Kalkulus I.

Járai Antal

Ezek a programok csak szemléltetésre szolgálnak

▶ 1. Halmazok

▶ 2. Számok

▶ 3. Határérték

▶ 4. Differenciálszámítás

▼ 5. Integrálszámítás

```
> restart;
```

▼ 5.1. Primitív függvények

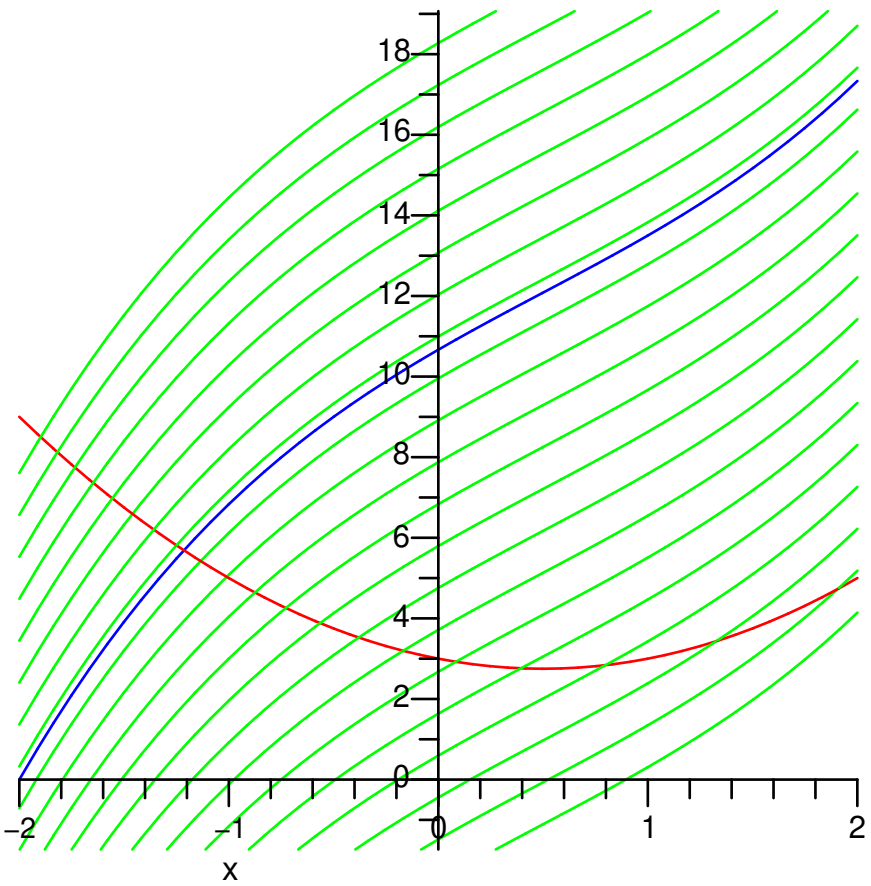
▼ *5.1.1. Primitív függvény.*

```
> int(cos(x),x); int(x/(x^3-1),x);
```

$$\sin(x)$$

$$-\frac{1}{6}\ln\left(x^2+x+1\right)+\frac{1}{3}\sqrt{3}\arctan\left(\frac{1}{3}\left(2x+1\right)\sqrt{3}\right)+\frac{1}{3}\ln\left(x-1\right) \quad (5.1.1.1)$$

```
> Student[Calculus1][AntiderivativeTutor]();
```



