Applications of Stochastics — Exercise sheet 3

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Let $\xi_1, \xi_2, \dots$ be the i.i.d. lifetimes in a renewal process, with non-arithmetic distribution function $F(s) = \mathbf{P}[\,\xi \le s\,]$ and mean $\mathbf{E}\xi = \mu \in (0,\infty)$. Then $T_k := \sum_{i=1}^k \xi_i$ are the renewal times, $N_t := \min\{k : T_k \ge t\}$, and $U(t) := \mathbf{E}N_t$. The excess lifetime (or overshoot) is $\gamma_t := T_{N_t} - t$, the current lifetime is $\delta_t := t - T_{N_t - 1}$, and the total lifetime is $\beta_t := \gamma_t + \delta_t$.

▷ **Exercise 1.**

(a) Find the renewal equation $H(t) = h(t) + H * F(t)$ for $H(t) := \mathbf{P}[\,\beta_t > x\,]$, where $x \ge 0$ is fixed arbitrarily.

(b) Find the renewal equation for $H(t) := \mathbf{P}[\,\gamma_t > x\,]$.

(c) Using the Renewal Theorem, find the limit distributions of β_t and γ_t as $t \to \infty$.

If you did the previous exercise correctly, you understand why we are interested in the next one:

▷ **Exercise 2.**

(a) Show that, for any distribution function $F(t)$,

$$\int_0^\infty 1 - F(\max\{x,t\})\,dt = \int_x^\infty s\,dF(s)\,.$$

(b) Show that if X has distribution function $F(t)$, then the size-biased version $\widehat{X}$ has distribution function $\frac{1}{\mathbf{E}X}\int_0^t s\,dF(s)$.

From the previous two exercises, conclude the following:

▷ **Exercise 3.**

(a) The limit distribution of the total lifetime β_t is the size-biased version of ξ.

(b) The limit distribution of the overshoot γ_t is the size-biased version $\widehat{\xi}$ multiplied with an independent $\mathsf{Unif}[0,1]$ variable.

▷ **Exercise 4.**

(a) Consider the renewal process with a non-arithmetic renewal distribution with finite mean. Show that $\lim_{t\to\infty} \mathbf{P}\big[\,\text{number of renewals in } [0,t] \text{ is odd}\,\big] = 1/2$.

(b)* Does this remain true if the renewal time has infinite mean?

Simple generalizations of the basic percolation arguments from class regarding $p_c(G) = p_c(G, \text{bond})$:

▷ **Exercise 5.**

(a) Show that in any graph $G(V,E)$ with maximal degree Δ, we have $p_c(G) \ge 1/(\Delta - 1)$.

(b) Show that if in a graph G the number of minimal edge-cutsets (a subset of edges whose removal disconnects a given vertex from infinity, minimal w.r.t. containment) of size n is at most $\exp(Cn)$ for some $C < \infty$, then $p_c(G) \le 1 - \epsilon(C) < 1$.

Recall from Exercise 9 of Sheet 1 that the function $p \log p + (1-p)\log(1-p)$ plays an important role in the large deviations theory of Bernoulli variables. That's just the tip of an iceberg of the relationship between large deviations and entropy theory, which we will not discuss in detail, but still, here is the definition and some basic properties of the **entropy** of a discrete random variable:

$$\mathsf{Ent}(X) := -\sum_{x \in \Omega} \mathbf{P}[\, X = x \,] \log \mathbf{P}[\, X = x \,] \,.$$

If X and Y are defined on the same probability space, then $\mathsf{Ent}(X, Y)$ is just the entropy of the variable (X, Y), while the **conditional entropy** $\mathsf{Ent}(X \mid Y)$ is defined as the Y-average of the entropies of the conditional distributions $X \mid Y = y$:

$$\mathsf{Ent}(X \mid Y) := \sum_{y \in \Omega} \left(-\sum_{x \in \Omega} \mathbf{P}[\, X = x \mid Y = y \,] \log \mathbf{P}[\, X = x \mid Y = y \,] \right) \mathbf{P}[\, Y = y \,] \,.$$

▷ **Exercise 6.**

(a) Show that if the probability space is finite, $|\Omega| = n$, then $\mathsf{Ent}(X) \le \log n$, with equality iff X is uniform on Ω. (Hint: use the concavity of $-x \log x$ on $x \in [0, 1]$.)

