# 1. Deterministic disease on the $k$-dimensional board

In this chapter we are discussing the behaviour of the function $G_{k,l}(n)$, what is the minimal number of initial black cubes needed for a contagious configuration, i.e. for the complete painting of the $k$-dimensional $n \times \ldots \times n$ cube board if we follow the $l$-neighbour painting rule, where $1 \leq l \leq 2k$. For more definitions and background see the *Introduction.*

First of all let us describe the solution of the initial exercise, i.e. the case $G_{2,2}(n) = G(n)$:

**Fact 1.1.** [Folklore] $G(n) = n$.

**Proof.** If we paint black the squares of a diagonal, it will be a contagious configuration, so $G(n) \leq n$. For a lower bound we can use the so-called *invariant method*: a suitable invariant is the perimeter of the black part of the board, which can never increase if we use our 2-neighbour painting rule. If we succeed in painting black the whole chessboard, we will have a perimeter $4n$, so we need at least $n$ black squares at the beginning. ∎

We also give a second proof, which can be applied for some other versions of this original problem, as well. In *Chapter 2* we will see that from some respect it is more natural to consider a torus of $n^2$ squares instead of the $n \times n$ square board, or, as a middle figure, a cylinder board. The corresponding functions are $G^T(n)$ and $G^C(n)$. It is clear that $G^T(n) \leq G^C(n) \leq G(n)$, as we have the same number of squares, but more adjacencies.

**Fact 1.2.** $G^T(n) = n - 1,\ G^C(n) = G(n) = n$.

**Proof.** Clearly, we can order the initial black squares of a contagious configuration in such a way that if we consider the final configuration determined by the first $k$ initial squares, this will always be connected until we reach the all-black board. Let us consider now the diameter of this growing black part of the board, where the metric is generated by the $L^1$-norm, i.e. $\text{dist}((a_1, a_2), (b_1, b_2)) = |a_1 - b_1| + |a_2 - b_2|$. (This metric on the square grid is sometimes called the *Manhattan-distance*, inspite of the tradition that for almost all the metropolises the antique Rome served as a model, so it would be more respectful to call it the *Rome-distance.*) Thus the all-black $n \times n$ square board has a diameter of $2n$, the cylinder of $2n - 1$ and the torus of $2n - 2$.

When we put in the initial black squares one by one according to the ordering described above, using our 2-neighbour painting rule the diameter of the resulting final black configuration can increase by at most 2 at each step. We start with an all-white board without initial black squares, so we need at least $n$ squares for the square board and the cylinder, and $n - 1$ for the torus. On the other hand, it is easy to see that this amount of initial black squares will suffice indeed. ∎

Now we are going to apply these simple methods and other ideas to determine the exact order of magnitude of our function $G_{k,l}(n)$.

## 1.1. General bounds

We can achieve exact values for the function $G_{k,l}(n)$ only in some very special cases. It is trivial that $G_{k,1}(n) = 1$, and that for $l = 2k$ we need a chessboard-like initial configuration of black cubes, thus $G_{k,2k}(n) \sim \frac{1}{2}n^k$. One may expect that the perimeter method can work exactly only for $l = k$ — what can we hope for the other cases? Generalizing the method of our starting exercise we get a relevant result for $l \geq k$:

**Perimeter Lemma.** $G_{k,l}(n) \geq \frac{l-k}{l}n^k + \frac{k}{l}n^{k-1}$. *Furthermore,* $G_{k,k}(n) = n^{k-1}$.

**Proof.** During the spreading of black cubes let us watch those $(k-1)$-dimensional faces of the cubes who are in between a black and a white cube. We call them "free black faces", and the number of them is exactly the surface of the black part in the board. If we follow an $l$-neighbour painting rule, when a white cube becomes black, we lose at least $l$ free black faces, and gain at most $2k - l$ of them, so we lose at least $2(l - k)$ of the surface altogether. Now let us suppose that we have $x$ initial black cubes, and they form a contagious configuration. They have at most $2kx$ free black faces, during the process $n^k - x$ new black cubes arise, and the all-black board has a surface of $2kn^{k-1}$, so

$$2kx - 2(l - k)(n^k - x) \geq 2kn^{k-1},$$

which gives the desired bound for $x$.