▶ *5.1.2. Tagonkénti integrálás határozatlan integrálokra.*

▶ *5.1.3. Parciális integrálás határozatlan integrálokra.*

▶ *5.1.4. Helyettesítéses integrálás határozatlan integrálokra.*

▶ *5.1.5. Megjegyzés.*

▶ *5.1.6. Tétel.*

▼ *5.1.7. Alapintegrálok.*

```
> int(x^n,x);
```

$$\frac{x^{n+1}}{n+1} \tag{5.1.7.1}$$

```
> int(1/(1+x^2),x);
```

$$\arctan(x) \tag{5.1.7.2}$$

```
> int(exp(x),x);
```

$$e^{x} \tag{5.1.7.3}$$

```
> int(sin(x),x);
```

$$-\cos(x) \tag{5.1.7.4}$$

```
> int(cos(x),x);
```

$$\sin(x) \tag{5.1.7.5}$$

```
> int(sinh(x),x);
```

$$\cosh(x) \tag{5.1.7.6}$$

```
> int(cosh(x),x);
```

$$\sinh(x) \tag{5.1.7.7}$$

```
> int(1/cosh(x)^2,x);
```

$$\frac{\sinh(x)}{\cosh(x)} \tag{5.1.7.8}$$

```
> int(x^n,x);
```

$$\frac{x^{n+1}}{n+1} \tag{5.1.7.9}$$

```
> int(1/x,x);
```

$$\ln(x) \tag{5.1.7.10}$$

```
> int(1/sinh(x)^2,x);
```

$$-\frac{\cosh(x)}{\sinh(x)} \tag{5.1.7.11}$$

```
> int(x^n,x);
```

$$\frac{x^{n+1}}{n+1} \tag{5.1.7.12}$$

```
> int(1/sqrt(1-x^2),x);
```

$$\arcsin(x) \tag{5.1.7.13}$$

```
> int(1/sin(x)^2,x);
```

$$-\frac{\cos(x)}{\sin(x)} \tag{5.1.7.14}$$

```
> int(1/cos(x)^2,x);
```

$$\frac{\sin(x)}{\cos(x)} \tag{5.1.7.15}$$

▶ ***5.1.8. Megjegyzés.***

▼ ***5.1.9. Példák.***

```
> Student[Calculus1][IntTutor]();
```

$$\int \sin(x)^2 \, dx \tag{5.1.9.1}$$

▶ ***5.1.10. Tétel.***

▼ ***5.1.11. Elemien integrálható függvények.***

```
> int(exp(x^2),x); int(cos(x^2),x); int(sin(x^2),x); int(sin
  (x)/x,x);
  int(cos(x)/x,x); int(exp(x)/x,x); int(sqrt(1+x^3),x); int
  (1/ln(x),x);
```

$$-\frac{1}{2}\,\mathrm{I}\sqrt{\pi}\,\mathrm{erf}(\mathrm{I}\,x)$$

$$\frac{1}{2}\sqrt{2}\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,x}{\sqrt{\pi}}\right)$$

$$\frac{1}{2}\sqrt{2}\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,x}{\sqrt{\pi}}\right)$$

$$\mathrm{Si}(x)$$

$$\mathrm{Ci}(x)$$

$$-\mathrm{Ei}(1,-x)$$

$$\frac{2}{5}\,x\sqrt{1+x^3}+\frac{6}{5}\,\frac{1}{\sqrt{1+x^3}}\left(\left(\frac{3}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3}\right)\sqrt{\frac{x+1}{\frac{3}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3}}}\,\sqrt{\frac{x-\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3}}{-\frac{3}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3}}}\,\sqrt{\frac{x-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\sqrt{3}}{-\frac{3}{2}+\frac{1}{2}\,\mathrm{I}\sqrt{3}}}\right.$$
$$\left.\mathrm{EllipticF}\left(\sqrt{\frac{x+1}{\frac{3}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3}}},\sqrt{\frac{-\frac{3}{2}+\frac{1}{2}\,\mathrm{I}\sqrt{3}}{-\frac{3}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3}}}\right)\right)$$

$$-\mathrm{Ei}(1,-\ln(x))$$ (5.1.11.1)

5.1.12. Segédtétel.

```
> f:=x^3+1; g:=x^2-3; gcdex(f,g,x,'u','v'); f*u+g*v; expand
  (%);
```

$$f := 1+x^3$$

$$g := x^2-3$$

$$1$$

$$\left(1+x^3\right)\left(-\frac{1}{26}+\frac{3}{26}\,x\right)+\left(x^2-3\right)\left(-\frac{9}{26}-\frac{3}{26}\,x^2+\frac{1}{26}\,x\right)$$

$$1$$ (5.1.12.1)

