Applications of Stochastics: Final Exam 1

NAME: ....

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The absolute maximum is 51 points, but 40 counts as 100%.

▷ **Exercise 1.**

(a) Compute the moment-generating function of $\mathsf{Expon}(\lambda)$. [**2 points**]

(b) Let $T_1, T_2, \dots$ be the iid lifetimes of light bulbs, having exponential distribution with mean 1/2 year. Give a good upper bound on the probability that 50 of these light bulbs (used one after the other) suffice for 50 years. [**5 points**]

▷ **Exercise 2.**

(a) Consider the Erdős-Rényi random graph $G(n,p)$. Show that $f(p) := \mathbf{P}\big[\, G(n,p) \text{ contains a triangle} \,\big]$ is strictly monotone increasing in p. [**3 points**]

(b) Define the clustering coefficient of a graph. Can it decrease when one edge is added to the graph? [**2 points**]

(c) Define the Barabási-Albert preferential attachment graph sequence with m incoming edges in each step. (More than one versions exist, any of them is good, as long as you are precise.) [**2 points**]

▷ **Exercise 3.** Let $X_1, X_2, \dots$ be iid integer-valued random variables, and let $S_n := X_1 + \cdots + X_n$.

(a) Define what it means that $S_n/n \to 0$ in probability. [**1.5 points**]

(b) Define what it means that the random walk $\{S_n\}_{n=1}^{\infty}$ is recurrent. [**1.5 points**]

(c) Prove the integer-valued Chung-Fuchs theorem: if (a) holds, then (b) holds. [**7 points**]

▷ **Exercise 4.** Let G be the triangle, an undirected graph on 3 vertices. Let A be its adjacency matrix.

(a) Find the eigenvalues of A and an orthonormal basis of eigenvectors. [**5 points**]

(b) Consider the iteration $\overline{x}_{t+1} := \overline{x}_t A$, with $\overline{x}_0 = (1,0,0)$. Is there a sequence of scalars c_t such that $c_t \overline{x}_t$ converges to a nonzero vector? If so, then find the limit. [**5 points**]

▷ **Exercise 5.** Let $\mathbf{P}[\,\xi = k\,] = p_k$, for $k = 1, 2\dots$ and $\sum_{k\geq 1} p_k = 1$. Let $N_t := \min\{n \geq 0 : T_n > t\}$, and let $\delta_t := t - T_{N_t - 1} \geq 0$ be the current lifetime. Note that $\delta_0 = 0$.

(a) Show that $(\delta_t)_{t=0}^{\infty}$ is an irreducible aperiodic Markov chain, and find its transition probabilities. [**5 points**]

(b) Show that δ_t converges in distribution to $\mathsf{Unif}\{0, 1, \dots, \hat{\xi} - 1\}$, where $\hat{\xi}$ is the size biased version of ξ. [**5 points**]

▷ **Exercise 6.**

(a) Define the copula of an n-dimensional joint distribution. [**3 points**]

(b) Does there exist a 2-dimensional distribution whose marginals are $\mathsf{Expon}(\lambda)$ and $\mathsf{Expon}(\mu)$, and whose copula is $C(u,v) = \min\{u,v\}$? Does it have a 2-dimensional density function? [**4 points**]