On near-critical SLE(6) and on the tail in Cardy's formula

Gábor Pete (Rényi Institute and TU Budapest)
http://www.math.bme.hu/∼gabor

Partly joint work with Christophe Garban (U Lyon)
and Oded Schramm

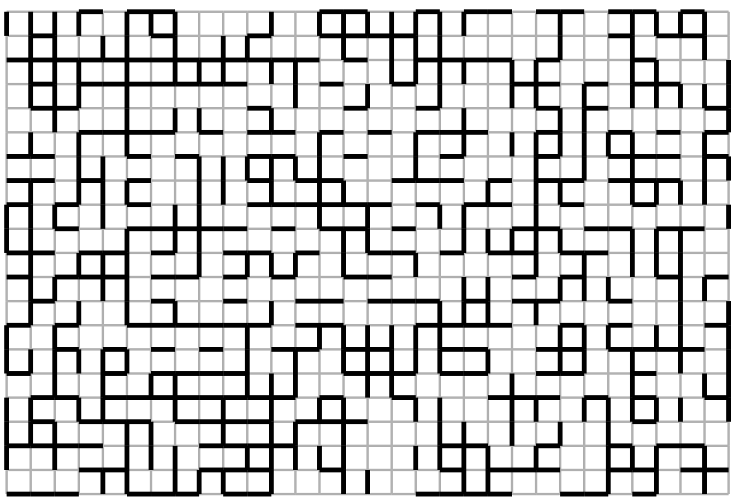
Pivotal, cluster and interface measures for critical planar percolation
[arXiv:1008.1378 math.PR], JAMS (2013);
The scaling limits of near-critical and dynamical percolation
[arXiv:1305.5526 math.PR];
and in preparation

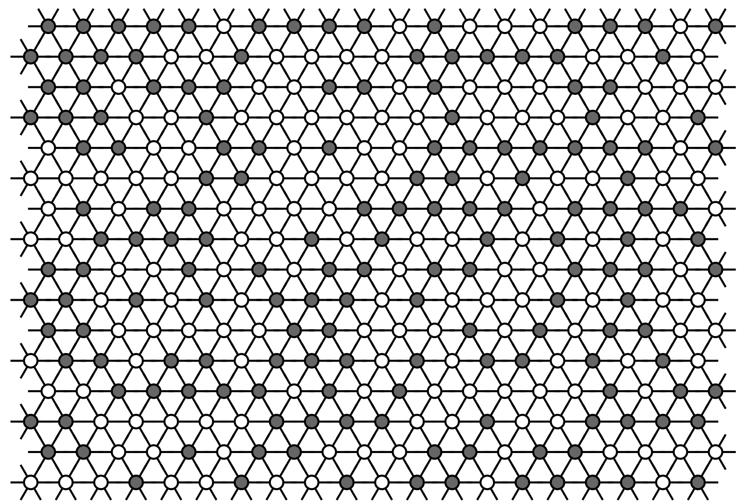
Cambridge, January 30, 2015

Phase transition in the percolation ensemble

Given labels $U(e) \sim \text{Unif}[0,1]$, the percolation p-clusters are the connected components of the random graph $\omega_p := \{e \in E : U(e) \leqslant p\}$.

For small p close to 0, expect small p-clusters only. For p close to 1, there is a unique giant p-cluster. Phase transition at some critical p_c density.





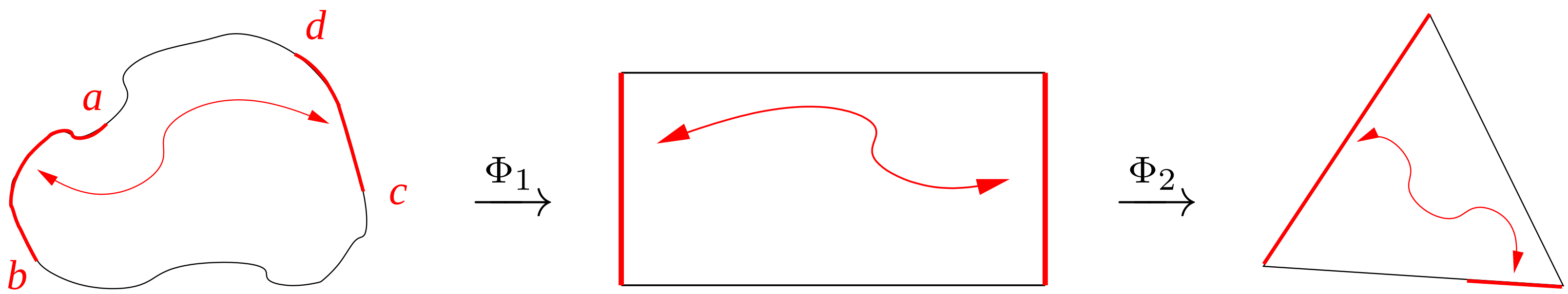
Harris '60 and Kesten '80: $p_c(\mathbb{Z}^2, \text{bond}) = p_c(\Delta, \text{site}) = 1/2$.

Conformal invariance at criticality

Theorem (Smirnov '01). For critical site percolation on $\Delta_{1/n}$, if $\mathcal{Q} \subset \mathbb{C}$ is a piecewise smooth quad, then

$$\lim_{n\to\infty} \mathbf{P}\Big[ab \longleftrightarrow cd \text{ inside } \mathcal{Q} \cap \Delta_{1/n} \Big]$$

exists, is strictly between 0 and 1, and conformally invariant. Value in $1 \times \rho$ rectangle is Cardy's formula.



