Applications of Stochastics — Exercise sheet 3

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The first exercise appeared in class verbatim, but I put it here for those who were not present.

▷ **Exercise 1.** Let $\xi_1, \xi_2, \ldots$ be the i.i.d. lifetimes of the light bulbs used by the university, with $\mathbf{E}\xi_i = \mu \in (0, \infty]$. We have a janitor who visits the corridor in front of my office at times given by a Poisson process with intensity λ, and if he sees that the bulb is dead, he replaces it by a new one. Thus the times $\tau_1, \tau_2, \ldots$ passing between the death of a light bulb and the next visit of the janitor are i.i.d. $\mathsf{Expon}(\lambda)$ variables.

(a) At what rate are bulbs replaced?

(b) What is the almost sure limiting fraction of visits by the janitor on which the bulb is working?

(c) What is the limiting fraction of time that the light works?

▷ **Exercise 2.** Mr Smith likes the brand UniCar. These cars break down after a uniform $\mathsf{Uni}[0, 2]$ years of use, independently of everything. Mr Smith wants to replace each of his old cars after a fixed T years of use, or the time of breakdown, whichever happens earlier. When a car breaks down, there is a cost of USD 1000 for towing it from the road and getting rid of it, and a new car costs USD 12000. If he replaces a car when it still works, he gets a discount at the store for the old car, so the new car costs only USD 10000 (and there is no extra cost of getting rid of the old car). How should Mr Smith choose T to optimize his spendings on the long run?

▷ **Exercise 3.** If X is a non-negative random variable with finite expectation, then its **size-biased version** $\widehat{X}$ is defined by

$$\mathbf{P}[\widehat{X} \in A] = \frac{\mathbf{E}[\, X\, \mathbf{1}_{\{X \in A\}}\,]}{\mathbf{E}[\, X\,]}.$$

If this looks incomprehensible to you, think of just two special cases (we will NOT use the general one): when X is discrete, with possible values $\{x_k\}_{k \geq 1}$, then $\mathbf{P}[\widehat{X} = x_k] = x_k \mathbf{P}[X = x_k]/\mathbf{E}X$; when X has a density function $f_X(x)$, then $\widehat{X}$ has density $f_{\widehat{X}}(x) = x f(x)/\mathbf{E}X$.

(a) Show that the size-biased version of $\mathsf{Poi}(\lambda)$ is just $\mathsf{Poi}(\lambda) + 1$.

(b) Show that the size-biased version of $\mathsf{Expon}(\lambda)$ is the sum of two independent $\mathsf{Expon}(\lambda)$'s.

(c)* Take a Poisson point process of intensity λ on $\mathbb{R}$. Condition on the interval $(-\epsilon, \epsilon)$ to contain at least one arrival. As $\epsilon \to 0$, what is the point process we obtain in the limit? What does this have to do with parts (a) and (b)?

Let $\xi_1, \xi_2, \ldots$ be the i.i.d. lifetimes in a renewal process, with non-arithmetic distribution function $F(s) = \mathbf{P}[\, \xi \leq s\,]$ and mean $\mathbf{E}\xi = \mu \in (0, \infty)$. Then $T_k := \sum_{i=1}^{k} \xi_i$ are the renewal times, $N_t := \min\{k : T_k \geq t\}$, and $U(t) := \mathbf{E}N_t$. The excess lifetime (or overshoot) is $\gamma_t := T_{N_t} - t$, the current lifetime is $\delta_t := t - T_{N_t - 1}$, and the total lifetime is $\beta_t := \gamma_t + \delta_t$.

▷ **Exercise 4.**

(a) Find the renewal equation $H(t) = h(t) + H * F(t)$ for $H(t) := \mathbf{P}[\, \beta_t > x\,]$, where $x \geq 0$ is fixed arbitrarily. (We actually did this in class.)

(b) Find the renewal equation for $H(t) := \mathbf{P}[\,\gamma_t > x\,]$.

(c) Using the Renewal Theorem, find the limit distributions of β_t and γ_t as $t \to \infty$.

(d) Identify the limit distribution of the total lifetime β_t as the size-biased version of ξ, and the limit distribution of the overshoot γ_t as the size-biased version $\widehat{\xi}$ multiplied with an independent $\mathsf{Unif}[0,1]$ variable. In order to avoid working with Stieltjes-integrals, you may assume that ξ has a density function.