5.1.13. Parciális törtekre bontás tétele.

```
> convert(x^2/(x^3+1),parfrac,x);
```

$$\frac{1}{3}\,\frac{2x-1}{x^2-x+1}+\frac{1}{3(x+1)} \tag{5.1.13.1}$$

5.1.14. Racionális törtfüggvények integrálása.

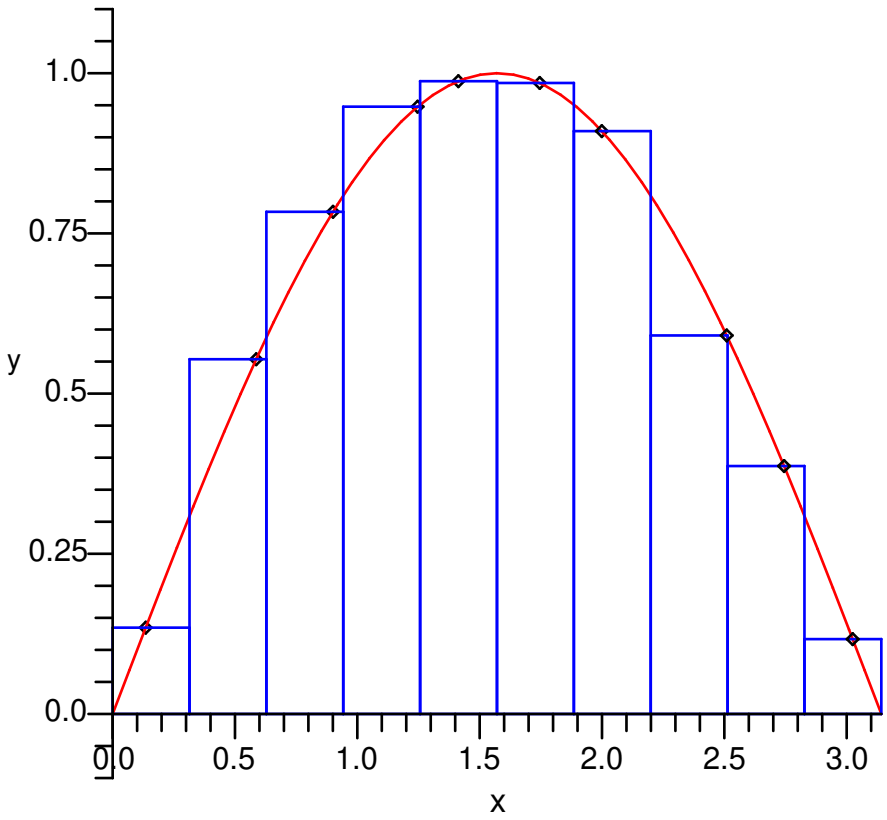
5.1.15. Racionális törtfüggvények integrálására visszavezethető integrálok.

5.2. Határozott integrál

```
> restart;
```

5.2.1. Definíció.

```
> Student[Calculus1][ApproximateIntTutor]();
```



▶ *5.2.2. Tétel.*

▶ *5.2.3. Tétel: az integrál egyértelműsége.*

▶ *5.2.4. Tétel: komplex értékű függvények integrálja.*

▶ *5.2.5. Tétel: az integrál linearitása.*

▶ *5.2.6. Tétel: az integrál nemnegativitása.*

▶ **5.2.7. Tétel: Cauchy–kritérium.*

▶ **5.2.8. Segédtétel.*

▼ *5.2.9. Tétel: az integrál intervallum–additivitása.*

```
> Student[Calculus1][IntTutor](x^3,x=0..3);
```

$$\int_0^3 x^3\,dx = \int_0^2 x^3\,dx + \int_2^3 x^3\,dx \qquad (5.2.9.1)$$

▶ *5.2.10. Definíció.*

▶ *5.2.11. Nullahalmazok.*

▶ *5.2.12. Lebesgue-feltétel.*

▶ *5.2.13. Következmény.*

▶ *5.2.14. Tétel.*

▶ *5.2.15. Megjegyzés.*

▶ *5.2.16. Az integrál, mint a felső határ függvénye.*

▶ *5.2.17. Következmény.*

▼ *5.2.18. Newton-Leibniz-formula.*

```
> Student[Calculus1][IntTutor](sin(x)^2,x=0..Pi);
```

$$\int_0^{\pi} \sin(x)^2 \, dx = \frac{1}{2} \pi \qquad (5.2.18.1)$$

▼ *5.2.19. Parciális integrálás határozott integrálokra.*

```
> Student[Calculus1][IntTutor](x*exp(x),x=-1..1);
```

$$\int_{-1}^{1} x \, e^{x} \, dx = 2 \, e^{-1} \qquad (5.2.19.1)$$

▼ *5.2.20. Helyettesítéses integrálás határozott integrálokra.*

```
> Student[Calculus1][IntTutor](1/(1+cos(x)),x=-Pi/2..Pi/2);
```

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{1+\cos(x)} \, dx = 2 \qquad (5.2.20.1)$$

▼ *5.2.21. Tétel.*

```
> int(1/sqrt(x),x=0..1);
```

$$2 \qquad (5.2.21.1)$$

▼ *5.2.22. Példák.*

```
> int(x^a,x=0..1) assuming a<0 and -1<a; int(x^a,x=0..1)
  assuming a<=-1;
```

$$\frac{1}{a+1}$$

$$\infty \tag{5.2.22.1}$$

```
> int(1/x,x=-1..1);
```

$$\mathit{undefined} \tag{5.2.22.2}$$

▶ *5.2.23. Megjegyzés.*

▶ *5.2.24. Abszolút integrálható függvények.*

▶ *5.2.25. Tétel.*

▶ *5.2.26. Improprius integrál.*

▼ *5.2.27. Példák.*

```
> int(x^a,x=1..infinity) assuming a<-1;
```

$$-\frac{1}{a+1} \tag{5.2.27.1}$$

```
> int(x^a,x=1..infinity) assuming a>=-1;
```