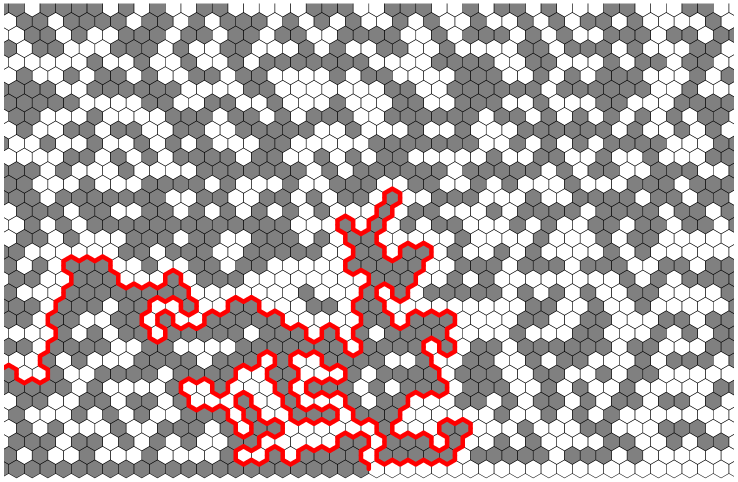
Conformal invariance at criticality

Theorem (Smirnov '01). For critical site percolation on $\Delta_{1/n}$, if $\mathcal{Q} \subset \mathbb{C}$ is a piecewise smooth quad, then

$$\lim_{n\to\infty} \mathbf{P}\Big[\, ab \longleftrightarrow cd \text{ inside } \mathcal{Q} \cap \Delta_{1/n} \Big]$$

exists, is strictly between 0 and 1, and conformally invariant. Value in $1 \times \rho$ rectangle is Cardy's formula.

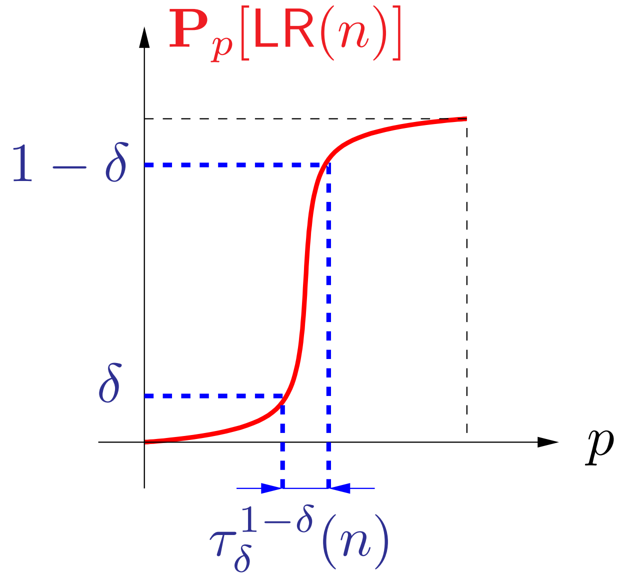
These quad-crossings are good enough to prove that exploration interface converges to Schramm-Loewner Evolution with $\kappa = 6$; Camia-Newman '06, Smirnov '06.

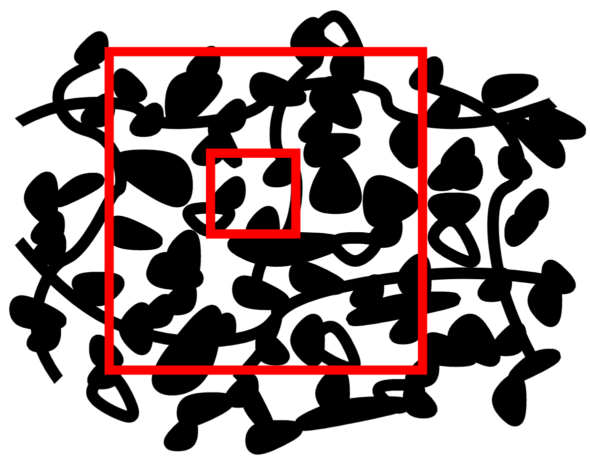


Moreover, there is a full scaling limit: quad-crossing topology by Schramm-Smirnov '10, and CLE(6) interface loop ensemble by Camia-Newman '06. Other suggestions by Aizenman '99 and Sheffield '09.

The near-critical percolation window

In a quad $\mathcal{Q} \cap \Delta_{1/n}$, how small $p_1 < p_c$ needs to be for all p_1-clusters to be small? For what $p_2 > p_c$ will the system be well-connected?

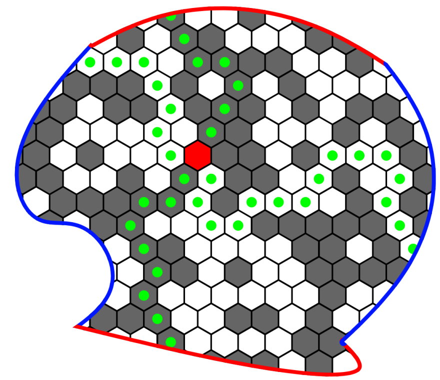




For $p > p_c$, correlation length: $L_\delta(p) := \min\{n : \mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big] > 1-\delta\}$.
For $p < p_c$, $L_\delta(p) := \min\{n : \mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big] < \delta\}$.

Kesten '87: Near-critical window for percolation is given by number of pivotal points at criticality: $|\tau(n)| \asymp 1/\mathbf{E}_{p_c}|\mathsf{Piv}_n| = n^{-3/4+o(1)}$.