The next exercise proves the renewal paradox in the case when ξ has arithmetic distribution.

▷ **Exercise 5.** Let $\mathbf{P}[\,\xi = k\,] = p_k$, for $k = 1, 2 \dots$ and $\sum_{k\geq 1} p_k = 1$. Let $N_t := \min\{n \geq 0 : T_n > t\}$, and, as before, let $\delta_t := t - T_{N_t - 1} \geq 0$ be the current lifetime. Note that $\delta_0 = 0$.

(a) Show that $(\delta_t)_{t=0}^{\infty}$ is an irreducible aperiodic Markov chain, and find its transition probabilities.

(b) Show that δ_t converges in distribution to $\mathsf{Unif}\{0, 1, \dots, \hat{\xi} - 1\}$, where $\hat{\xi}$ is the size biased version of ξ.

▷ **Exercise 6.**

(a) Recall (or prove now again) that if $\xi_1 + \eta_1 + \xi_2 + \eta_2 + \dots$ is an alternating renewal process with expectations $\mathbf{E}\xi_i = \mu \in (0, \infty)$ and $\mathbf{E}\eta_i = \lambda \in (0, \infty)$, then the asymptotic proportion of time spent in ξ-intervals is $\mu/(\mu + \lambda)$.

(b) A harder, local version can be proved using an appropriate renewal equation and the Renewal Theorem: if the distribution of the independent sum $\xi_i + \eta_i$ is non-arithmetic, then the probability that moment t is in a ξ-interval converges to $\mu/(\mu + \lambda)$ as $t \to \infty$.

(c) As a special case, show that in a renewal process with a non-arithmetic renewal distribution with finite mean, $\lim_{t\to\infty} \mathbf{P}\big[\,\text{number of renewals in } [0,t] \text{ is odd}\,\big] = 1/2$.

(d) ** Does the last conclusion remain true if the renewal time has infinite mean?

▷ **Exercise 7.** Let $S_n := X_1 + \dots + X_n$ be a random walk on $\mathbb{R}$, with iid increments satisfying $\mathbf{E}X_i < 0$.

(a) Recall (or prove now again) that S_n is transient, and $S_{\max} = \max\{0, S_1, S_2, \dots\}$ is an almost surely finite variable. Moreover, if $\mathbf{E}[\,e^{t_0 X_i}\,] < \infty$ for some $t_0 > 0$, then $\mathbf{P}\big[\,S_{\max} > m\,\big] < C\exp(-cm)$ for some $0 < c, C < \infty$, for all $m > 0$. In particular, $\mathbf{E}S_{\max} < \infty$.

(b) Let $(W_n)_{n\geq 0}$ be random variables on a single probability space with marginal distributions $W_n \stackrel{d}{=} \max\{0, S_1, \dots, S_n\}$, but arbitrary joint distribution otherwise. Assuming $\mathbf{E}S_{\max} < \infty$ from the previous item, show that $W_n/n \to 0$ almost surely.

(c) Give an example of a sequence of random variables V_n on a single probability space so that they converge in distribution to an almost surely finite variable V_∞, but V_n/n does not converge almost surely to 0.

(d) ** If we do not assume the finite moment generating function for X_i, is it still always the case that $\mathbf{E}S_{\max} < \infty$?

The next exercise is just a repetition of what we did in class, but I don't want you to forget these.

▷ **Exercise 8.** Consider a queueing process with iid inter-arrival times $(\mathcal{A}_n)_{n\geq 1}$ and iid service times $(\mathcal{B}_n)_{n\geq 0}$, with $\mathbf{E}\mathcal{A}_n = 1/\lambda$ and $\mathbf{E}\mathcal{B}_n = 1/\mu$. Assume that $\lambda < \mu$, moreover, that the walk $S_n := X_1 + \dots + X_n$ with jumps $X_n := \mathcal{B}_{n-1} - \mathcal{A}_n$, $n = 1, 2, \dots$, satisfies the condition $\mathbf{E}S_{\max} < \infty$ from the previous exercise.

(a) Combine the argument on the scan from Feller's book with part (c) of the previous exercise and with Durrett's EOSP Theorem 3.5 to get that the limiting utilization ratio is λ/μ.

