The exact noise and dynamical sensitivity of critical percolation, via the Fourier spectrum

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Joint work with Christophe Garban (Université Paris-Sud and ENS)
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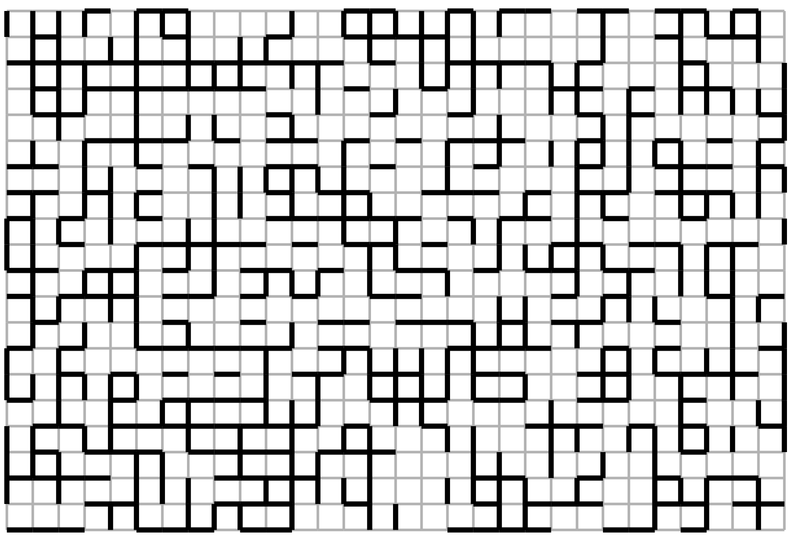
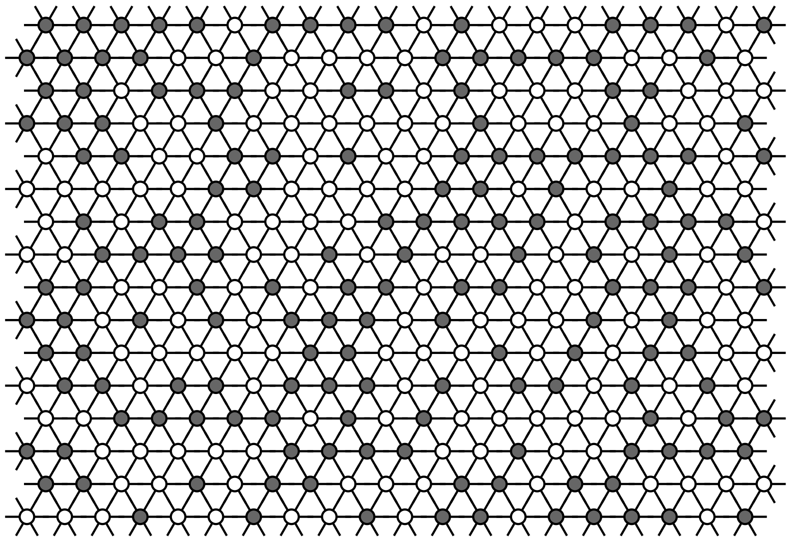
Plan of the talk

- Critical percolation: RSW, conformal invariance, SLE_6 exponents
- Noise sensitivity of critical percolation
- Dynamical percolation
- Why is the Fourier spectrum useful?
- The Fourier spectrum of critical percolation
- Strategy of proof
- Further results and questions

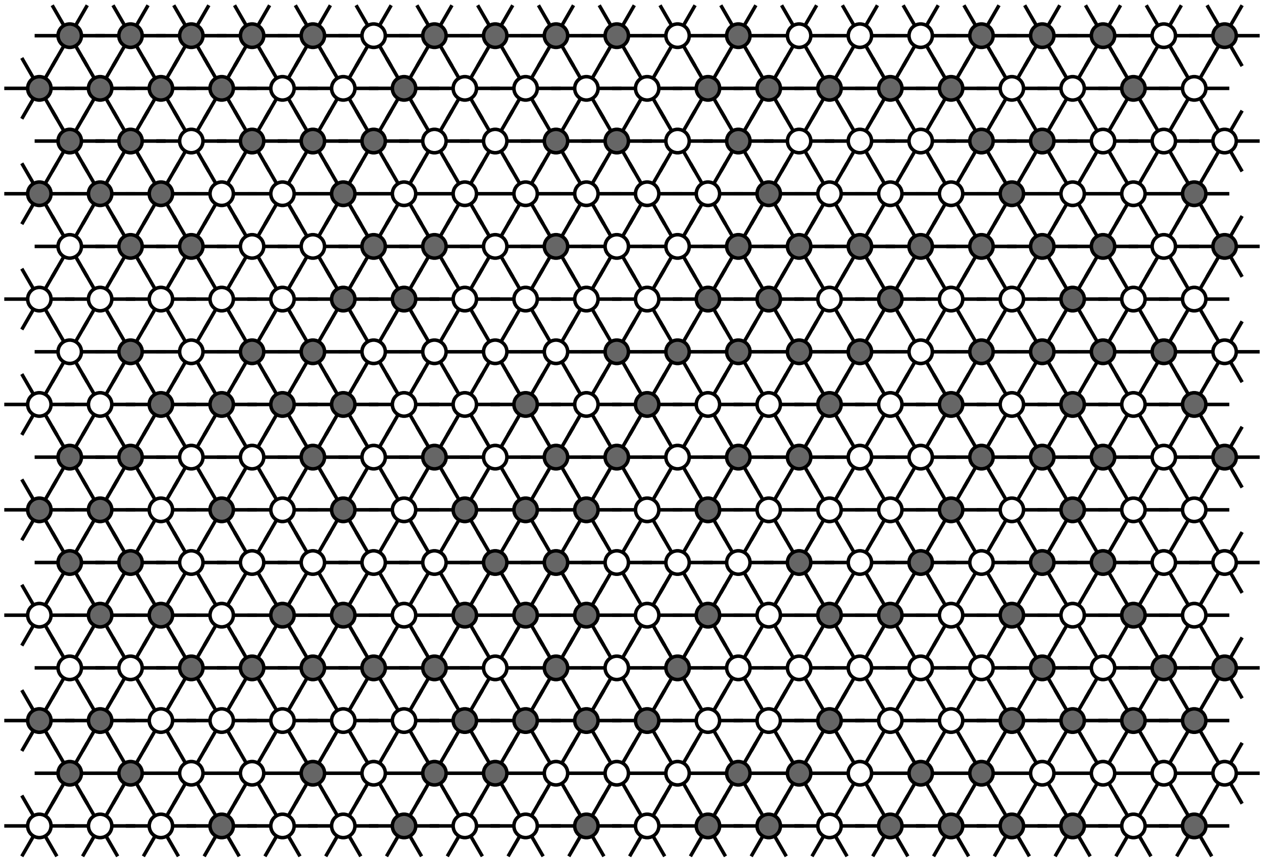
Bernoulli(p) site and bond percolation

Given an (infinite) graph $G = (V, E)$ and $p \in [0, 1]$. Each site (or bond) is chosen open with probability p, closed with $1 - p$, independently of each other. Consider the open connected clusters.