From critical to near-critical percolation

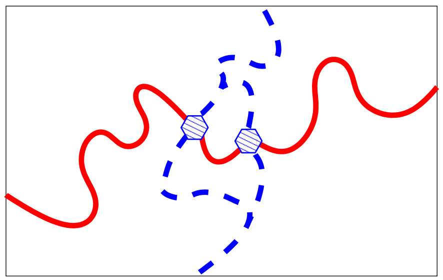


A site is pivotal in ω if flipping it changes the existence of a left-right crossing. Equivalent to having alternating 4 arms. For nice quads, there are not many pivotals close to $\partial \mathcal{Q}$, hence

$$\mathbf{E}_{p_c}|\mathrm{Piv}_n| \asymp n^2 \, \alpha_4(n) = n^{3/4+o(1)} \text{ on } \Delta_{1/n}.$$

If $p - p_c \gg r(n) := 1/\mathbf{E}_{p_c}|\mathrm{Piv}_n| = n^{-3/4+o(1)}$, we have opened many critical pivotals (clear in expectation, but also true in probability) — hence already supercritical. But maybe many *new* pivotals appeared on the way, so a pivotal switch happens earlier?

New pivotals do appear. But will they be switched as p is raised?



Stability by Kesten '87: multi-arm probabilities stay comparable inside this regime, thus changes are not faster, $r(n)$ is indeed the critical window.

Digression: near-critical FK Ising

Kesten's stability in the FK(p, q) random cluster model in the $q = 2$ Ising case is completely false:

Duminil-Copin, Garban & P. (2013): expected number of pivotal edges at p_c is $\mathbf{E}|\mathrm{Piv}_n| = n^{13/24+o(1)}$, but the critical window around p_c is n^{-1} only.

Changes are faster because in any monotone coupling, pivotals are $much$ $more\ likely$ to get opened, moreover, there are $atoms$: at certain p values many edges get opened at once.

The Near-Critical Ensemble Scaling Limit

Unif$[0,1]$ labels, percolation at level p on $\Delta_{1/n}$ with $p = p_c + \lambda r(n)$, $\lambda \in (-\infty, \infty)$, coupled together.

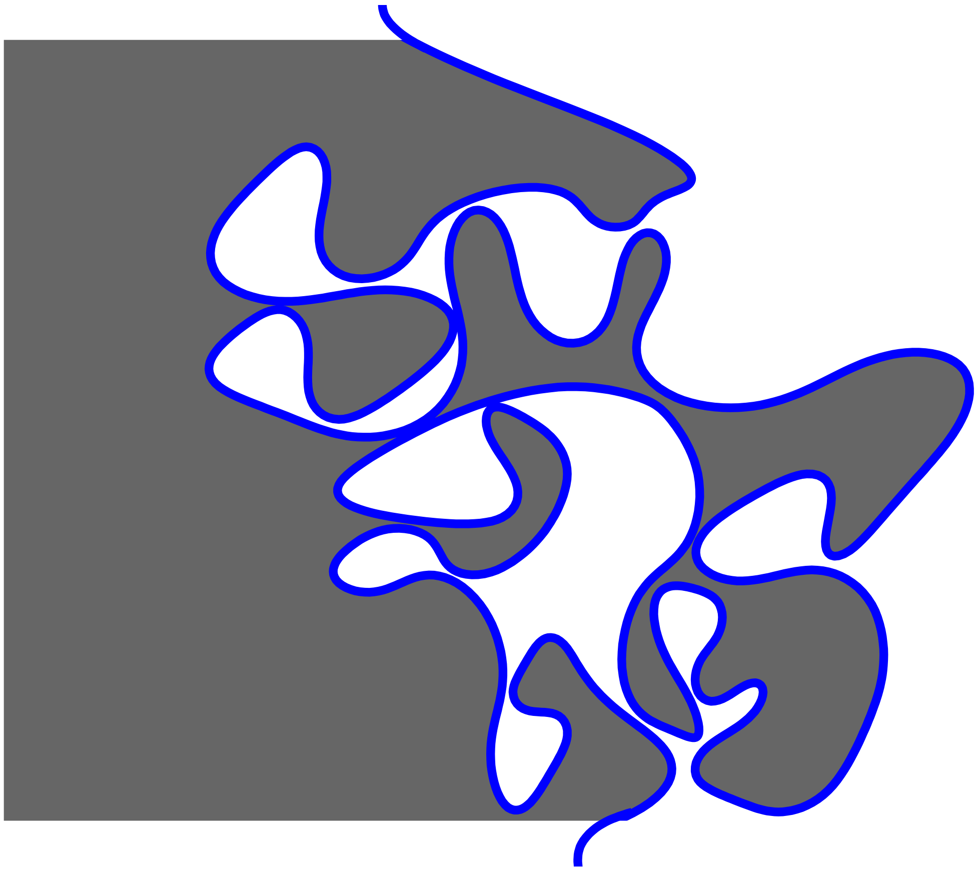
Theorem (GPS 2010, 13). On $\Delta_{1/n}$, as $n \to \infty$, the NCESL exists in the quad-crossing topology, is Markovian in λ, and is conformally covariant: if the domain is changed by $\phi(z)$, then time is scaled locally by $|\phi'(z)|^{3/4}$.