$$\infty \tag{5.2.27.2}$$

```
> int(exp(a*x),x=0..infinity) assuming a<0;
```

$$-\frac{1}{a} \tag{5.2.27.3}$$

```
> int(exp(a*x),x=0..infinity) assuming a>=0;
```

$$\infty \tag{5.2.27.4}$$

▶ *5.2.28. Tétel.*

▶ *5.2.29. További improprius integrálok.*

▼ 5.3. Alkalmazások

```
> restart;
```

▶ *5.3.1. Végtelen kicsinyek.*

▼ *5.3.2. Görbe hossza, heurisztikusan.*

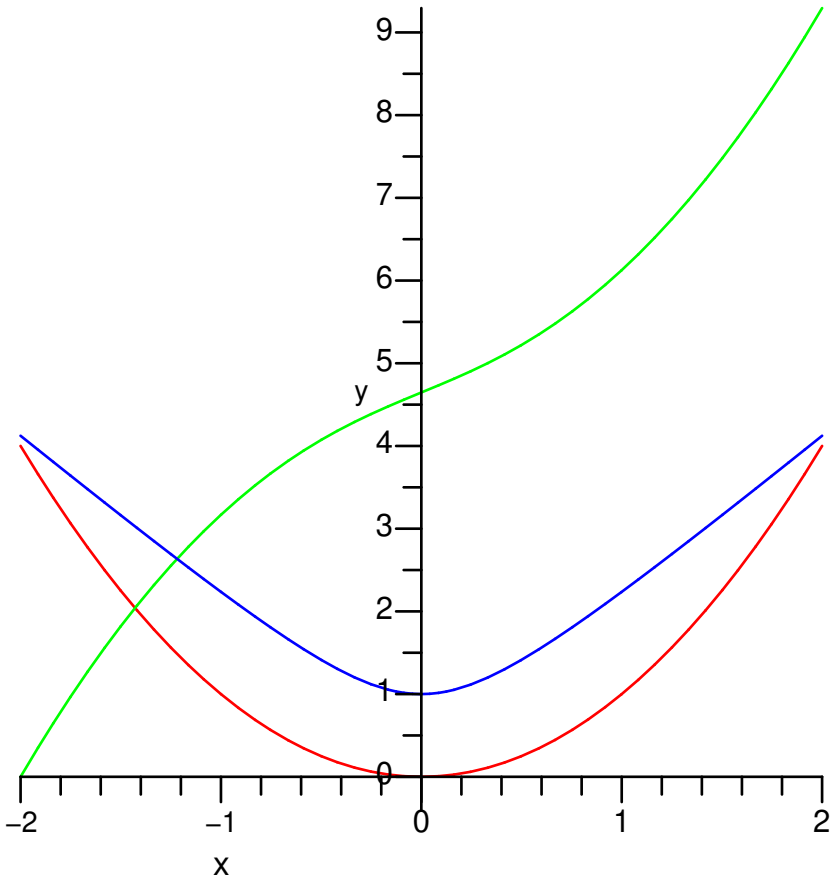
```
> z:=exp(I*t); zp:=diff(z,t); int(evalc(abs(zp)),t=0..2*Pi);
```

$$z := \mathrm{e}^{\mathrm{I}t}$$

$$zp := \mathrm{I}\,\mathrm{e}^{\mathrm{I}t}$$

$$2\,\pi \tag{5.3.2.1}$$

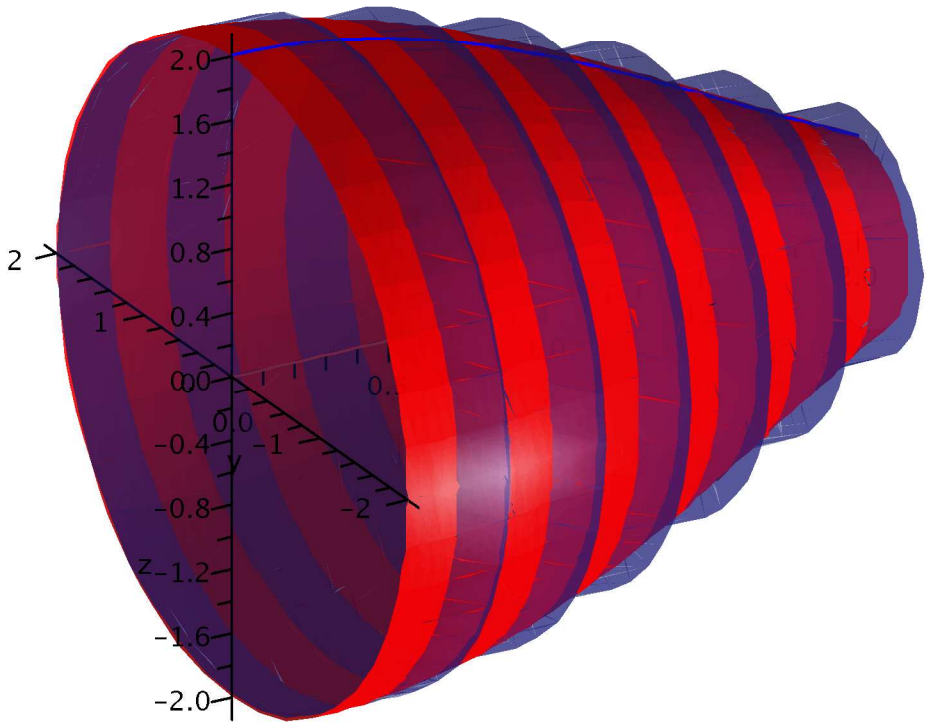
```
> Student[Calculus1][ArcLengthTutor]();
```