(b) Show that $\mathsf{Ent}(X \mid Y) \le \mathsf{Ent}(X)$, with equality iff X and Y are independent.

(c) Show that $\mathsf{Ent}(X \mid Y) = \mathsf{Ent}(X, Y) - \mathsf{Ent}(Y)$. Deduce that $\mathsf{Ent}(X, Y) \le \mathsf{Ent}(X) + \mathsf{Ent}(Y)$, with equality iff X and Y are independent.

As in class, the **Ising model** on a finite graph $G(V, E)$ is the random spin configuration $\sigma : V \longrightarrow \{\pm 1\}$ defined as follows. Given an external magnetic field $h \in \mathbb{R}$, the Hamiltonian is

$$H_h(\sigma) := -h \sum_{x \in V(G)} \sigma(x) - \sum_{(x,y) \in E(G)} \sigma(x)\sigma(y) \,,$$

and then the measure, at inverse temperature $\beta = 1/T \ge 0$, is

$$\mathbf{P}_{\beta,h}[\sigma] := \frac{\exp(-\beta H_h(\sigma))}{Z_{\beta,h}} \,, \quad \text{where} \quad Z_{\beta,h} := \sum_{\sigma} \exp(-\beta H_h(\sigma)) \,.$$

▷ **Exercise 7.** Using the method of Lagrange multipliers, show that, for any finite graph $G(V, E)$, any external field $h \in \mathbb{R}$, and any given energy level $E \ge 0$, among all probability measures μ on $\{\pm 1\}^{V(G)}$ that have $\mathbf{E}_\mu[\, H(\sigma) \,] = E$, the measures that maximize the entropy $\mathsf{Ent}(\mu)$ are all of the above form $\mathbf{P}_{\beta,h}$ for some $\beta > 0$. (For some values of E, there may exist no measure μ with $\mathbf{E}_\mu[\, H(\sigma) \,] = E$, but that is OK.)

▷ **Exercise 8.** The partition function $Z_{\beta,h}$ contains a lot of information about the model:

(a) Show that the **expected total energy** is

$$\mathbf{E}_{\beta,h}[\, H \,] = -\frac{\partial}{\partial \beta} \ln Z_{\beta,h} \,, \text{ with variance } \mathrm{Var}_{\beta,h}[H] = -\frac{\partial}{\partial \beta} \mathbf{E}_{\beta,h}[\, H \,] \,.$$

(b) The **average free energy** or **pressure** is defined by $f(\beta, h) := (\beta |V|)^{-1} \ln Z_{\beta,h}$. Show that for the **average total magnetization** $M(\sigma) := |V|^{-1} \sum_{x \in V} \sigma(x)$, we have

$$m(\beta, h) := \mathbf{E}_{\beta,h}[\, M \,] = \frac{\partial}{\partial h} f(\beta, h) \,.$$

(c) The **susceptibility** of the total magnetization to a change in the external magnetic field is

$$\chi(\beta, h) := \frac{1}{\beta} \frac{\partial}{\partial h} m(\beta, h) = \frac{1}{\beta} \frac{\partial^2}{\partial h^2} f(\beta, h) \,.$$

Relate this quantity to $\mathrm{Var}_{\beta,h}[M]$. Deduce that $f(\beta, h)$ is convex in h.