Construction of limit process partially follows suggestion by Camia-Fontes-Newman (2006). Built from the scaling limit of critical percolation, in two main steps:

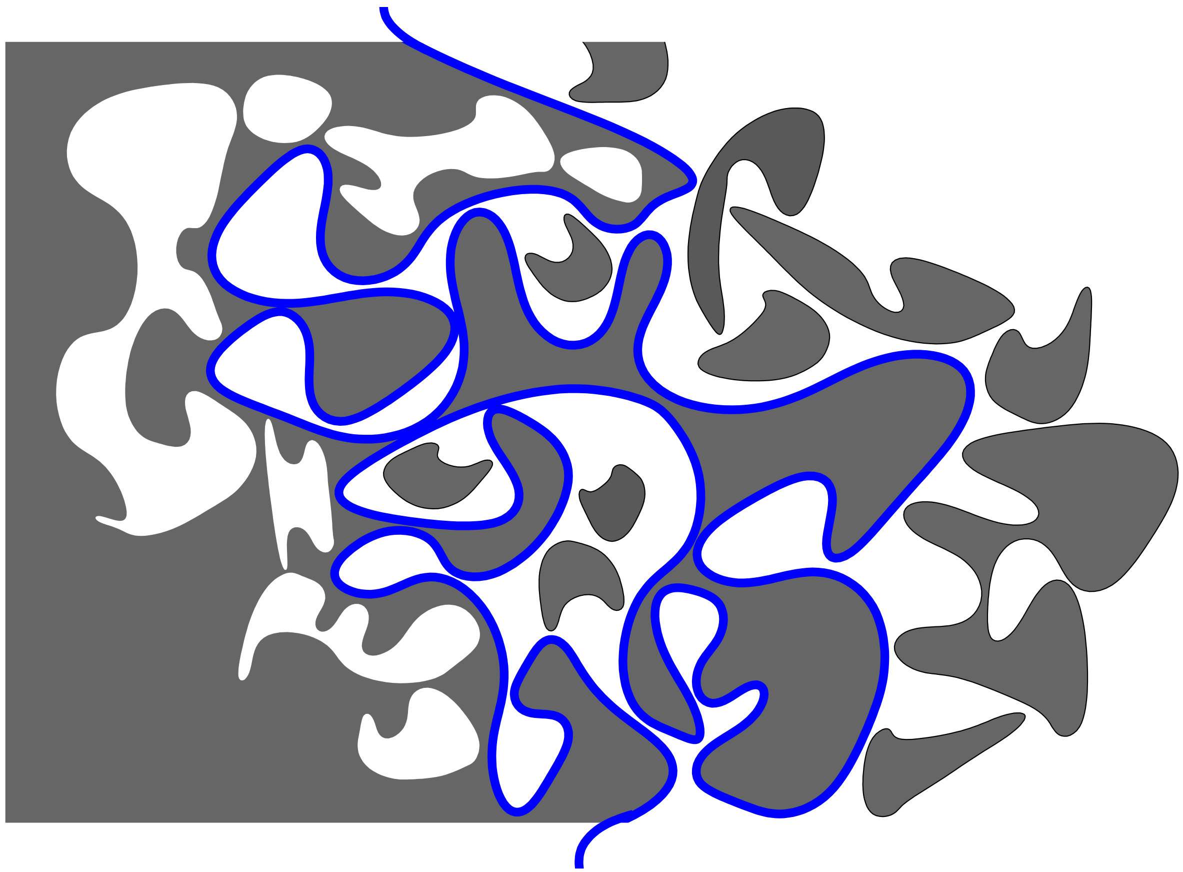
1) In critical percolation, can tell from quad-crossings how many ϵ-macroscopic pivotals there are at different places. Get ϵ-pivotal measure, measurable w.r.t. quad crossing topology.

2) Stability: can describe dynamics in λ by following how *initial* $(\lambda = 0)$ macroscopic pivotals change their color, using independent randomness for these switches, with intensity measure being the ϵ-pivotal measures.

