Applications of Stochastics — Exercise sheet 4: Renewal equations. Queueing. Copulas. Percolation.

Gábor Pete
http://www.math.bme.hu/~gabor

December 18, 2021

Let $\xi_1, \xi_2, \dots$ be the i.i.d. lifetimes in a renewal process, with non-arithmetic distribution function $F(s) = \mathbf{P}[\,\xi \le s\,]$ and mean $\mathbf{E}\xi = \mu \in (0, \infty)$. Then $T_k := \sum_{i=1}^k \xi_i$ are the renewal times, $N_t := \min\{k : T_k \ge t\}$, and $U(t) := \mathbf{E}N_t$. The excess lifetime (or overshoot) is $\gamma_t := T_{N_t} - t$, the current lifetime is $\delta_t := t - T_{N_t - 1}$, and the total lifetime is $\beta_t := \gamma_t + \delta_t$.

▷ **Exercise 1.**

(a) Find the renewal equation $H(t) = h(t) + H * F(t)$ for $H(t) := \mathbf{P}[\,\beta_t > x\,]$, where $x \ge 0$ is fixed arbitrarily. (We actually did this in class.)

(b) Find the renewal equation for $H(t) := \mathbf{P}[\,\gamma_t > x\,]$.

(c) Using the Renewal Theorem, find the limit distributions of β_t and γ_t as $t \to \infty$.

(d) Identify the limit distribution of the total lifetime β_t as the size-biased version of ξ, and the limit distribution of the overshoot γ_t as the size-biased version $\widehat{\xi}$ multiplied with an independent $\mathsf{Unif}[0,1]$ variable. In order to avoid working with Stieltjes-integrals, you may assume that ξ has a density function.

▷ **Exercise 2.**

(a) Recall (or prove now again) that if $\xi_1 + \eta_1 + \xi_2 + \eta_2 + \dots$ is an alternating renewal process with expectations $\mathbf{E}\xi_i = \mu \in (0,\infty)$ and $\mathbf{E}\eta_i = \lambda \in (0,\infty)$, then the asymptotic proportion of time spent in ξ-intervals is $\mu/(\mu+\lambda)$.

(b) A harder, local version can be proved using an appropriate renewal equation and the Renewal Theorem: if the distribution of the independent sum $\xi_i + \eta_i$ is non-arithmetic, then the probability that moment t is in a ξ-interval converges to $\mu/(\mu+\lambda)$ as $t \to \infty$.

(c) As a special case, show that in a renewal process with a non-arithmetic renewal distribution with finite mean, $\lim_{t\to\infty} \mathbf{P}\big[\text{number of renewals in } [0,t] \text{ is odd}\big] = 1/2$.

(d) ** Does the last conclusion remain true if the renewal time has infinite mean? (The double star means that I do not actually know how to solve this. I have not tried hard.)

▷ **Exercise 3.** Let $S_n := X_1 + \dots + X_n$ be a random walk on $\mathbb{R}$, with iid increments satisfying $\mathbf{E}X_i < 0$.

(a) Recall (or prove now again) that S_n is transient, and $S_{\max} = \max\{0, S_1, S_2, \dots\}$ is an almost surely finite variable.

(b) Assume that there exists some $t_0 > 0$ such that $\mathbf{E}[\,e^{tX_i}\,] < \infty$ for all $t \in [0, t_0]$. Prove that $\mathbf{P}\big[\,S_{\max} > m\,\big] < C\exp(-cm)$ for some $0 < c, C < \infty$, for all $m > 0$. In particular, $\mathbf{E}S_{\max} < \infty$.

(c) Let $(W_n)_{n\ge 0}$ be random variables on a single probability space with marginal distributions $W_n \overset{d}{=} \max\{0, S_1, \dots, S_n\}$, but arbitrary joint distribution otherwise. Assuming $\mathbf{E}S_{\max} < \infty$ from the previous item, show that $W_n/n \to 0$ almost surely.

(d) ** Without the condition $\mathbf{E}S_{\max} < \infty$, can it happen that $W_n/n \to 0$ fails?