Site percolation on triangular grid Δ
= face percolation on hexagonal grid:



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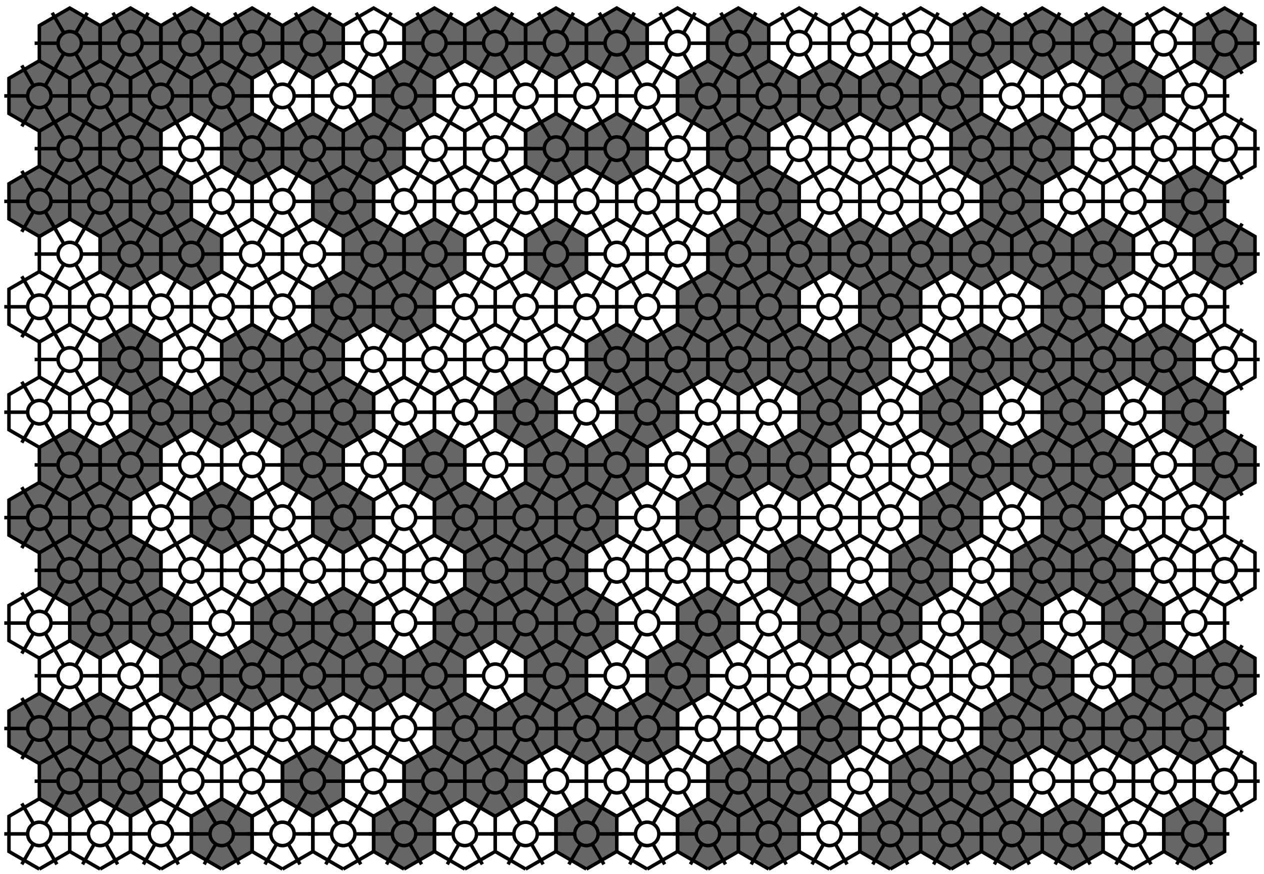

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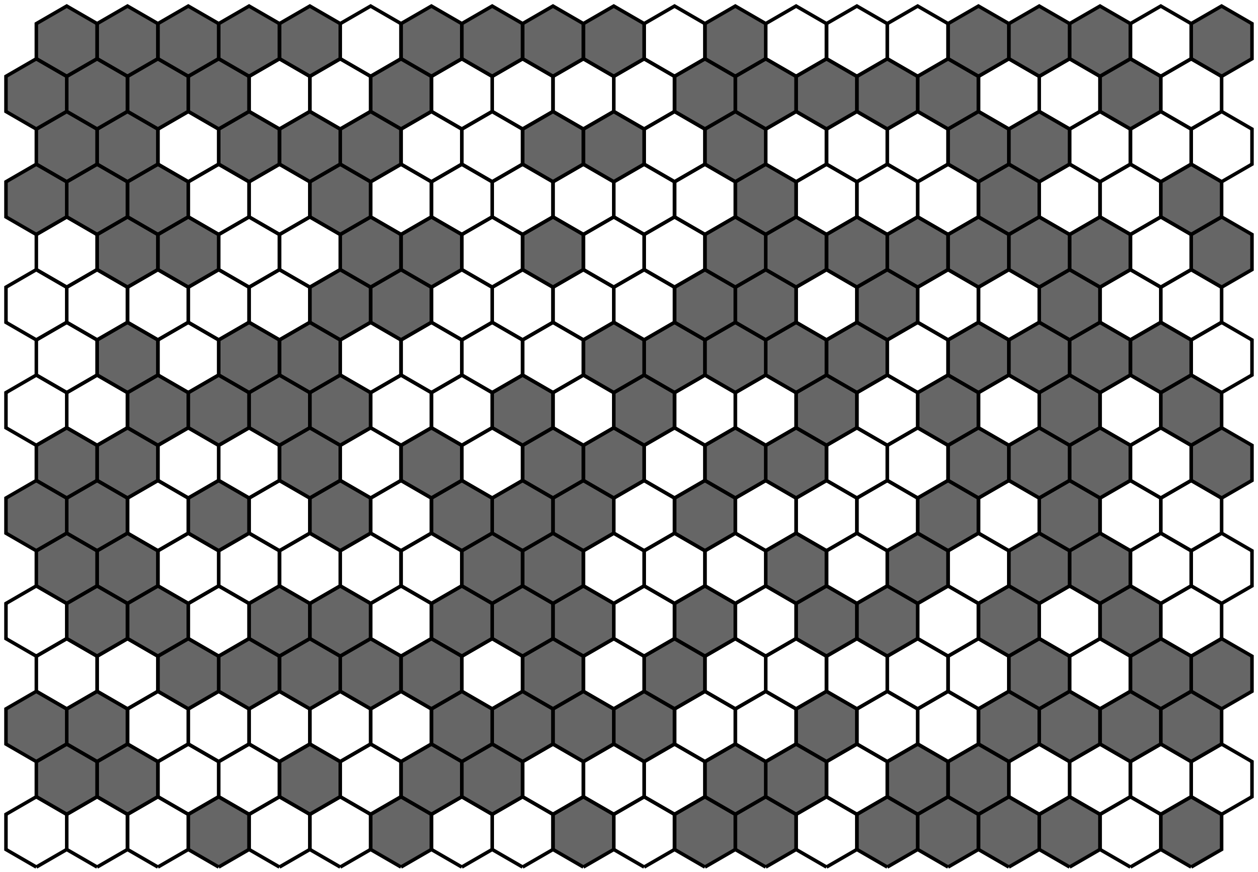