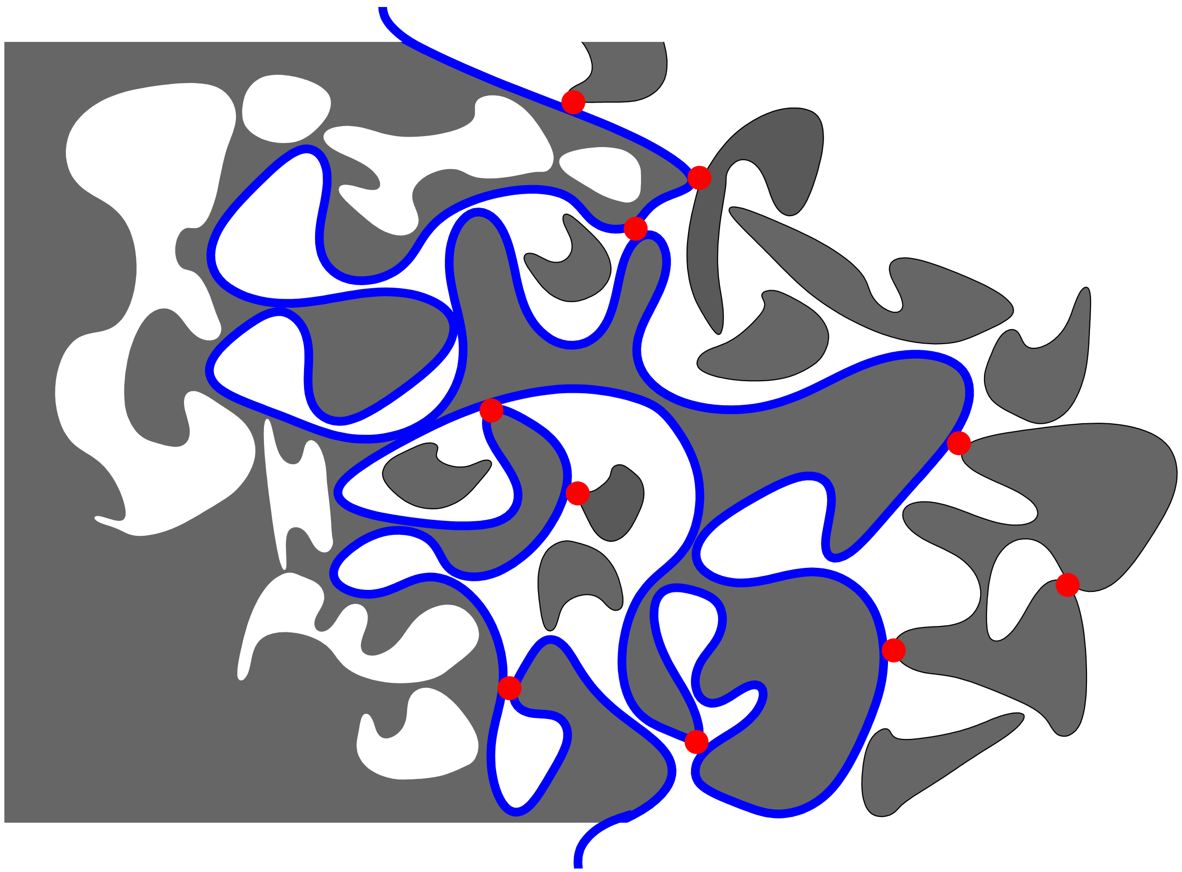
The near-critical exploration interface



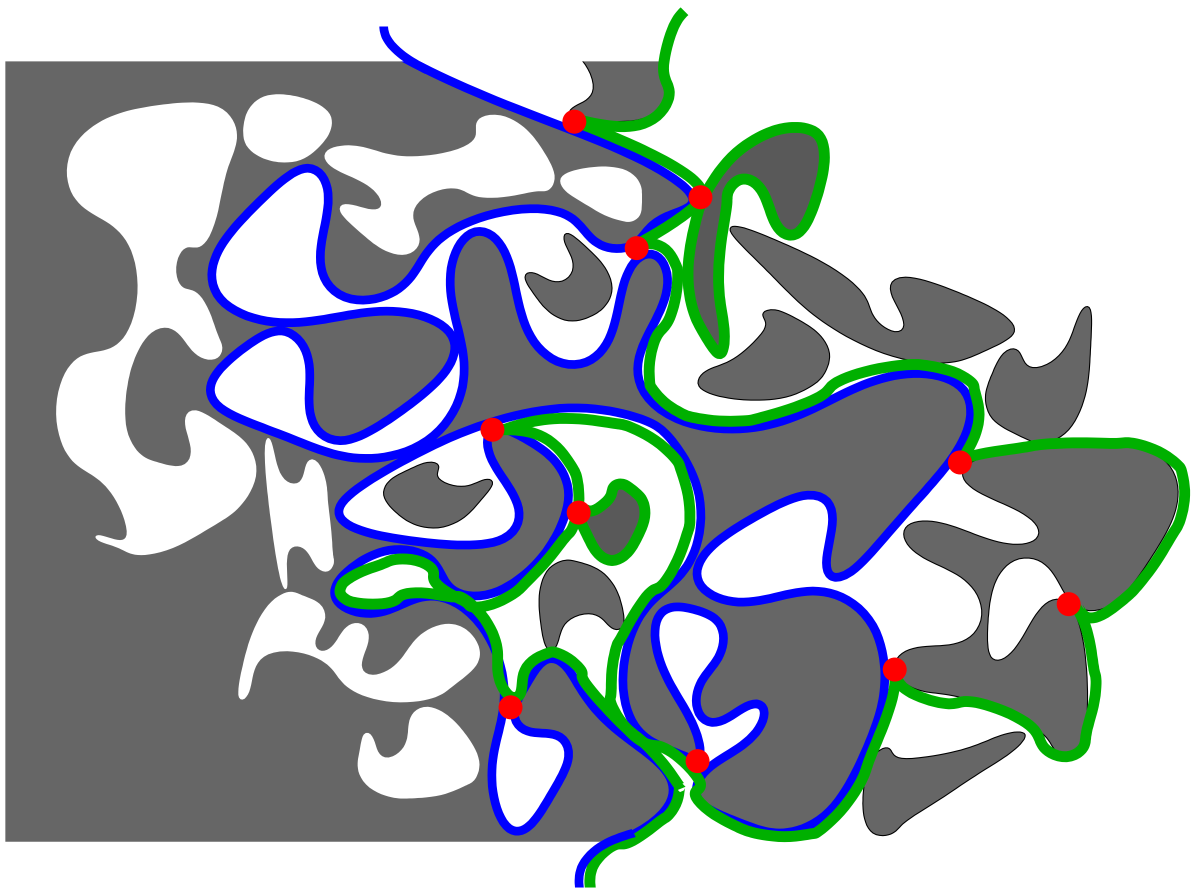
The near-critical exploration interface



The near-critical exploration interface



The near-critical exploration interface



Singularity of the massive scaling limit

Since having an average number of pivotals and switching one of them is enough to establish a connection, we have:

Lemma. For $\lambda > 0$, exists $c_\lambda > 0$ s.t. $\mathbf{P}_\lambda\big[\mathsf{LR}([0,u]^2)\big] \geqslant 1/2 + c_\lambda u^{3/4}$.

Now, divide $[0,1]^2$ into small $\frac{1}{k} \times \frac{1}{k}$ squares. Let

$$A_k := \left\{ \frac{k^2}{2} + \frac{c_\lambda}{2} k^{5/4} \leqslant \text{ small squares are crossed} \right\}.$$

Then $\mathbf{P}_\lambda[A_k] = 1 - o(1)$, while $\mathbf{P}_0[A_k] = o(1)$, since drift $k^{5/4}$ is larger than the normal fluctuation $\sqrt{k^2}$. Hence singularity of quad-crossing limit.