For $l = k$ this lower bound is $n^{k-1}$. For the same upper bound we need a generalization of the diagonal construction used for $G(n) = n$.

For k=l=1 there exist $n$ contagious configuration of size 1 (here n=4):

Figure 1.1.a

For k=l=2 we can find $n$ solutions again, with a cyclic combination of the 1-dimensional constructions:

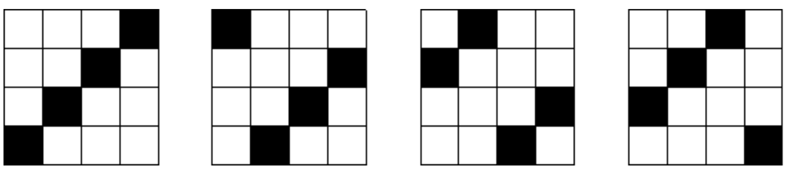

Figure 1.1.b

For three dimensions we use the cyclic combinations of the $k = 2$ configurations, and so on, we always can build up the $n$ appropriate solutions. (On the figure we can see two possibilities from the $k = l = 3,\ n = 4$ case.)

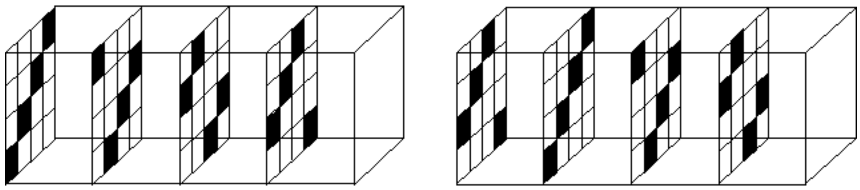

Figure 1.1.c

These constructions consist of $n^{k-1}$ black cubes, so we are done. ∎

Thus for fixed $l \geq k$ we have $G_{k,l}(n) = \Omega(n^k)$. With a simple geometric trick we can get a lower bound even for the cases $l < k$:

**Projection Lemma.** $G_{k,l}(n) \geq G_{k-1,l}(n)$.

**Proof.** Let us consider a $k$-dimensional board with a contagious configuration of $G_{k,l}(n)$ black cubes, and observe the process projected down to a fixed $(k-1)$-dimensional face of the board. In this projected image a $(k-1)$-cube is black iff at least one of its $n$ originals was black. If such a cube in the image becomes black, then it has an original becoming black, and the images of the at least $l$ black neighbours are distinct neighbours of our $(k-1)$-cube in the image. So the projected process itself follows an $l$-neighbour painting rule, and the image of the initial black cubes was contagious for this $(k-1)$-dimensional process, so $G_{k,l}(n) \geq G_{k-1,l}(n)$. ∎

Iterating this result for $k \geq l-1$ we get $G_{k,l}(n) \geq G_{l-1,l}(n)$, where the right hand side can just be estimated by the *Perimeter Lemma*: $G_{k,l}(n) = \Omega(n^{l-1})$.

Now we need a good upper bound, which will come from the following recursive painting technique:

**Recursion Lemma.** $G_{k,l}(n) \leq f_k(k)G_{k,l}(n-2) + f_k(k-1)G_{k-1,l-1}(n-2) + f_k(k-2)G_{k-2,l-2}(n-2) + \ldots + f_k(0)G_{0,l-k}(n-2)$, *where* $f_k(m)$ *is the number of* $m$*-faces of the* $k$*-dimensional cube.*

