼ *2.3.21. Definíció.*

```
> int(x^2+y^2,[x,y]=Ellipse(2*x^2+3*y^2-1));
```

$$\frac{5}{144}\sqrt{6}\,\pi \tag{2.3.21.1}$$

▼ *2.3.22. Impropius integrál.*

```
> int(exp(-x^2-y^2),[x,y]=Circle(<0,0>,R));
```

$$\pi - \mathrm{e}^{-R^2}\,\pi \tag{2.3.22.1}$$

```
> limit(%,R=infinity);
```

$$\pi \tag{2.3.22.2}$$

▶ *2.3.23. Mérték.*

▶ *2.3.24. Tétel.*

▼ *2.3.25. Normál integrációs tartomány.*

```
> int(x^2+y^2-z^2,[x,y,z]=Region(-R..R,-sqrt(R^2-x^2)..sqrt
(R^2-x^2),
```

```
-sqrt(R^2-x^2-y^2)..sqrt(R^2-x^2-y^2)));
```

$$\frac{4}{15} R^5 \pi \tag{2.3.25.1}$$

▶ 2.3.26. Tétel.

▶ 2.3.27. Példa: forgástest térfogata.

▼ 2.3.28. Példa: tehetetlenségi nyomaték.

```
> int(x^2+y^2,[x,y,z]=Region(-R..R,-sqrt(R^2-x^2)..sqrt(R^2-
x^2),
-sqrt(R^2-x^2-y^2)..sqrt(R^2-x^2-y^2)));
```

$$\frac{8}{15} R^5 \pi \tag{2.3.28.1}$$

▶ 2.3.29. Rademacher tétele.

▶ 2.3.30. Integráltranszformációs formula.

▼ 2.3.31. Példák: polár-, henger- és gömbi koordináták.

```
> Jacobian(<r*cos(phi),r*sin(phi)>,[r,phi],'determinant'=
true);
```

$$\begin{bmatrix} \cos(\phi) & -r\sin(\phi) \\ \sin(\phi) & r\cos(\phi) \end{bmatrix}, \cos(\phi)^2 r + r\sin(\phi)^2 \tag{2.3.31.1}$$

```
> simplify(%[2],trig);
```

$$r \tag{2.3.31.2}$$

```
> Jacobian(<r*cos(phi),r*sin(phi),z>,[r,phi,z],'determinant'=
true);
```

$$\begin{bmatrix} \cos(\phi) & -r\sin(\phi) & 0 \\ \sin(\phi) & r\cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \cos(\phi)^2 r + r\sin(\phi)^2 \tag{2.3.31.3}$$

```
> simplify(%[2],trig);
```

$$r \tag{2.3.31.4}$$

```
> Jacobian(<r*cos(phi)*cos(theta),r*sin(phi)*cos(theta),r*sin
(theta)>,
[r,phi,theta],'determinant'=true);
```

$$\begin{bmatrix} \cos(\phi)\cos(\theta) & -r\sin(\phi)\cos(\theta) & -r\cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & r\cos(\phi)\cos(\theta) & -r\sin(\phi)\sin(\theta) \\ \sin(\theta) & 0 & r\cos(\theta) \end{bmatrix}, \tag{2.3.31.5}$$

$$\cos(\phi)^2\cos(\theta)^3 r^2+\sin(\phi)^2\cos(\theta)^3 r^2$$
$$+r^2\sin(\phi)^2\cos(\theta)\sin(\theta)^2+r^2\cos(\phi)^2\cos(\theta)\sin(\theta)^2$$

```
> simplify(%[2],trig);
```

$$\cos(\theta)\, r^2 \tag{2.3.31.6}$$

```
> with(Student[MultivariateCalculus]);
```

$$[\mathit{Add2DCoordinateNames}, \mathit{Add3DCoordinateNames}, \mathit{ApproximateInt}, \mathit{ApproximateIntTutor}, \mathit{CenterOfMass}, \mathit{ChangeOfVariables}, \mathit{CrossSection}, \mathit{CrossSectionTutor}, \mathit{DirectionalDerivative}, \mathit{DirectionalDerivativeTutor}, \mathit{FunctionAverage}, \mathit{Gradient}, \mathit{GradientTutor}, \mathit{Jacobian}, \mathit{LagrangeMultipliers}, \mathit{MultiInt}, \mathit{Revert}, \mathit{SecondDerivativeTest}, \mathit{SurfaceArea}, \mathit{TaylorApproximation}, \mathit{TaylorApproximationTutor}] \tag{2.3.31.7}$$

```
> ChangeOfVariables(x^2+y^2,['cartesian'[x,y],'polar'[r,phi]]
);
```