Critical percolation

For any G there is a $p_c \in [0,1]$, s.t. $\mathbf{P}_p[\exists\ \infty\ \text{cluster}] = 0$ for $p < p_c$, but $\mathbf{P}_p[\exists\ \infty\ \text{cluster}] = 1$ for $p > p_c$, because of Kolmogorov's 0-1 law.

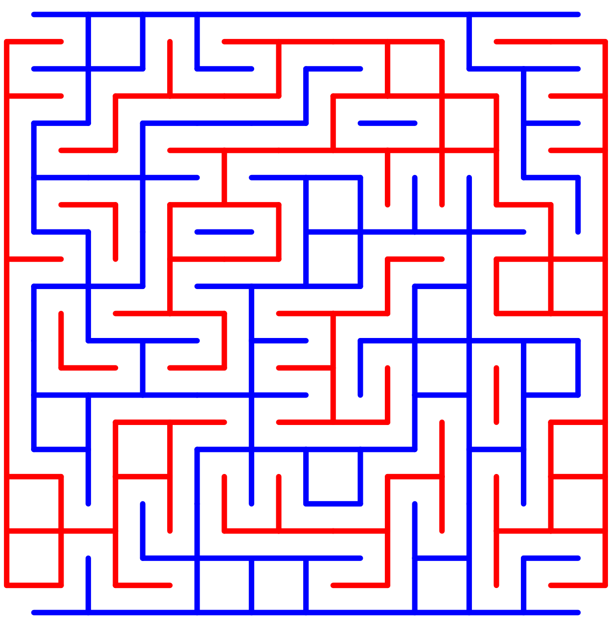
Simplest model of phase transition.

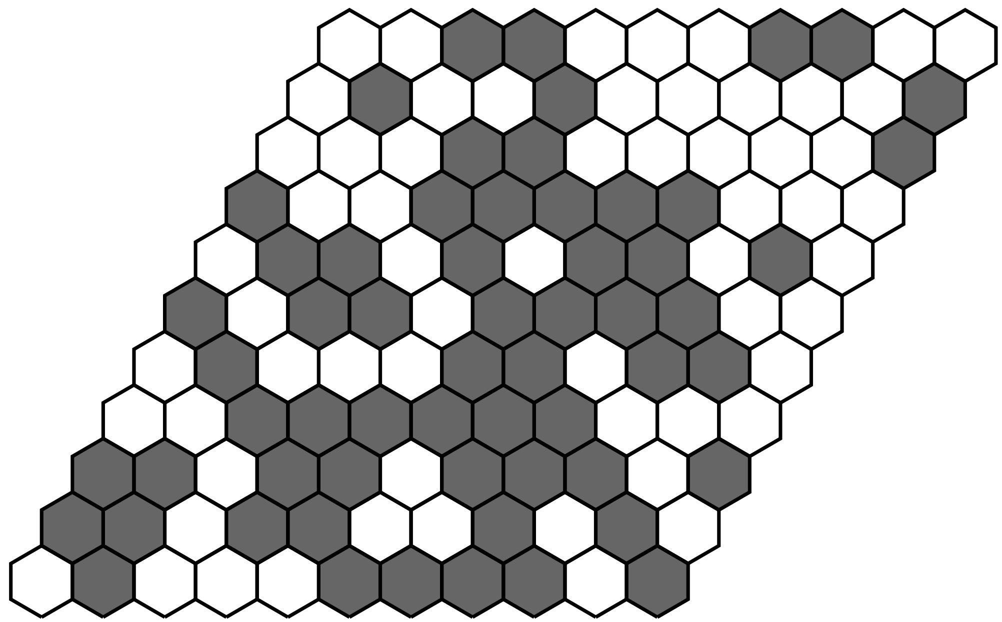
The case of planar lattices and trees is understood best. E.g.:

Theorem (Harris 1960 and Kesten 1980).
$p_c(\mathbb{Z}^2, \text{bond}) = p_c(\Delta, \text{site}) = 1/2$.
At $p = 1/2$, there is a.s. no infinite cluster.
For $p > 1/2$, there is a.s. exactly one infinite cluster.

Why is $p_c = 1/2$? Duality!

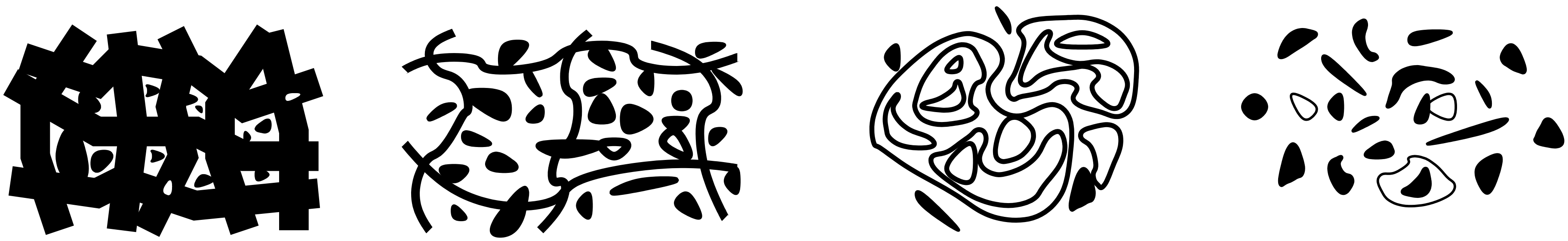
$\mathbb{Z}^2$ bond percolation at $p = 1/2$: in an $(n+1) \times n$ rectangle, left-right crossing has probability exactly $1/2$.
For site percolation on Δ, same on an $n \times n$ rhombus.





Crossing probabilities and criticality

$p \approx 0.8$ $p \approx 0.55$ $p = 0.5$ $p \approx 0.45$



Theorem (Russo 1978 and Seymour-Welsh 1978). In critical percolation on almost any planar lattice, for $n, L > 0$,

$$0 < a_L < \mathbf{P}[\,\text{left-right crossing in } n \times Ln\,] < b_L < 1.$$

Same holds for annulus-crossings.