Similar but harder argument by Nolin-Werner '08 proves singularity of the exploration interface:

The interface meets $k^{7/4}$ small squares, each with drift $k^{-3/4}$ "to the right". Resulting drift k is larger than normal fluctuation $k^{7/8}$.

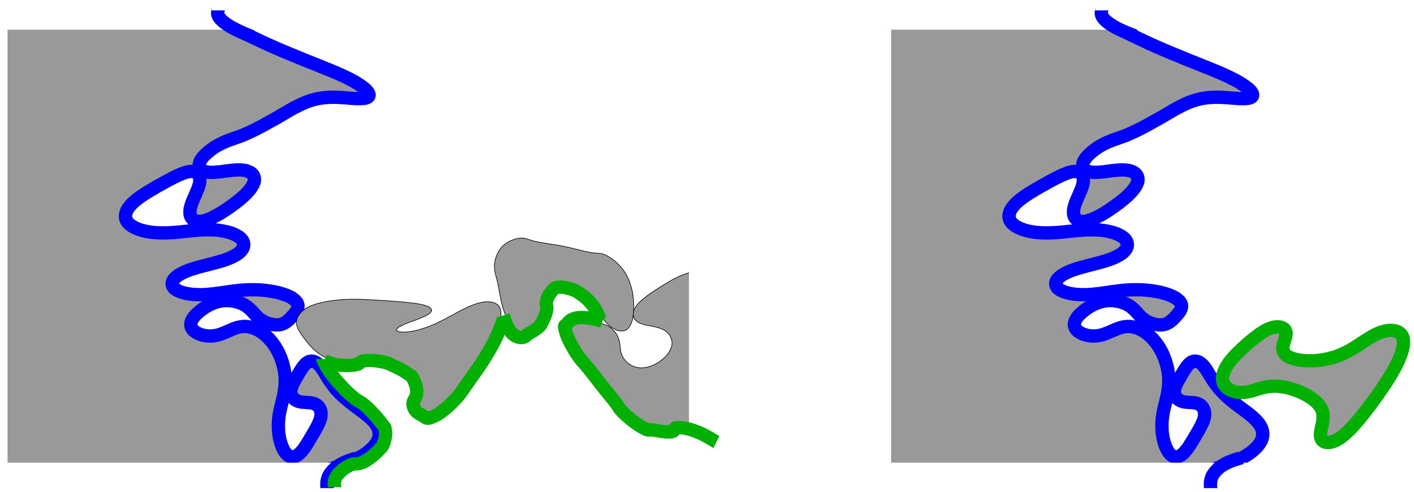
Note. Singularity is expected for $\kappa > 4$, absolute continuity for $\kappa \leqslant 4$.

Guess for the Loewner driving function

Expect "of course"

$$dW_t = \sqrt{6}\, dB_t + dA_t \,,$$

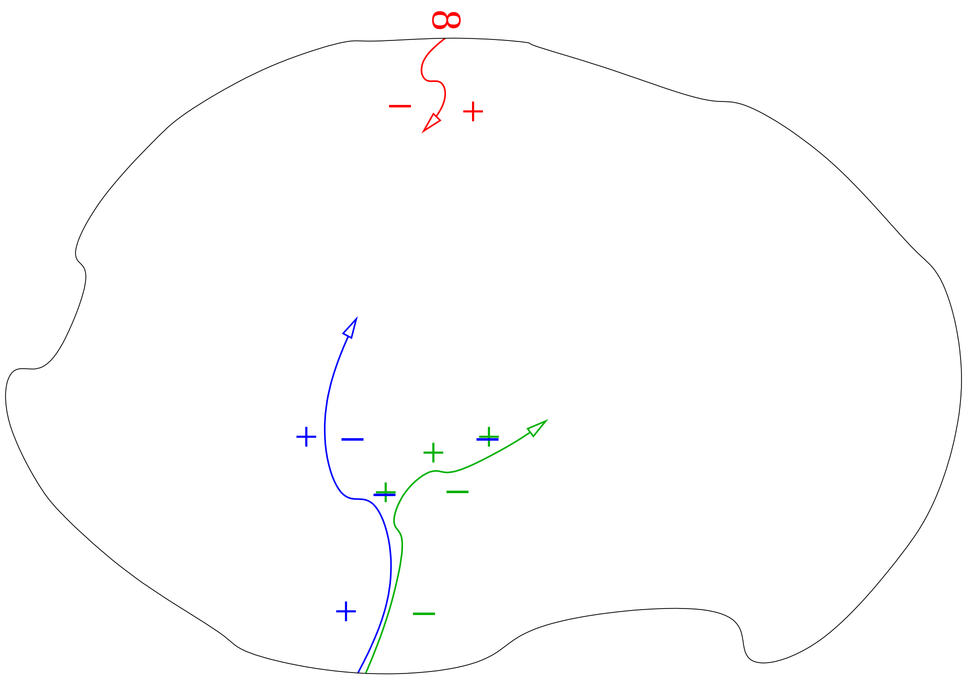
where B_t is Brownian motion and A_t is a monotone drift, increasing for $\lambda > 0$, decreasing for $\lambda < 0$. $\sqrt{6}\, dB_t$ because zooming in spatially is equivalent to moving λ closer to 0, while monotone A_t seems natural.



The left boundary to right boundary ratio, in terms of half-plane harmonic measure from infinity, is typically larger for the near-critical interface than for the critical. But not always!