**Proof.** Into the middle of a $k$-dimensional board of size $n$ place a board of size $n-2$, with a contagious configuration of $G_{k,l}(n-2)$ black cubes. Still we have to paint black the cubes next to the surface of the board, these cubes form $f_k(k-1)$ $(k-1)$-dimensional, $f_k(k-2)$ $(k-2)$-dimensional, etc., $f_k(0)$ 0-dimensional subboards of size $n-2$ each. In the $(k-1)$-dimensional subboards all the cubes have already a black neighbour because of the black $k$-dimensional subboard in the middle, so we need only $G_{k-1,l-1}(n-2)$ initial black cubes for each these $(k-1)$-dimensional subboards to paint them black all. In the $(k-2)$-dimensional subboards the cubes have already two black neighbours, so we need only $G_{k-2,l-2}(n-2)$ initial black cubes for each, and so on. (Note that for $l \leq k$ we have $G_{i,l-k}(m) = 0$.) Summing up these numbers of initial black cubes we get the result.

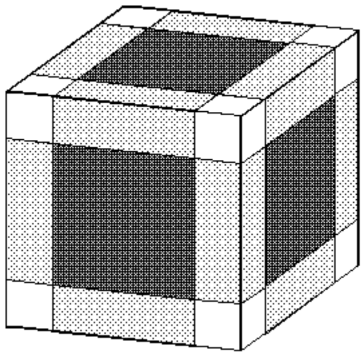

Figure 1.2

∎

This result roughly says that our function does not grow too quickly. We can make this observation more precise by a double induction on $k$ and $l$. Let us suppose that for some fixed $k, l$ we have $G_{k,l}(n) = O(n^{l-1})$, what is trivial for $k = 0$, $l$ arbitrary, and for $l = 1$, $k$ arbitrary. Plugging this inductional hypothesis for less than $k$ and $l$ in the *Recursion Lemma* we get that the difference $G_{k,l}(n) - G_{k,l}(n-2)$ is smaller than the sum of $k$ (that is, a fixed number of) $O(n^{l-2})$ terms, so it is $O(n^{l-2})$ itself. Thus, as known about the *discrete differential operator* of sequences, $G_{k,l}(n) = O(n^{l-1})$, and our upper bound has been proved.

Combining these results we have

**Theorem 1.1.** *For fixed $k, l$ we have*

$$G_{k,l}(n) = \begin{cases} \Theta(n^{l-1}), & \text{if } 1 \le l \le k \\ \Theta(n^k), & \text{if } k+1 \le l \le 2k \end{cases} \tag{1}$$

∎

Our next task is to determine exact asymptotics for as many special cases as possible.

## 1.2. Asymptotics for two and three dimensions

First we consider the disease problem on the two-dimensional square.

**Theorem 1.2.**

**(a)** $G_{2,1}(n) = 1$

**(b)** $G_{2,2}(n) = n$

**(c)** $G_{2,3}(n) \sim \frac{1}{3}n^2$,
*or more exactly,* $\frac{1}{3}n^2 + \frac{2}{3}n \le G_{2,3}(n) \le \frac{1}{3}n^2 + \frac{4}{3}n + O(1)$.

**(d)** $G_{2,4}(n) \sim \frac{1}{2}n^2$, *or more exactly,* $\frac{1}{2}n^2 + \frac{1}{2}n \le G_{2,4}(n) \le \frac{1}{2}n^2 + 2n + O(1)$.

**Proof.** The parts (a) and (b) have already been proved. For (c) and (d) the lower bounds come from the *Perimeter Lemma.* The construction for the upper bound of (c):

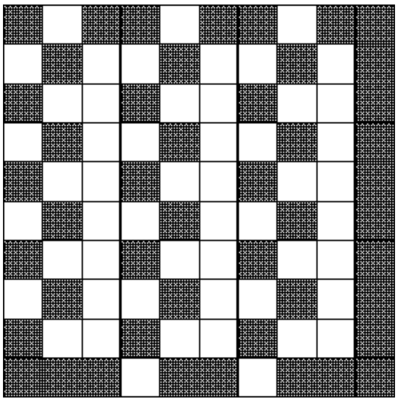

Figure 1.3.a

The exact realization depends on the mod 6 residue of $n$. The result we get is the best for the case $n \equiv 2 \pmod 6$, namely $\frac{1}{3}n^2 + n + O(1)$, and the worst for $n \equiv 0, 1 \pmod 6$, namely $\frac{1}{3}n^2 + \frac{4}{3}n + O(1)$.