By repeating this on all scales, and gluing the pieces by FKG:

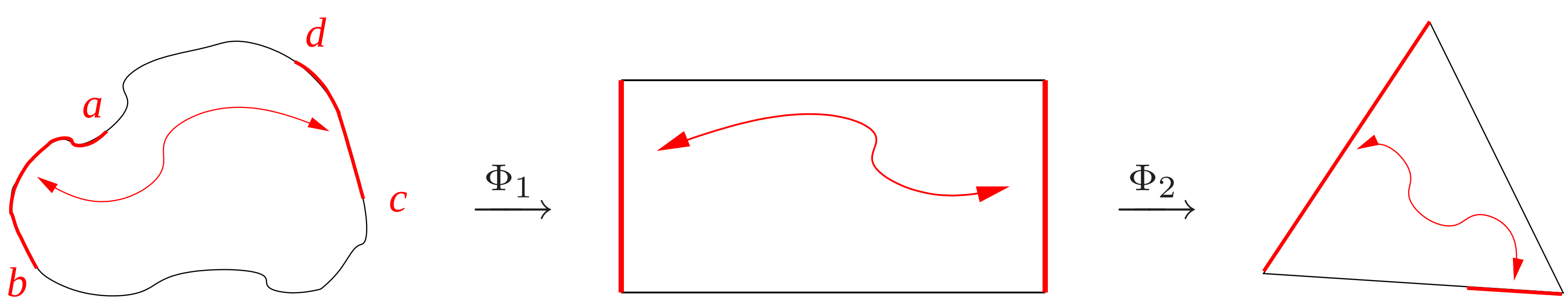
$$(r/R)^{\alpha} < \mathbf{P}[\,\partial B_r \longleftrightarrow \partial B_R\,] < (r/R)^{\beta}.$$

Conformal invariance on Δ

Theorem (Smirnov 2001). For p=1/2 bond percolation on Δ_ϵ, and $D \subset \mathbb{R}$ simply connected domain with four boundary points $\{a, b, c, d\}$,

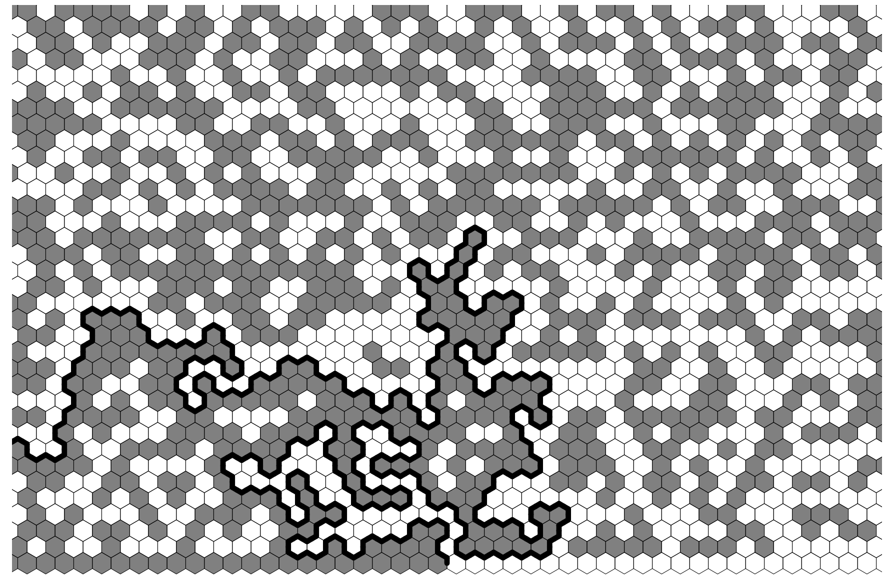
$$\lim_{\epsilon \longrightarrow 0} \mathbf{P}\Big[\, ab \longleftrightarrow cd \text{ inside the discrete approximation } D_\epsilon \,\Big]$$

exists, is strictly between 0 and 1, and is conformally invariant.



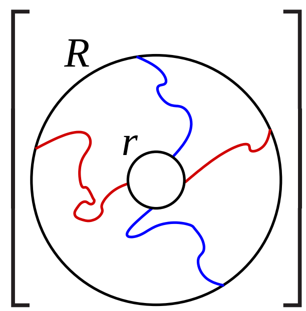
SLE_6 exponents

Given the conformal invariance, the exploration path converges to the Stochastic Loewner Evolution with $\kappa = 6$ (Schramm 2000).



Using the SLE_6 curve, several critical exponents can be computed (Lawler-Schramm-Werner, Smirnov-Werner 2001), e.g.:

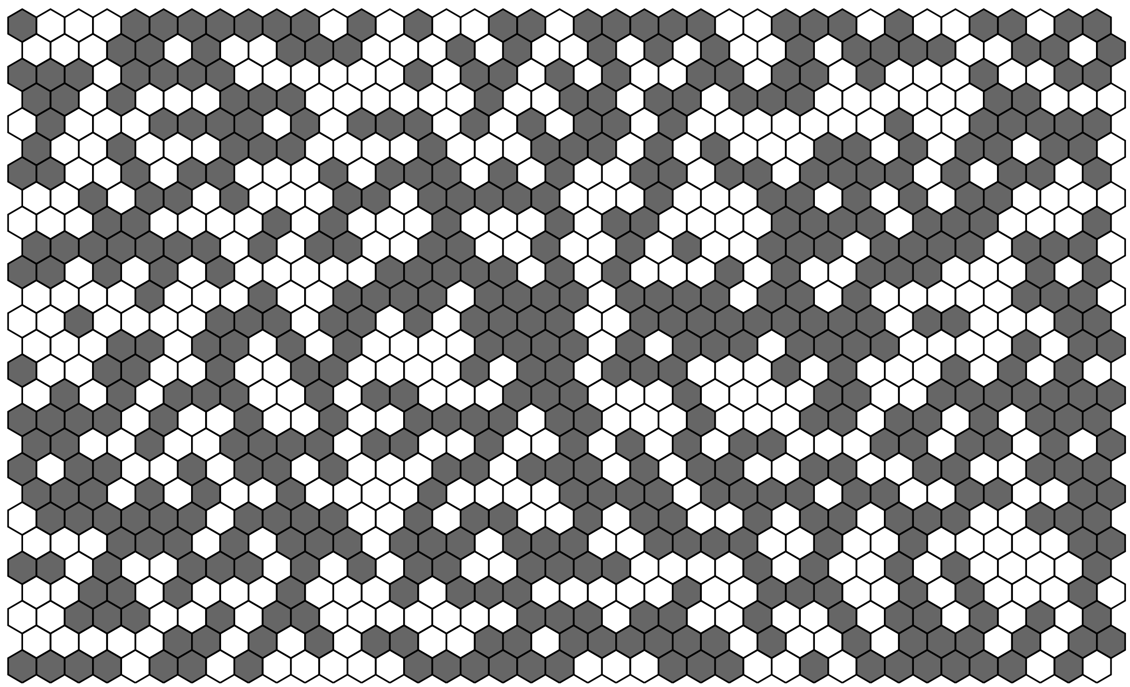
$$\alpha_4(r, R) := \mathbf{P}\left[\ \ \right] = (r/R)^{5/4+o(1)},$$



while $\alpha_1(r, R) = (r/R)^{5/48+o(1)}$ and $\mathbf{P}_{p_c+\epsilon}[0 \longleftrightarrow \infty] = \epsilon^{5/36+o(1)}$.