The **Curie-Weiss model** is the Ising model on the complete graph K_n, with edge weights $1/n$, so that the Hamiltonian is

$$H_{n,h}(\sigma) := -h \sum_{i=1}^{n} \sigma_i - \frac{1}{2n} \sum_{i,j=1}^{n} \sigma_i \sigma_j \,.$$

(The 1/2 factor is to make up for having each pair $\{i,j\}$ with $i \neq j$ twice in the sum. The appearance of the terms $i = j$ causes just a shift of H by a constant, which is not visible in $\mathbf{P}_{\beta,h}$.) In terms of the average magnetization $M(\sigma) = \sum_i \sigma_i / n$, note that we can write

$$H_{n,h}(\sigma) = -\big(hM(\sigma) + M(\sigma)^2/2\big)n \,,$$

and the number of σ's with $M(\sigma) = x \in \{-1, \frac{-n+2}{n}, \dots, \frac{n-2}{n}, 1\}$ is $\binom{n}{n(1+x)/2}$. Thus,

$$Z_{n,\beta,h} = \sum_x c_{n,\beta,h}(x) \,, \quad \text{where} \quad c_{n,\beta,h}(x) := \binom{n}{n(1+x)/2} \exp\Big(\beta n \big(hx + x^2/2\big)\Big) \,.$$

▷ **Exercise 9.**

(a) Show that $f(\beta,h) := \lim_{n\to\infty} f_n(\beta,h) = \lim_{n\to\infty} \frac{\max_x \ln c_{n,\beta,h}(x)}{\beta n}$.

(b) Similarly to Exercise 9 from Sheet 1, show that $\ln c_{n,\beta,h}(x) = n\big(\beta h x - \Phi_\beta(x)\big) + o(n)$, where

$$\Phi_\beta(x) = \frac{1-x}{2} \ln \frac{1-x}{2} + \frac{1+x}{2} \ln \frac{1+x}{2} - \frac{\beta x^2}{2} \qquad \text{for } x \in [-1,1] \,.$$

(c) Sketch the curves $\Phi_\beta(x)$ and $\Phi'_\beta(x)$ on $x \in [-1,1]$, for some parameters $\beta < 1$, $\beta = 1$, and $\beta > 1$.

(d) By choosing the appropriate root $x = x_0(\beta,h)$ of $\Phi'_\beta(x) = \beta h$, find $\lim_{n\to\infty} \arg\max_x \ln c_{n,\beta,h}(x)$. Note that part (a) gives

$$\frac{\partial}{\partial h} f(\beta,h) = \frac{\partial}{\partial h} \left(h x_0(\beta,h) - \frac{\Phi_\beta(x_0(\beta,h))}{\beta} \right) = x_0(\beta,h) \,.$$

(e) By part (b) of the previous exercise, $m_n(\beta,h) = \frac{\partial}{\partial h} f_n(\beta,h)$. Assuming that $m(\beta,h) := \lim_{n\to\infty} m_n(\beta,h) = \frac{\partial}{\partial h} f(\beta,h)$ holds for $h \neq 0$ (which is indeed the case), deduce from the above that

$$\lim_{h\to 0+} m(\beta,h) > 0 \qquad \text{and} \qquad \lim_{h\to 0-} m(\beta,h) < 0 \qquad \text{for } \beta > 1 \,,$$

while these limits equal 0 for $\beta \leq 1$. Hence $m(\beta,h)$ is discontinuous at $h = 0$ iff $\beta > 1$.

(f) Show that

$$\frac{1}{\beta} \frac{\partial^2}{\partial h^2} f(\beta,h) = \frac{1}{\beta} \frac{\partial}{\partial h} x_0(\beta,h) = \frac{1 - x_0(\beta,h)^2}{1 - \beta(1 - x_0(\beta,h)^2)} \,.$$

For $\beta = 1$, deduce that $\frac{\partial}{\partial h} x_0(\beta,h) = \infty$. That is, $m(1,h)$ is continuous but not analytic at $h = 0$. Assuming that the limiting susceptibility $\chi(\beta,h) := \lim_{n\to\infty} \chi_n(\beta,h)$ equals $\frac{1}{\beta} \frac{\partial^2}{\partial h^2} f(\beta,h)$, we get that the limiting susceptibility is $\chi(1,0) = \infty$. What does that mean for the variance of the average magnetization?

(g)* Show that $\frac{\partial}{\partial h} x_0(\beta, 0+) < \infty$ for $\beta > 1$, so that the limiting susceptibility is finite.