$$r^2\cos(\phi)^2+r^2\sin(\phi)^2 \tag{2.3.31.8}$$

```
> simplify(%,trig);
```

$$r^2 \tag{2.3.31.9}$$

```
> ChangeOfVariables(Int(Int(x^2+y^2,x),y),
['cartesian'[x,y],'polar'[r,phi]]);
```

$$\iint r^3\, \mathrm{d}\phi\, \mathrm{d}r \tag{2.3.31.10}$$

```
> ChangeOfVariables(x^2+y^2-z^2,['cartesian'[x,y,z],
'cylindrical'[r,phi,z]]);
```

$$r^2\cos(\phi)^2+r^2\sin(\phi)^2-z^2 \tag{2.3.31.11}$$

```
> simplify(%,trig);
```

$$-z^2+r^2 \tag{2.3.31.12}$$

```
> ChangeOfVariables(Int(Int(Int(x^2+y^2-z^2,x),y),z),
['cartesian'[x,y,z],'cylindrical'[r,phi,z]]);
```

$$\iiint\left(-z^2+r^2\right) r\, \mathrm{d}\phi\, \mathrm{d}r\, \mathrm{d}z \tag{2.3.31.13}$$

▶ ***2.3.32. Tétel: paraméteres integrálok differenciálása.***

▶ ***2.3.33. Pályák.***

▶ ***2.3.34. Tétel.***

▼ ***2.3.35. Tétel.***

```
> v:=<cos(t),sin(t),t>; diff(v,t); %.%; simplify(%,trig);
```

$$v := (\cos(t))e_x + (\sin(t))e_y + (t)e_z$$

$$-\sin(t)e_x + (\cos(t))e_y + e_z$$

$$1 + \sin(t)^2 + \cos(t)^2$$

$$2 \tag{2.3.35.1}$$

2.3.36. Példák.

2.3.37. Sima pályák természetes paraméterezése.

```
> s=int(sqrt(%),t=0..T); solve(%,T); subs(t=%,v);
  diff(%,s); %.%; simplify(%,trig);
```

$$s = \sqrt{2}\,T$$

$$\frac{1}{2}\,s\sqrt{2}$$

$$\left(\cos\left(\frac{1}{2}\,s\sqrt{2}\right)\right)e_x + \left(\sin\left(\frac{1}{2}\,s\sqrt{2}\right)\right)e_y + \frac{1}{2}\,s\sqrt{2}\,e_z$$

$$-\frac{1}{2}\sin\left(\frac{1}{2}\,s\sqrt{2}\right)\sqrt{2}\,e_x + \frac{1}{2}\cos\left(\frac{1}{2}\,s\sqrt{2}\right)\sqrt{2}\,e_y + \frac{1}{2}\sqrt{2}\,e_z$$

$$\frac{1}{2} + \frac{1}{2}\sin\left(\frac{1}{2}\,s\sqrt{2}\right)^2 + \frac{1}{2}\cos\left(\frac{1}{2}\,s\sqrt{2}\right)^2$$

$$1 \tag{2.3.37.1}$$

2.3.38. Integrálás felületeken és felszín.

```
> PathInt(x^2+y^2+z^2,[x,y,z]=Path(<t,t^2,t^3>,t=0..2));
  evalf(%);
```

$$\frac{2}{36971235}\,\frac{1}{\sqrt{-2+\mathrm{I}\sqrt{5}}\,\left(2+\mathrm{I}\sqrt{5}\right)}\left(\sqrt{161}\,\left(\right.\right.$$

$$768697076\sqrt{-2+\mathrm{I}\sqrt{5}}$$

$$+\,384348538\,\mathrm{I}\sqrt{-2+\mathrm{I}\sqrt{5}}\,\sqrt{5}$$

$$-\,20010\sqrt{9-4\,\mathrm{I}\sqrt{5}}\,\sqrt{9+4\,\mathrm{I}\sqrt{5}}\;\mathrm{EllipticF}\left(2\sqrt{-2+\mathrm{I}\sqrt{5}}\,,\right.$$

$$\left.\frac{1}{3}\sqrt{-1+4\,\mathrm{I}\sqrt{5}}\right)$$

$$-3523\,\mathrm{I}\sqrt{9-4\,\mathrm{I}\sqrt{5}}\,\sqrt{9+4\,\mathrm{I}\sqrt{5}}\,\mathrm{EllipticF}\left(2\sqrt{-2+\mathrm{I}\sqrt{5}},\right.$$