Percolation and noise

Take an ω critical percolation configuration. Let ω^ϵ be a new configuration, where each site (or bond) is resampled with probability ϵ, independently. (The ϵ-noised version of ω.)



For how large an ϵ can we still predict from ω whether there is a left-right crossing in ω^ϵ?

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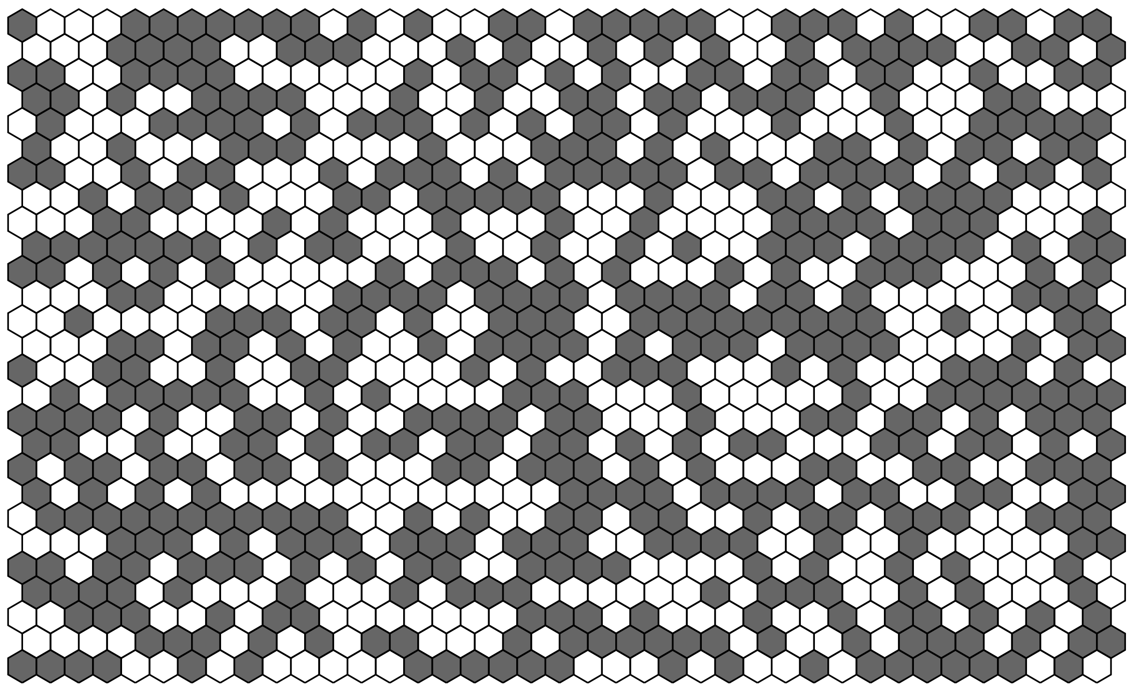


For how large an ϵ can we still predict from ω whether there is a left-right crossing in ω^ϵ?

Noise sensitivity of percolation

Theorem (Benjamini, Kalai & Schramm 1998). If $\epsilon > 0$ is fixed, and f_n is the indicator function for a left-right percolation crossing in an $n \times n$ square, then as $n \to \infty$

$$\mathbf{E}\big[\, f_n(\omega)\, f_n(\omega^\epsilon)\,\big] - \mathbf{E}\big[\, f_n(\omega)\,\big]^2 \to 0.$$

This holds for all $\epsilon = \epsilon_n > c/\log n$.

Theorem (Steif & Schramm 2005). Same if $\epsilon_n > n^{-a}$ for some positive $a > 0$. If triangular lattice, may take any $a < 1/8$.

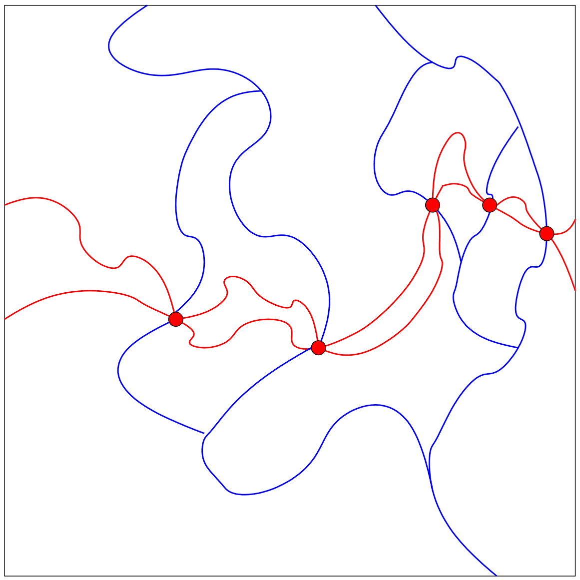
Theorem (Garban, P & Schramm 2008). Same holds if and only if $\epsilon_n\, \mathbf{E}\big[\, |\text{pivotals}|\,\big] \to \infty$. For triangular lattice, this threshold is $\epsilon_n = n^{-3/4+o(1)}$.

Naive idea: how many pivotals are there?

A site (or bond) is pivotal in ω, if flipping it changes the existence of a left-right crossing. $\mathbf{E}|\mathrm{Piv}_n| \asymp n^2\,\alpha_4(n) \quad (= n^{3/4+o(1)})$.

Furthermore, $\mathbf{E}[\,|\mathrm{Piv}_n|^2\,] \leqslant C\,(\mathbf{E}|\mathrm{Piv}_n|)^2$. So, $\mathbf{P}\big[\,|\mathrm{Piv}_n| > \lambda\mathbf{E}|\mathrm{Piv}_n|\,\big] < C/\lambda^2$, any λ.

Concentration around mean also from below: $\mathbf{P}\big[\,0 < |\mathrm{Piv}_n| < \lambda\mathbf{E}|\mathrm{Piv}_n|\,\big] \asymp \lambda^{11/9+o(1)}$, as $\lambda \to 0$ (exponent only for Δ).



Cannot have many pivotals. $\Longrightarrow$ If $\epsilon_n\mathbf{E}[\,|\mathrm{Piv}_n|\,] \to 0$, then we don't hit any pivotals. $\Longrightarrow$ Asymptotically full correlation.