▶ *5.3.3. Polárkoordinátákkal adott szektor területe, heurisztikusan.*

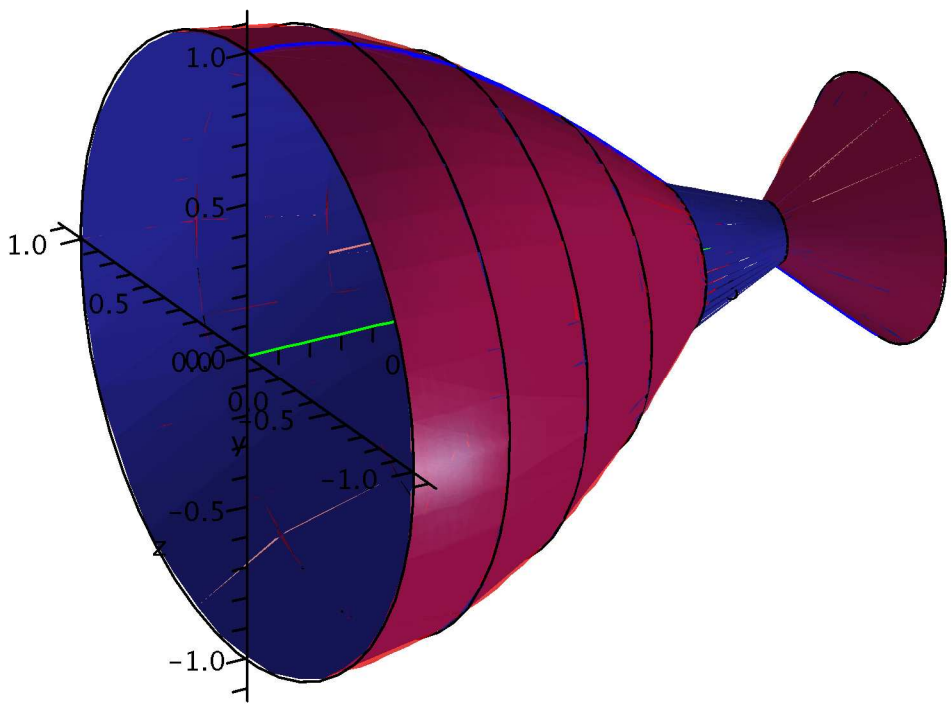
▼ *5.3.4. Forgástest térfogata, heurisztikusan.*

```
> Student[Calculus1][VolumeOfRevolutionTutor]();
```