(b) Deduce the same result from comparing the two versions of Little's law (which we proved only on an intuitive level, but never mind).

▷ **Exercise 9.** Consider an **M/M/1 queuing system**: the interarrival times are iid $\mathsf{Expon}(\lambda)$, the service times are iid $\mathsf{Expon}(\mu)$. Assume $\lambda < \mu$. Let's start at time 0 with nobody in the system.

Let $N_0 = 0, N_1, N_2, \dots$ be the time moments when a customer arrives at the system or leaves it (having been just served). Let Y_i be the number of people in the system (including the one currently being served, if there is any), at time N_i.

(a) Show that $(Y_i)_{i\geq 0}$ is an irreducible aperiodic Markov chain. Find its transition probabilities and stationary distribution.

(b) Assume that μ and λ are such that the utilization ratio in the queueing process is 99%. On the long run, what is the average number of people in the system?

(c) Now assume that the expected service time increases by 1%, from λ to 1.01λ. How does the average number of people in the system change?

▷ **Exercise 10.** Consider an **M/G/1 queuing system**: the arrival process is Markovian, with rate λ, the service is general, with rate μ. Let $H_1 := \inf\{t > 0 : Q_t^+ = 0\}$ be the length of the first busy period. Assume $\lambda < \mu$.

Show that the busy and idle periods form an alternating renewal process. Using Exercise 6 (a) and the limiting utilization ratio from Exercise 8, show that $\mathbf{E}H_1 = \frac{1}{\mu-\lambda}$.

▷ **Exercise 11.** Show that the copula $C(u_1, \dots, u_n)$ of any n-dimensional joint distribution satisfies

$$\max\left\{1 - n + \sum_{i=1}^{n} u_i \,,\, 0\right\} \leq C(u_1, \dots, u_n) \leq \min\{u_1, \dots, u_n\}\,.$$

Show by examples that the upper bound is sharp for any $n \geq 1$, while the lower bound is sharp for $n = 1, 2$.

▷ **Exercise 12.** Consider site percolation on $\mathbb{Z}^2$. Show that $1/3 \leq p_c(\mathbb{Z}^2) \leq 5/6$.

As in class, the **Ising model** on a finite graph $G(V, E)$ is the random spin configuration $\sigma : V \longrightarrow \{\pm 1\}$ defined as follows. Given an external magnetic field $h \in \mathbb{R}$, the Hamiltonian is

$$H_h(\sigma) := -h \sum_{x\in V(G)} \sigma(x) - \sum_{(x,y)\in E(G)} \sigma(x)\sigma(y)\,,$$

and then the measure, at inverse temperature $\beta = 1/T \geq 0$, is

$$\mathbf{P}_{\beta,h}[\sigma] := \frac{\exp(-\beta H_h(\sigma))}{Z_{\beta,h}}\,, \quad \text{where} \quad Z_{\beta,h} := \sum_{\sigma} \exp(-\beta H_h(\sigma))\,.$$

▷ **Exercise 13.** The partition function $Z_{\beta,h}$ contains a lot of information about the model:

(a) Show that the **expected total energy** is

$$\mathbf{E}_{\beta,h}[\,H\,] = -\frac{\partial}{\partial\beta} \ln Z_{\beta,h}\,, \text{ with variance } \mathrm{Var}_{\beta,h}[H] = -\frac{\partial}{\partial\beta}\mathbf{E}_{\beta,h}[\,H\,]\,.$$

(b) The **average free energy** or **pressure** is defined by $f(\beta, h) := (\beta|V|)^{-1} \ln Z_{\beta,h}$. Show that for the **average total magnetization** $M(\sigma) := |V|^{-1} \sum_{x\in V} \sigma(x)$, we have

$$m(\beta, h) := \mathbf{E}_{\beta,h}[\,M\,] = \frac{\partial}{\partial h} f(\beta, h)\,.$$

(c) The **susceptibility** of the total magnetization to a change in the external magnetic field is

$$\chi(\beta, h) := \frac{1}{\beta}\frac{\partial}{\partial h} m(\beta, h) = \frac{1}{\beta}\frac{\partial^2}{\partial h^2} f(\beta, h)\,.$$

Relate this quantity to $\mathrm{Var}_{\beta,h}[M]$. Deduce that $f(\beta, h)$ is convex in h.