The construction for (d), slightly depending on whether $n$ is odd or even:

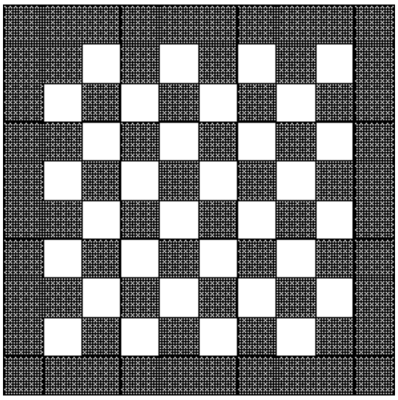

Figure 1.3.b

∎

Now let us see the three-dimensional cube board.

**Theorem 1.3.**

**(a)** $G_{3,1}(n) = 1$

**(b)** $G_{3,2}(n) \sim \frac{3}{2}n$,

*or more exactly,* $G_{3,2}(n) = \lfloor \frac{3}{2}n \rfloor$

**(c)** $G_{3,3}(n) = n^2$

**(d)** $G_{3,4}(n) \sim \frac{1}{4}n^3$, *or more exactly*, $\frac{1}{4}n^3 + \frac{3}{4}n^2 \leq G_{3,4}(n) \leq \frac{1}{4}n^3 + 2n^2$

**(e)** $\frac{2}{5}n^3 + \frac{3}{5}n^2 \leq G_{3,5}(n) \leq \frac{3}{7}n^3 + O(n^2)$

**(f)** $G_{3,6}(n) \sim \frac{1}{2}n^3$,
*or more exactly*, $\frac{1}{2}n^3 + \frac{1}{2}n^2 \leq G_{3,6}(n) \leq \frac{1}{2}n^3 + 3n^2$.

**Proof.** The cases (a) and (c) are already known. The lower bounds for (d), (e) and (f) come from the *Perimeter Lemma.* The lower bound for (b) can be verified by the method of *Fact 1.2.*: in the $L^1$-norm the $k$-dimensional cube board of size $n$ has a diameter of $kn$, the first initial black cube has a diameter of $k$, each of the other initial black cubes can increase the diameter by at most 2, so for $k = 3$ we need $\lfloor \frac{3}{2}n \rfloor$ initial black cubes at least. And this is enough, as the figure shows:

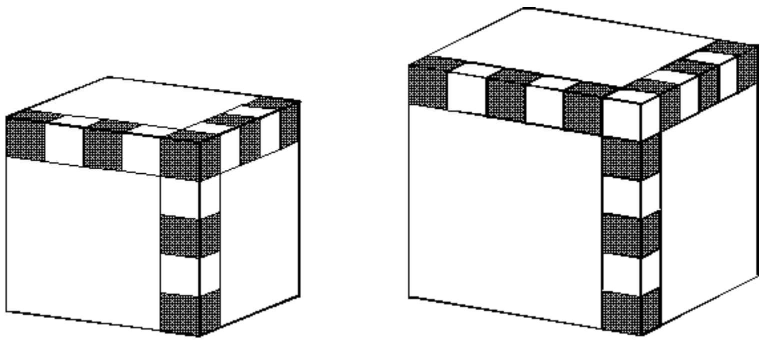

Figure 1.4.a

For the upper bound of (d) let us divide our board into $n$ slabs; each of these slabs can be considered as a two-dimensional subboard. If into each odd slab we place a contagious configuration corresponding to $G_{2,4}(n)$, these slabs become black by themselves, and for the even slabs it is enough to start with $G_{2,2}(n)$-configurations. This is approximately $\frac{n}{2}(\frac{1}{2}n^2 + 2n) + \frac{n}{2}n \sim \frac{1}{4}n^3$ altogether, which can be calculated exactly for even and odd values of $n$.