5.3.5. Forgástest felszíne, heurisztikusan.

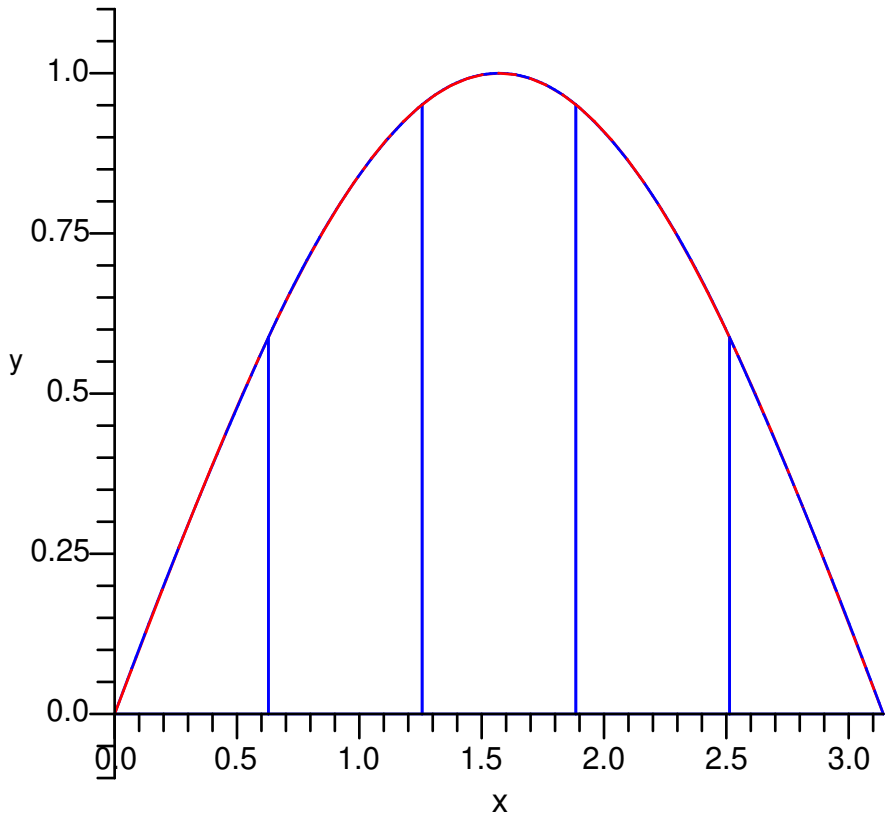
```
> Student[Calculus1][SurfaceOfRevolutionTutor]();
```



▶ *5.3.6. Tömeg, tömegközéppont, tehetetlenségi nyomaték.*

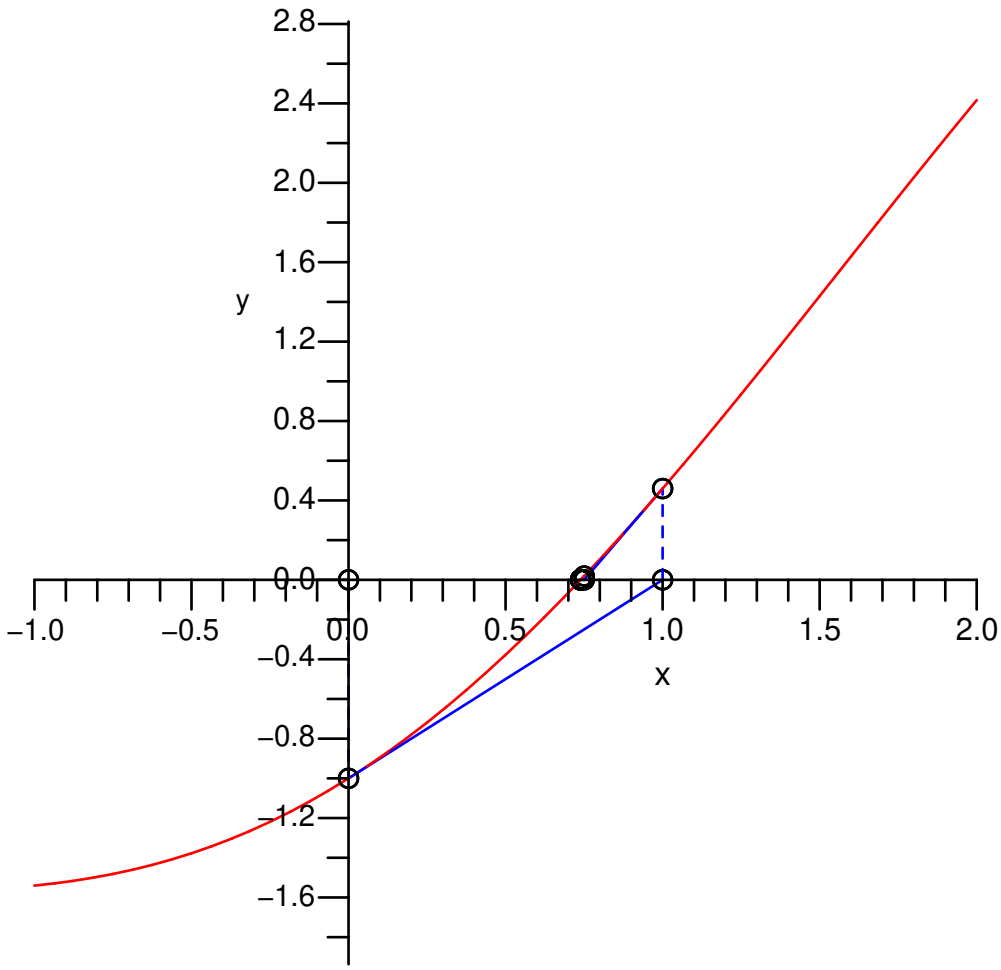
▼ *5.3.7. Közelítő számítások.*

```
> Student[Calculus1][ApproximateIntTutor]();
```