$$\left.\frac{1}{3}\sqrt{-1+4\,\mathrm{I}\sqrt{5}}\right)\sqrt{5}$$

$$+12964\sqrt{9-4\,\mathrm{I}\sqrt{5}}\,\sqrt{9+4\,\mathrm{I}\sqrt{5}}\,\mathrm{EllipticE}\left(2\sqrt{-2+\mathrm{I}\sqrt{5}},\right.$$

$$\left.\left.\left.\frac{1}{3}\sqrt{-1+4\,\mathrm{I}\sqrt{5}}\right)\right)\right)$$

$$263.8435852-7.095521418\,10^{-8}\,\mathrm{I} \qquad (2.3.38.1)$$

```
> SurfaceInt(x+y+z,[x,y,z]=Surface(<s,t,4-s*t>,
  [s,t]=Triangle(<0,0>,<0,1>,<1,0>))); evalf(%);
```

$$\int_0^1 \left(\frac{1}{3}\left(1+s^2\right)^{3/2} s-\frac{1}{4}\,s\ln\left(1+s^2\right)-\frac{1}{3}\left(1+s^2\right)^{3/2}-\frac{1}{4}\ln\left(1\right.\right.$$

$$\left.+s^2\right)s^3-\ln\left(1+s^2\right)s^2-\ln\left(1+s^2\right)-\frac{1}{3}\left(2+2\,s^2-2\,s\right)^{3/2} s$$

$$+2\ln\left(-s+1+\sqrt{2+2\,s^2-2\,s}\right)s^2$$

$$+\frac{1}{3}\left(2+2\,s^2-2\,s\right)^{3/2}-\frac{1}{2}\,s^2\sqrt{2+2\,s^2-2\,s}$$

$$-\frac{3}{2}\,s\sqrt{2+2\,s^2-2\,s}+\frac{1}{2}\,s\ln\left(-s+1+\sqrt{2+2\,s^2-2\,s}\right)$$

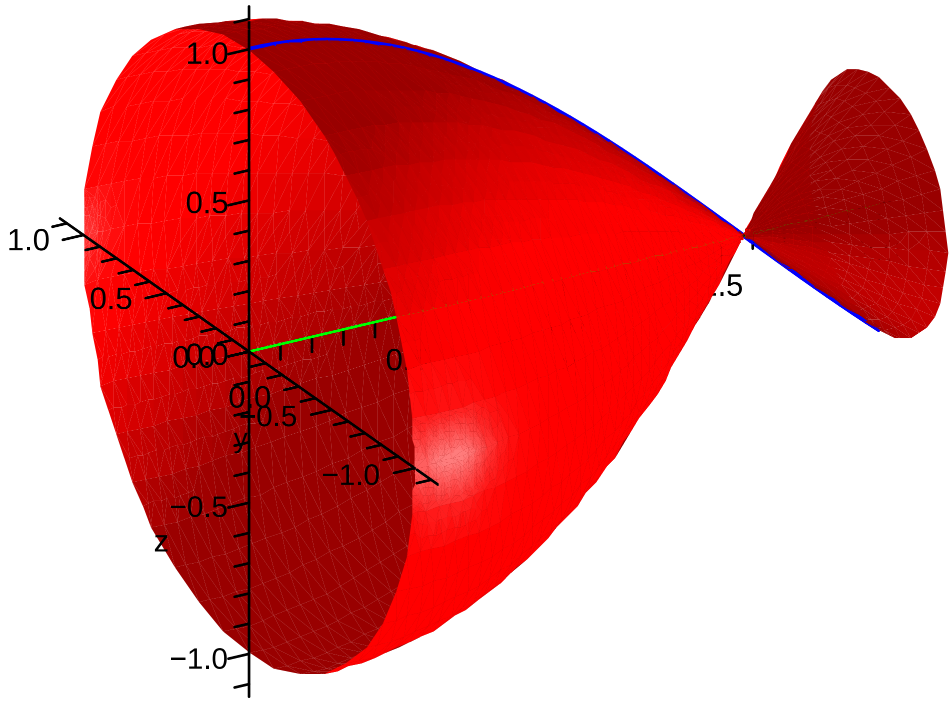
$$+\frac{1}{2}\ln\left(-s+1+\sqrt{2+2\,s^2-2\,s}\right)s^3+2\sqrt{2+2\,s^2-2\,s}+2\ln\left(-s\right.$$

$$\left.\left.+1+\sqrt{2+2\,s^2-2\,s}\right)\right)\mathrm{d}s$$

$$2.646495329 \qquad (2.3.38.2)$$

2.3.39. Példa: forgásfelület felszíne.

```
> Student[Calculus1][SurfaceOfRevolutionTutor]();
```



▶ 3. Függvénysorozatok és függvénysorok