Cannot have few pivotals (if there is any). $\Longrightarrow$ If $\epsilon_n\mathbf{E}[\,|\mathrm{Piv}_n|\,] \to \infty$, then we do hit many pivotals. But this $\not\Longrightarrow$ asymptotic independence!

Dynamical percolation

Each variable is resampled according to an independent Poisson(1) clock. This is a Markov process $\{\omega(t) : t \in [0, \infty)\}$, in which $\omega(t+s)$ is an ϵ-noised version of $\omega(t)$, with $\epsilon = 1 - \exp(-s)$.

An exceptional time is such a (random) t, at which an almost sure property of the static process fails for $\omega(t)$.

Main example: (Non-)existence of an infinite cluster in percolation.

Toy example: Brownian motion on the circle does sometimes hit a fix point, as opposed to its static version: a uniform random point.

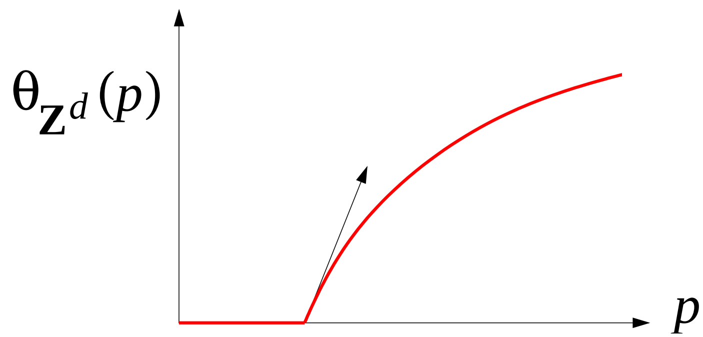
In this toy example, the set of exceptional times is a random Cantor set of Lebesgue measure zero (because of Fubini) and Hausdorff-dimension $1/2$.

Dynamical percolation results

Theorem (Häggström, Peres & Steif 1997).

- No exceptional times when $p \neq p_c$.
- No exceptional times when $p = p_c$ for bond percolation on $\mathbb{Z}^d$, $d \geqslant 19$.

The second fact is essentially due to:



Theorem (Steif & Schramm 2005).

- There are exceptional times (a.s.) for critical site percolation on the triangular lattice.
- They have Hausdorff dimension in $[1/6, 31/36]$.

Theorem (Garban, P & Schramm 2008).

- There are exceptional times also on $\mathbb{Z}^2$.
- On the triangular grid they have Hausdorff dimension $31/36$.
- On the triangular grid, there are exceptional times with an infinite white **and** an infinite black cluster simultaneously. $(1/9 \leqslant \dim \leqslant 2/3)$

What is the Fourier spectrum and why is it useful?

$f_n : \{\pm1\}^{V_n} \longrightarrow \{\pm1\}$ indicator function of left-right crossing. Element of the space $L^2(\Omega,\mu)$, where $\Omega = \{\pm1\}^{V_n}$, μ uniform probability measure, inner product $\mathbf{E}[\, fg\,]$, having a nice orthonormal basis:

For $S \subset V_n$, let $\chi_S(\omega) := \prod_{v\in S}\omega(v)$, the parity inside S.

Any function $f \in L^2(\Omega,\mu)$ in this basis (Fourier-Walsh series):

$$\hat f(S) := \mathbf{E}[\, f\chi_S\,]\,; \qquad f = \sum_{S\subset V} \hat f(S)\,\chi_S\,.$$

By Parseval, $\sum_S \hat f(S)^2 = \mathbf{E}[\, f^2\,]$. So $\nu_f(S) := \hat f(S)^2/\mathbf{E}[\, f^2\,]$ is a probability measure, and may take a random sample from it:

the spectral sample $\mathscr{S}_f \subset V_n$, a random set with law ν_f.

For the crossing function, $\mathbf{E}[\, f_n^2\,] = 1$. Get $\mathscr{S}_n$, a strange random set of bits in the plane. $\mathbf{P}[\, x,y \in \mathrm{Piv}_n\,] = \nu[x,y\in\mathscr{S}_n]$, but not for more points.

$\mathbf{E}[\,\omega^\epsilon(v)\omega(v)\,] = 1-\epsilon$, so $\mathbf{E}[\,\chi_S(\omega^\epsilon)\chi_S(\omega)\,] = (1-\epsilon)^{|S|}$. Therefore,

$$\mathbf{E}[\,f(\omega^\epsilon)f(\omega)\,] = \sum_{S\subseteq V} \hat{f}(S)^2(1-\epsilon)^{|S|} = \mathbf{E}_\nu\big[(1-\epsilon)^{|\mathscr{S}_f|}\big].$$

(In other words: the χ_S are eigenfunctions of the noise operator $(N_\epsilon f)(\omega) := \mathbf{E}[\,f(\omega^\epsilon)\mid\omega\,]$ with eigenvalues $(1-\epsilon)^{|S|}$, while $\mathbf{E}[\,f(\omega^\epsilon)f(\omega)\,] = \mathbf{E}[\,f N_\epsilon f\,]$.)

And the correlation is:

$$\mathbf{E}[\,f(\omega^\epsilon)f(\omega)\,]-\mathbf{E}[\,f\,]^2 = \sum_{\emptyset\neq S\subset V} \hat{f}(S)^2\,(1-\epsilon)^{|S|} = \sum_{k=1}^{|V_n|}(1-\epsilon)^k \sum_{|S|=k} \hat{f}(S)^2.$$

If, for some sequence k_n, we have $\nu\big[0 < |\mathscr{S}_n| < tk_n\big] \to 0$ as $t \to 0$, uniformly in n, then $(1-\epsilon)^k \sim \exp(-\epsilon k)$ implies that for $\epsilon_n \gg 1/k_n$ we have asymptotic independence.

But this concentration is much harder to prove than for $\mathrm{Piv}_n \ldots$

Proving existence of exceptional times

Second Moment Method:

Let $Q_R := \{t \in [0,1] : 0 \longleftrightarrow R\}$ and $Z_R := Leb(Q_R)$.

$$\mathbf{P}\big[\,Q_R \neq \emptyset\,\big] = \mathbf{P}\big[\,Z_R > 0\,\big] \geqslant \frac{\mathbf{E}[\,Z_R\,]^2}{\mathbf{E}\big[\,Z_R^2\,\big]}\,.$$

$\mathbf{E}[\,Z_R\,] = \int_0^1 \mathbf{P}[\,0 \longleftrightarrow_t R\,]\,dt = \mathbf{P}[\,0 \longleftrightarrow R\,]$.

$\mathbf{E}\big[\,Z_R^2\,\big] = \int_0^1 \int_0^1 \mathbf{P}[\,0 \longleftrightarrow_t R,\, 0 \longleftrightarrow_s R\,]\,ds\,dt \asymp \int_0^1 \mathbf{E}\big[\,f_R(\omega_0)\,f_R(\omega_s)\,\big]\,ds$.