▷ **Exercise 4.** Consider a **G/G/1 queueing process** with iid inter-arrival times $(\mathcal{A}_n)_{n\geq 1}$ and iid service times $(\mathcal{B}_n)_{n\geq 0}$, with $\mathbf{E}\mathcal{A}_n = 1/\lambda$ and $\mathbf{E}\mathcal{B}_n = 1/\mu$. (As in class, we are starting to serve the zeroth customer at time 0.) Assume that $\lambda < \mu$; moreover, assume that the walk $S_n := X_1 + \cdots + X_n$ with jumps $X_n := \mathcal{B}_{n-1} - \mathcal{A}_n$, $n = 1, 2, \dots$, satisfies the condition $\mathbf{E}S_{\max} < \infty$ from the previous exercise.

(a) Recall from class (or from the scan from Feller's book) that the waiting time W_n of the nth customer has the same distribution as $\max\{0, S_1, \dots, S_n\}$.

(b) Let $\mathcal{B}(t)$ be the total time in $[0, t]$ while the system is busy. Show that

$$\mathcal{B}_0 + \cdots + \mathcal{B}_{N_t - 1} - W_{N_t} \leq \mathcal{B}(t) \leq \mathcal{B}_0 + \cdots + \mathcal{B}_{N_t},$$

where N_t is the index of the last customer arriving before time t.

(c) Using the SLLN and part (c) of the previous exercise, show that the limiting **utilization ratio** is

$$\lim_{t\to\infty} \frac{\mathcal{B}(t)}{t} = \frac{\lambda}{\mu} \quad \text{almost surely.}$$

▷ **Exercise 5.** Consider an **M/M/1 queuing system**: the interarrival times are iid $\mathsf{Expon}(\lambda)$, the service times are iid $\mathsf{Expon}(\mu)$. Assume $\lambda < \mu$. Let's start at time 0 with nobody in the system.

Let $N_0 = 0, N_1, N_2, \dots$ be the time moments when a customer arrives at the system or leaves it (having been just served). Let Y_i be the number of people in the system (including the one currently being served, if there is any), at time N_i.

(a) Show that $(Y_i)_{i\geq 0}$ is an irreducible aperiodic Markov chain. Find its transition probabilities and stationary distribution.

(b) Assume that μ and λ are such that the utilization ratio in the queueing process is 99%. On the long run, what is the average number of people in the system?

(c) Now assume that the expected service time increases by 1%, from λ to 1.01λ. How does the average number of people in the system change?

▷ **Exercise 6.** Show that the copula of any n-dimensional joint distribution is invariant under scalings and shifts:

$$C_{(X_1,\dots,X_n)}(u_1, \dots, u_n) = C_{(\sigma_1 X_1+\mu_1,\dots,\sigma_n X_n+\mu_n)}(u_1, \dots, u_n),$$

for any μ_i's and positive σ_i's.

In particular, for $n = 2$, show that the copula of the 2-dimensional joint Gaussian $\mathsf{N}(\bar{\mu}, \Sigma)$ depends only on the correlation coefficient between the two coordinates.

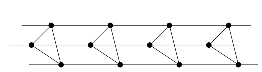
▷ **Exercise 7.** Show that the copula $C(u_1, \dots, u_n)$ of any satisfies

$$\max\left\{1 - n + \sum_{i=1}^{n} u_i\,,\, 0\right\} \leq C(u_1, \dots, u_n) \leq \min\{u_1, \dots, u_n\}\,.$$

Show by examples that the upper bound is sharp for any $n \geq 1$, while the lower bound is sharp for $n = 1, 2$.

In particular, does there exist a 2-dimensional distribution whose marginals are $\mathsf{Expon}(\lambda)$ and $\mathsf{Expon}(\mu)$ distributions, and whose copula is $C(u, v) = \min\{u, v\}$? Does it have a 2-dimensional density function?

▷ **Exercise 8.** What is the critical bond percolation density for the infinite triangular ladder?



▷ **Exercise 9.** Consider site percolation on $\mathbb{Z}^2$. Show that $1/3 \leq p_c(\mathbb{Z}^2) \leq 5/6$.