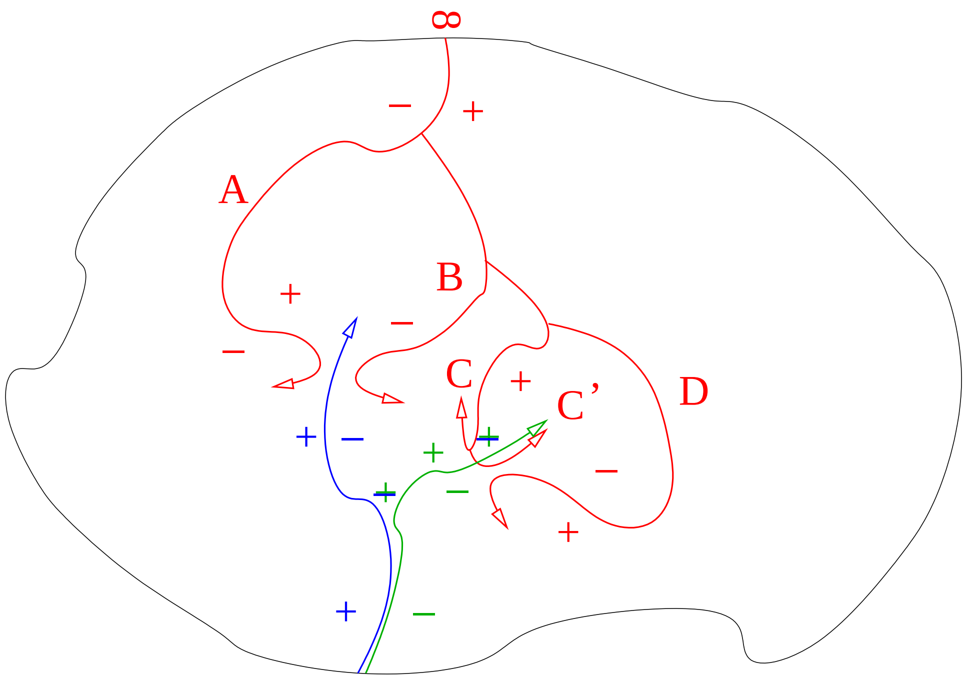
W_t is indeed a sub-martingale

$\mathbf{E}W_t$ is expected difference between harmonic measure of left side and right side. Measure it with percolation instead of random walk, with reversed sides!



W_t is indeed a sub-martingale

$\mathbf{E}W_t$ is expected difference between harmonic measure of left side and right side. Measure it with percolation instead of random walk, with reversed sides!



	γ_1	γ_2
A	+	+
B	−	+/−
C	+/−	+
C'	−	+
D	−	−

That is, $W(\gamma_1) \mid \gamma_1, \gamma_2, \omega \leqslant W(\gamma_2) \mid \gamma_1, \gamma_2, \omega$

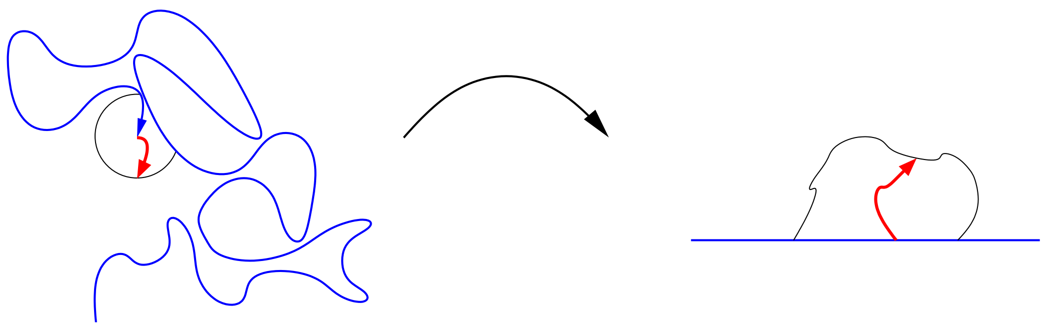
Implies $\mathbf{E}W(\gamma_1) \leqslant \mathbf{E}W(\gamma_2)$, but not $\mathbf{E}[W(\gamma_1) \mid \gamma_1] \leqslant \mathbf{E}[W(\gamma_2) \mid \gamma_2]$.

Guess for the Loewner drift term

So,

$$dW_t = \sqrt{6}\, dB_t + dA_t\,,$$

where A_t is a monotone drift, increasing for $\lambda > 0$, decreasing for $\lambda < 0$.



In ρ-neighborhood of the tip $\gamma(t)$, expected number of pivotals is $\asymp r(n\rho)^{-1}$.
So, expected change in crossing probability from p_c to $p_c + \lambda r(n)$ is $\asymp \lambda \rho^{3/4}$.
So, expected exit position $\gamma(t+dt)$ deviates by $\asymp \lambda \rho^{3/4}$ degrees.
Under Loewner map g_t, radius ρ becomes roughly ρ', on the order of $(dt)^{1/2}$.
After a LLN:

$$dA_t = c\,\lambda\,\rho^{3/4}\rho' = c'\,\lambda\,|d\gamma_t|^{3/4}\,|dt|^{1/2}\,.$$

Could this SDE make sense?

For what d_1 and d_2 could $\sum_i |\gamma(t_{i+1}) - \gamma(t_i)|^{d_1} |t_{i+1} - t_i|^{d_2}$ converge, where step-size $|t_{i+1} - t_i| = \delta \to 0$?