Thus we again want to estimate the correlation $\mathbf{E}\big[\,f_R(\omega_0)\,f_R(\omega_s)\,\big] = \mathbf{E}\big[\,f_R\,T_s f_R\,\big]$ from above, where

$$T_s f(\omega) := \mathbf{E}\big[\,f(\omega_s)\;\big|\;\omega_0 = \omega\,\big] = N_{1-\exp(-s)} f(\omega).$$

Three very simple examples

$\text{Dictator}_n(x_1, \dots, x_n) := x_1$.
Here $\mathbf{E}[\,\text{Dic}_n\, N_\epsilon \text{Dic}_n\,] = 1 - \epsilon$, so noise-stable.
And $\nu[\mathscr{S}_n = \{x_1\}] = 1$.

$\text{Majority}_n(x_1, \dots, x_n) := \operatorname{sgn}(x_1 + \cdots + x_n) \approx \frac{1}{\sqrt{n}}(x_1 + \cdots + x_n)$.
Here $\mathbf{E}[\,\text{Maj}_n\, N_\epsilon \text{Maj}_n\,] = 1 - O(\epsilon)$, so noise-stable.
And $\nu[\mathscr{S}_n = \{x_i\}] \asymp 1/n$, most of the weight is on singletons.

$\text{Parity}_n(x_1, \dots, x_n) := x_1 \cdots x_n$
Here $\mathbf{E}[\,\text{Par}_n\, N_\epsilon \text{Par}_n\,] = (1 - \epsilon)^n$, the most sensitive to noise.
And $\nu[\mathscr{S}_n = \{x_1, \dots, x_n\}] = 1$.

Basic properties of the spectral sample

Inclusion formula: $\nu_f\big[\mathscr{S} \subset U\big] = \mathbf{E}\Big[\mathbf{E}\big[\,f \mid U\,\big]^2\Big]$.

Proof:

$$\mathbf{E}\big[\,\chi_S \mid U\,\big] = \begin{cases} \chi_S & S \subset U\,, \\ 0 & S \not\subset U\,. \end{cases}$$

Thus $\quad \mathbf{E}\Big[\mathbf{E}\big[\,f \mid U\,\big]^2\Big] = \mathbf{E}\Big[\Big(\sum_{S\subset U} \hat{f}(S)\,\chi_S\Big)^2\Big] = \sum_{S\subset U} \hat{f}(S)^2.$ ∎

From this, for disjoint subsets A and B,

$$\begin{aligned} \nu\big[\mathscr{S} \cap B \neq \emptyset,\ \mathscr{S} \cap A = \emptyset\big] &= \nu\big[\mathscr{S} \subseteq A^c\big] - \nu\big[\mathscr{S} \subseteq (A \cup B)^c\big] \\ &= \mathbf{E}\Big[\mathbf{E}[\,f \mid A^c\,]^2 - \mathbf{E}[\,f \mid (A \cup B)^c\,]^2\Big] \\ &= \mathbf{E}\Big[\big(\mathbf{E}[\,f \mid A^c\,] - \mathbf{E}[\,f \mid (A \cup B)^c\,]\big)^2\Big]. \end{aligned}$$

For the spectral sample $\mathscr{S}_n$ of the $n \times n$ crossing:

With $A := \emptyset$ we get: $\nu\big[\mathscr{S}_n \cap B \neq \emptyset\big] \leqslant C\,\alpha_4(B, V_n)$;

with $A := B^c$ we get: $\nu\big[\emptyset \neq \mathscr{S}_n \subseteq B\big] \leqslant C\,\alpha_4(B, V_n)^2$.

If $B = \{x\}$: equality in both cases, $\nu[x \in \mathscr{S}_n] = \mathbf{P}\big[\, x \in \mathrm{Piv}_n \,\big]$, and

$$\mathbf{E}_\nu[|\mathscr{S}_n|] = \mathbf{E}[\,|\mathrm{Piv_n}|\,] =: m_n \quad \left(= n^{3/4+o(1)}\right).$$

If B is a sub-square of side r, and $B' = B/3$, then

$$\mathbf{E}_\nu\Big[|\mathscr{S} \cap B'| \,\Big|\, \mathscr{S} \cap B \neq \emptyset\Big] = \sum_{x \in B'} \frac{\nu[x \in \mathscr{S}]}{\nu[\mathscr{S} \cap B \neq \emptyset]} \geqslant \sum_{x \in B'} \frac{\alpha_4(x, V_n)}{C\,\alpha_4(B, V_n)}$$
$$\asymp |B'|\alpha_4(r) \asymp m_r,$$

as we would expect from a random fractal-like set. But we need something stronger: with good probability, and conditioned on other sub-squares.

Main results for the spectral sample (GPS)

If $r \in [1, n]$, then $\{|\mathscr{S}_n| < m_r\}$ is basically equivalent to being contained inside some $r \times r$ sub-square:

$$\mathbf{P}\big[\,|\mathscr{S}_n| < m_r\,\big] \asymp \alpha_4(r,n)^2 \left(\frac{n}{r}\right)^2.$$

In particular, on the triangular lattice Δ,

$$\mathbf{P}\big[\,|\mathscr{S}_n| < \lambda\, m_n\,\big] \asymp \lambda^{2/3+o(1)}.$$

The *scaling limit* of $\mathscr{S}_n$ is a conformally invariant Cantor-set with Hausdorff-dimension $3/4$.

The existence of the scaling limit follows from Schramm & Smirnov: *Percolation is black noise*, answering a question of Tsirelson.

The strategy of proof

Tile the $n \times n$ square with $(n/r)^2$ boxes of size r. Let $X = X_{r,n}$ be the number of boxes intersecting $\mathscr{S}_n$. We already know that

$$\mathbf{E}[\, X \,] \geqslant \alpha_4(r,n)(n/r)^2 \asymp (n/r)^{3/4+o(1)}.$$

1st step: X is smaller than $C \log(n/r)$ with only very small probability.

2nd step: In a non-empty r-box, with positive probability $|\mathscr{S}_n| \geqslant c\, m_r$.

If we could repeat this step for each of the X nonempty boxes, $\mathscr{S}_n$ would be large almost surely.