5.3.8. Newton-módszer.

```
> Student[Calculus1][NewtonsMethodTutor](x-cos(x),0);
```




```
> relaxnewton:=proc(f::procedure,fp::procedure,x0,epsilon)
  local x,xx,t; x:=x0; t:=1.; print(x,t,f(x),abs(f(x)));
  do if abs(f(x))<epsilon then return(x) fi;
    t:=1.;
    do xx:=x-t*f(x)/fp(x); print(xx,t,f(xx),abs(f(xx)));
      if abs(f(xx))<abs(f(x)) then break else t:=t/2 fi;
    od; x:=xx; od; end;

  relaxnewton(z->z^5-z+3,z->5*z^4-1,3.+10.*I,0.0000001);

  relaxnewton(z->z^5-z+3,z->5*z^4-1,(1/5.)^(1/4)+0.01,
  0.0000001);
```

relaxnewton := **proc**(*f*::*procedure*, *fp*::*procedure*, *x0*, ε)

local *x*, *xx*, *t*;

x := *x0*;

t := 1.;

print(*x*, *t*, *f*(*x*), abs(*f*(*x*)));

```
    do
        if abs(f(x)) < ε then
            return x
        end if;
        t := 1.;
        do
            xx := x − t* f(x) / fp(x);
            print(xx, t, f(xx), abs(f(xx)));
            if abs(f(xx)) < abs(f(x)) then
                break
            else
                t := 1 / 2 * t
            end if
        end do;
        x := xx
    end do
end proc
```

$$3. + 10.\,\mathrm{I},\ 1.,\ 1.23243\,10^{5} + 14040.\,\mathrm{I},\ 1.240401493\,10^{5}$$

$$2.399872243 + 8.000043773\,\mathrm{I},\ 1.,\ 40384.62044 + 4599.180768\,\mathrm{I},\ 40645.66437$$

$$1.919638552 + 6.400097857\,\mathrm{I},\ 1.,\ 13233.67341 + 1505.902344\,\mathrm{I},\ 13319.07856$$

$$1.535180765 + 5.120145726\,\mathrm{I},\ 1.,\ 4336.920528 + 492.5281458\,\mathrm{I},\ 4364.798236$$

$$1.227051191 + 4.096113408\,\mathrm{I},\ 1.,\ 1421.688163 + 160.6504428\,\mathrm{I},\ 1430.736103$$

$$0.9793624175 + 3.276555586\,\mathrm{I},\ 1.,\ 466.4696379 + 52.04789824\,\mathrm{I},\ 469.3643646$$

$$0.7786832399 + 2.619787728\,\mathrm{I},\ 1.,\ 153.5006812 + 16.57703671\,\mathrm{I},\ 154.3931905$$

$$0.6126365287 + 2.091022469\,\mathrm{I},\ 1.,\ 50.98085391 + 5.042396308\,\mathrm{I},\ 51.22961278$$

$$0.4673540873 + 1.658543942\,\mathrm{I},\ 1.,\ 17.42867285 + 1.321921898\,\mathrm{I},\ 17.47873322$$

$$0.3202538500 + 1.285961515\,\mathrm{I},\ 1.,\ 6.518954023 + 0.117325617\,\mathrm{I},\ 6.520009728$$

$$0.1022977568 + 0.9060193817\,\mathrm{I},\ 1.,\ 3.233581963 - 0.3728503660\,\mathrm{I},\ 3.255006867$$

$$\begin{array}{l}
-0.9977750502 + 0.3000912412\,I,\ 1.,\\
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\quad 2.921980382 - 0.6347627568\,I,\ 2.990132624\\
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\qquad 188.2298311\\
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\quad 1.739041227 + 1.931106675\,I,\ 2.598737651\\
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\qquad 0.8387912964\\
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\quad -0.1236837561 + 0.019656712\,I,\ 0.1252360086\\
-0.3084767808 + 1.250051687\,I,\ 1.,\\
\quad -0.0018307092 + 0.000149634\,I,\ 0.001836814228\\
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\quad 4.099508385\,10^{-7}\\
-0.3084151033 + 1.249926914\,I,\ 1.,\ -2.7\,10^{-9} + 2.\,10^{-9}\,I,\\
\quad 3.360059523\,10^{-9}\\
\qquad -0.3084151033 + 1.249926914\,I\\
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\qquad -9.397076885,\ 0.2500000000,\ -73263.96516,\ 73263.96516\\
\qquad -4.359168288,\ 0.1250000000,\ -1566.689577,\ 1566.689577\\
\qquad -1.840213992,\ 0.06250000000,\ -16.26266179,\ 16.26266179\\
\qquad -.5807368430,\ 0.03125000000,\ 3.514683182,\ 3.514683182\\
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\end{array}$$