The hull created from t to $t + \delta$ is of size $\asymp \sqrt{\delta}$. Under the inverse Loewner map f_t, size is roughly $\sqrt{\delta} |f_t'(W_t + i\sqrt{\delta})|$. Hence the sum of the δ^{-1} steps is about

$$\delta^{-1} \mathbf{E}\big[\, |f_1'(W_1 + i\sqrt{\delta})|^{d_1} \,\big]\, \delta^{d_1/2}\, \delta^{d_2}\,.$$

Assuming that derivative exponents are the same as for SLE(6), the sum will be of constant order iff

$$14 + 4(d_1 + d_2)^2 = 15 d_1 + 18 d_2\,.$$

Also, the dimension count should be fine: $1 = -3/4 + d_1 + 2d_2$.

These two equations have two solutions: $(d_1, d_2) = (3/4, 1/2)$ and $(d_1, d_2) = (7/4, 0)$. We had the first. What is the second?

Open problems on massive limits

1. Prove that the Loewner driving function formula holds for the scaling limit curve. Prove uniqueness for this self-interacting SDE.

2. Is it useful for anything? E.g., near-critical Cardy's formula? Tail is found in second part of this talk.

3. Do locality + rotation and translation invariance + Markovian property characterize the near critical interface up to a choice of λ?

4. Does $(d_1, d_2) = (7/4, 0)$ describe anything meaningful? Maybe related to natural parameterization of SLE(6)?

5. Relationship of our formula to the Makarov-Smirnov (ICMP 2009) formulas obtained from massive harmonic observables?

6. We are *very far* from building a near-critical scaling limit for FK Random Cluster models using the critical scaling limit.

7. How many massive versions of SLE(κ) could are there be?

The tail of the near-critical crossing probability

By NCESL established by GPS (2013),

$$f(\lambda, \mathcal{Q}) := \lim_{n\to\infty} \mathbf{P}_{p_c+\lambda r(n)}[\mathsf{LR}_{n\mathcal{Q}}]$$

exists, and is conformally covariant. In particular, for any scaling factor $\rho > 0$,

$$f(\rho\lambda, \mathcal{Q}) = f(\lambda, \rho^{4/3}\mathcal{Q}) \, .$$

Already from Kesten (1987):

$$\lim_{\lambda\to-\infty} f(\lambda, \mathcal{Q}) = 0 \, , \quad \text{and} \quad \lim_{\lambda\to\infty} f(\lambda, \mathcal{Q}) = 1 \, .$$

Theorem. As $\lambda \to -\infty$, we have $f(\lambda, [0,1]^2) = \exp\left(-\Theta\left(|\lambda|^{4/3}\right)\right)$.

Asked by Ahlberg & Steif (2014), who studied what kind of scaling limits arise for threshold functions of monotone Boolean functions.

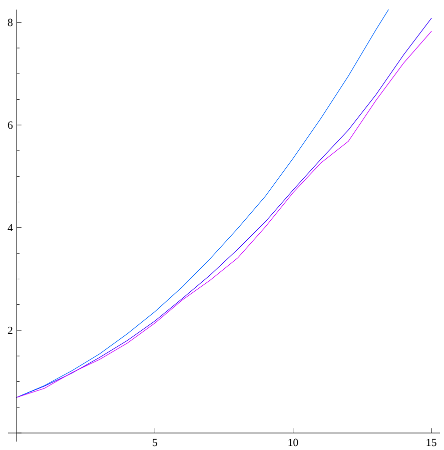
The tail of the dynamical crossing probability?

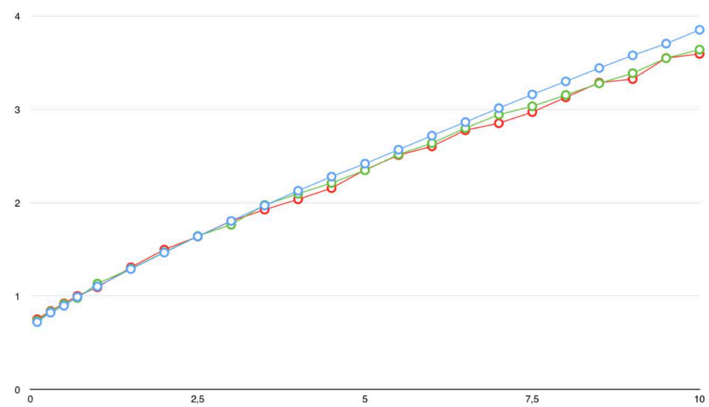
Another motivation is Hammond, Mossel & P. (2012): resample each site at rate $r(n)$, keeping the configuration stationary, and look at

$$g(t,\mathcal{Q}) := \lim_{n\to\infty} \mathbf{P}\Big[\,\mathsf{LR}_{n\mathcal{Q}} \text{ does not hold at any moment in } [0,t]\,\Big]\,.$$

Again, this limit exists and is conformally covariant by GPS (2013).

Using spectral computations and a dynamical FKG-inequality: there exists $c > 0$, and for every $K > 0$ some $c_K > 0$, such that for all $t > 0$, $\exp(-c\,t) \leqslant g(t,[0,1]^2) \leqslant c_K t^{-K}$.





Proof of the near-critical tail

Quite similar to Duminil-Copin's proof (2013) that planar percolation Wulff crystal is asymptotically circular as $p \searrow p_c$.

By the scaling covariance, need to show

$$f\big(-1, [0, \lambda^{4/3}]^2\big) = \exp\Big(-\Theta\big(\lambda^{4/3}\big)\Big) ,$$

as $\lambda \to \infty$. For this, the main step is to prove in the scaling limit measure $\mathbf{P}_{\lambda=-1}$ that there exist some $L > 0$ such that, for any $\underline{x} \in \mathbb{Z}^2$,

$$\mathbf{P}_{\lambda=-1}\Big[B_L(\underline{0}) \longleftrightarrow B_L(L\underline{x})\Big] = \exp\big(-\Theta(\|\underline{x}\|)\big) .$$