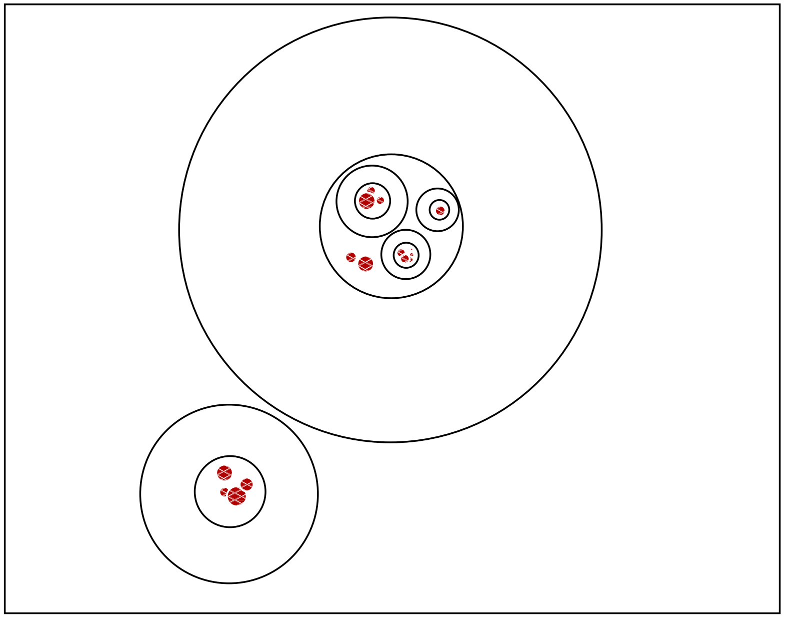
But we can prove Step 2 only in the presence of negative information about $\mathscr{S}_n$ everywhere else! (Partial independence.)

3rd step: Using a sampling trick and a strange large deviation result, 1+2 turns out to be enough.

Annulus structures

Proposition 1. $\nu[X \leqslant k] \leqslant k^{C \log k}\,(n/r)^2\,\alpha_4(r,n)^2.$

An annulus structure $\mathcal{A}$ compatible with a set S:

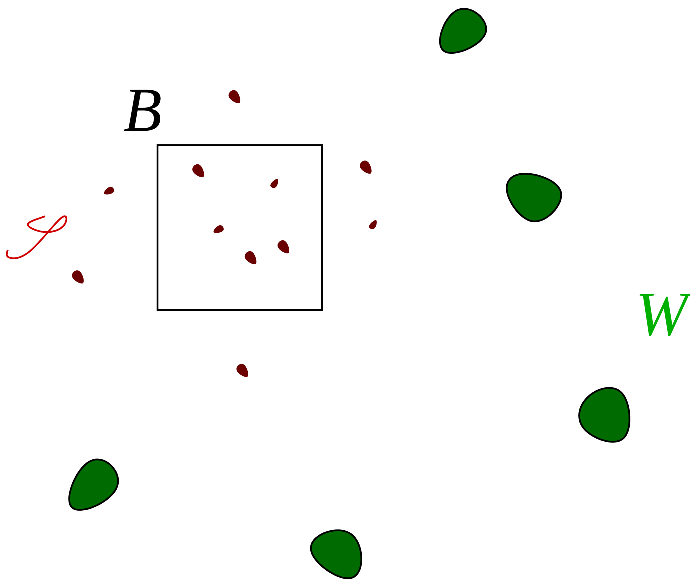


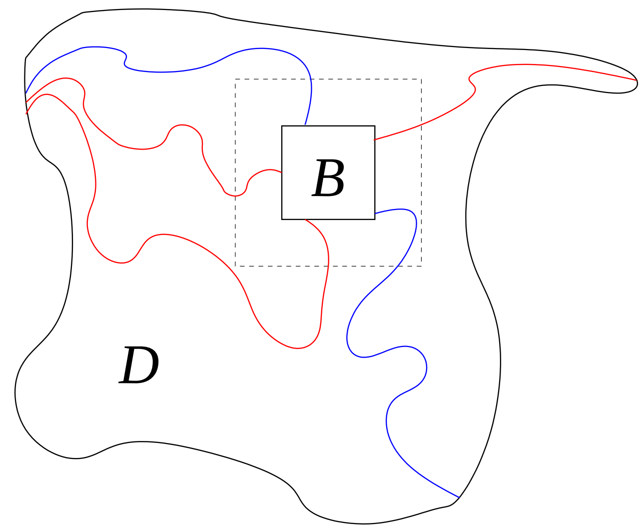
Lemma. $\nu\big[\mathscr{S} \text{ compatible with } \mathcal{A}\big] \leqslant \prod_{A\in\mathcal{A}} \alpha_4(A)^2.$

Thus, we need to construct a family of annulus structures that has some member compatible with any k-element set, but $\sum_{\mathcal{A}} \prod_{A\in\mathcal{A}} \alpha_4(A)^2$ is still small. This is done recursively.

Partial independence

Proposition 2. If B is an r-box in $[0,n]^2$, and $W \cap (3B) = \emptyset$, then $\mathbf{P}\Big[\, |\mathscr{S} \cap B| > c\, r^2\, \alpha_4(r) \,\Big|\, \mathscr{S} \cap W = \emptyset \neq \mathscr{S} \cap (2B) \Big] \geqslant c\,.$





Separation Lemma. If $\mathrm{dist}(B, \partial D) > \mathrm{diam}(B)$, then conditioned on the k-arm event in $D \setminus B$ with fixed endpoints on ∂D, then with a uniformly positive conditional probability the k arms are "well-separated" around B.

Large deviation lemma

Proposition 3. Suppose $X_i, Y_i \in \{0,1\}$, $i = 1, \ldots, n$, and that $\forall J \subset [n]$ and $\forall i \in [n] \setminus J$

$$\mathbf{P}\big[\, Y_i = 1 \,\big|\, \forall_{j \in J} Y_j = 0 \,\big] \geqslant c\, \mathbf{P}\big[\, X_i = 1 \,\big|\, \forall_{j \in J} Y_j = 0 \,\big].$$

Then

$$\mathbf{P}\Big[\forall_i Y_i = 0\Big] \leqslant c^{-1}\, \mathbf{E}\Big[\exp\Big(-(c/e)\sum_i X_i\Big)\Big].$$

We use this with $X_j := 1_{\mathscr{S} \cap B_j \neq \emptyset}$ and $Y_j := 1_{\mathscr{S} \cap B_j \cap Q \neq \emptyset}$ for a random Bernoulli set Q, independent from everything else, with density so that it meets with probability $1/2$ a fixed set of cardinality m_r in B_j.

Some related results and questions

Theorem (Hammond, P & Schramm 2008). There is a natural local time measure μ on the set of exceptional times. At a μ-typical time, the configuration has the law of Kesten's Incipient Infinite Cluster (1986).

Theorem (Garban, P & Schramm 2008). The *scaling limits* of the dynamical percolation process and near-critical percolation exist, are governed by macroscopic pivotals, and are *conformally covariant*. The scaling limit of the Minimal Spanning Tree exists and is translationally, rotationally, and scale invariant, but *probably not* conformally.

Question1: Can one build similar proofs for other Boolean functions?
Question2: What about crossing functions, but non-uniform measure, e.g., Random Cluster measures? Ising model is expected to be stable, because of non-existence of pivotals ($\kappa < 4$ versus $\kappa > 4$ in SLE_κ).
Question3 (G. Kalai): Is the Universe noise sensitive? Current particle physics focuses on stable phenomena, low-eigenvalue representations. What about dark matter/energy?