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0.8089276006, 0.0002441406250, 2.537448779, 2.537448779

0.7384763473, 0.0001220703125, 2.481149247, 2.481149247

0.7032507206, 0.00006103515625, 2.468758184, 2.468758184

0.6856379073, 0.00003051757812, 2.465883538, 2.465883538

0.6768315007, 0.00001525878906, 2.465205933, 2.465205933

0.6724282973, 0.000007629394530, 2.465048658, 2.465048658

0.6702266957, 0.000003814697265, 2.465014378, 2.465014378

0.6691258948, 0.000001907348632, 2.465008201, 2.465008201

−1067.192127, 1., $-1.384245318\ 10^{15}$, $1.384245318\ 10^{15}$

−533.2615002, 0.5000000000, $-4.312222394\ 10^{13}$, $4.312222394\ 10^{13}$

−266.2961872, 0.2500000000, $-1.339136173\ 10^{12}$, $1.339136173\ 10^{12}$

−132.8135307, 0.1250000000, $-4.132488076\ 10^{10}$, $4.132488076\ 10^{10}$

−66.07220241, 0.06250000000, $-1.259197619\ 10^{9}$, $1.259197619\ 10^{9}$

−32.70153825, 0.03125000000, $-3.739732126\ 10^{7}$, $3.739732126\ 10^{7}$

−16.01620618, 0.01562500000, $-1.053878194\ 10^{6}$, $1.053878194\ 10^{6}$

−7.673540141, 0.007812500000, −26595.28111, 26595.28111

−3.502207123, 0.003906250000, −520.3746648, 520.3746648

−1.416540614, 0.001953125000, −1.287008084, 1.287008084

$$-1.349270566,\ 1.,\ -.122661857,\ 0.122661857$$
$$-1.341393320,\ 1.,\ -0.001515350,\ 0.001515350$$
$$-1.341293547,\ 1.,\ -2.32\ 10^{-7},\ 2.32\ 10^{-7}$$
$$-1.341293532,\ 1.,\ -5.\ 10^{-9},\ 5.\ 10^{-9}$$
$$-1.341293532 \tag{5.3.8.1}$$

5.3.9. Differenciálegyenletek.

```
> de:=diff(y(x),x,x)=2*y(x)+1; dsolve(de);
```

$$de := \frac{d^2}{dx^2} y(x) = 2\,y(x) + 1$$

$$y(x) = e^{\sqrt{2}\,x}_C2 + e^{-\sqrt{2}\,x}_C1 - \frac{1}{2} \tag{5.3.9.1}$$

5.3.10. Szeparábilsi differenciálegyenletek megoldása.

5.3.11. Elsőrendű lineáris differenciálegyenletek megoldása.

*5.3.12. Variációszámítás.

```
> with(VariationalCalculus); g:='g';
```

[*ConjugateEquation*, *Convex*, *EulerLagrange*, *Jacobi*, *Weierstrass*]

$$g := g \tag{5.3.12.1}$$

```
> L:=g*rho*y(x)*sqrt(1+diff(y(x),x)^2);
```

$$L := g\,\rho\,y(x)\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2} \tag{5.3.12.2}$$

```
> EulerLagrange(L,x,y(x)); simplify(%); %[2];
```

$$\left\{ g\,\rho\,y(x)\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2} - \frac{\left(\frac{d}{dx}y(x)\right)^2 g\,\rho\,y(x)}{\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2}} = K_1, \right.$$

$$\left. g\,\rho\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2} - \frac{g\,\rho\left(\frac{d}{dx}y(x)\right)^2}{\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2}} \right\}$$

$$+\frac{g\,\rho\,y(x)\left(\frac{d}{dx}y(x)\right)^2\left(\frac{d^2}{dx^2}y(x)\right)}{\left(1+\left(\frac{d}{dx}y(x)\right)^2\right)^{3/2}}-\frac{g\,\rho\,y(x)\left(\frac{d^2}{dx^2}y(x)\right)}{\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2}}\Bigg\}$$

$$\left\{-\frac{g\,\rho\left(-1-\left(\frac{d}{dx}y(x)\right)^2+y(x)\left(\frac{d^2}{dx^2}y(x)\right)\right)}{\left(1+\left(\frac{d}{dx}y(x)\right)^2\right)^{3/2}},\ \frac{g\,\rho\,y(x)}{\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2}}=K_1\right\}$$

$$\frac{g\,\rho\,y(x)}{\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2}}=K_1 \qquad (5.3.12.3)$$

*5.3.13. Integrálkritérium.

```
> sum(n^a,n=1..infinity) assuming a<-1;
```

$$\zeta(-a) \qquad (5.3.13.1)$$