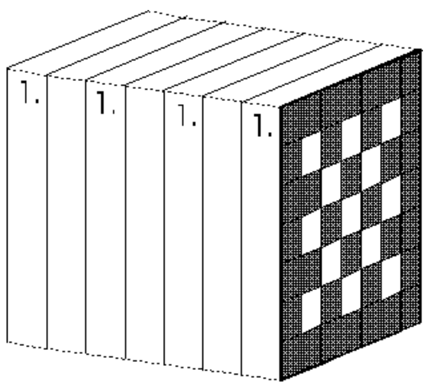


Figure 1.4.b

The case (f) is very similar to *Theorem 1.2. (d).* All the cubes on the boundary have to be black initially, and we need a chessboard-like configuration inside the board. This is approximately $\frac{1}{2}(n-2)^3 + 6n^2 \sim \frac{1}{2}n^3$ initial black squares.

Finally, here is the first unsolved problem, the case of $G_{3,5}(n)$. The lower bound is around $\frac{2}{5}n^3$, but, if we generalize the construction of *Theorem 1.2 (c)* to the case $l = k-1$, we get only $\frac{4}{9}n^3$:

Let us divide the board into slabs again. The slabs with number 1 and 2 get a chessboard-like initial configuration, this is enough to paint them black all. Thus we need only $G_{2,3}(n)$-configurations for the slabs with number 3, the final result is $\sim \frac{2}{3}n\frac{1}{2}n^2 + \frac{1}{3}n\frac{1}{3}n^2 = \frac{4}{9}n^3$ initial black cubes.

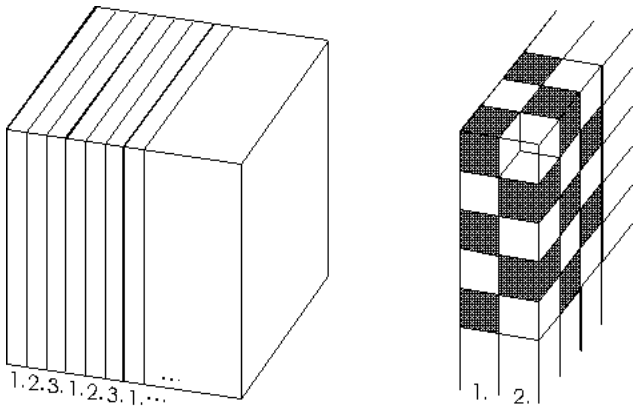


Figure 1.4.c

We may consider the generalization of another construction for $G_{2,3}(n) \le \frac{1}{3}n^2 + O(n)$ to get $G_{3,5}(n) \le \frac{3}{7}n^3 + O(n^2)$. But we think that the truth is $\frac{2}{5}n^3$ (see *Conjecture 1.1* in the next section), and the exact description of this construction would be rather complicated, so we save the Reader it. This construction is inductive: in two dimensions we pick four rotated copies of an already existing $n \times n$ construction, and glue them together into a $(2n-1) \times (2n-1)$ construction, deleting a lot of black squares becoming unnecessary. In three dimensions we do the same with eight rotated copies, but here we already do not get the desired result, only $\sim \frac{3}{7}n^3$. ∎

## 1.3. Higher dimensions

For the higher dimensional cases we have the generalizations of our previous results.

**Theorem 1.4.**

**(a)** $G_{k,2}(n) \sim \frac{k}{2}n$,
*or more exactly,* $G_{k,2}(n) = \left\lceil \frac{k(n-1)}{2} \right\rceil + 1$.

**(b)** $\left(G_{k,k+2}(n) \sim \frac{2}{k+2}n^k\right) \Rightarrow \left(G_{k+1,k+2}(n) \sim \frac{1}{k+2}n^{k+1}\right)$.

**(c)** $G_{k,2k}(n) \sim \frac{1}{2}n^k$,
*or more exactly,* $\frac{1}{2}n^k + \frac{1}{2}n^{k-1} \le G_{k,2k}(n) \le \frac{1}{2}n^k + kn^{k-1}$.

**Proof.** One can easily apply the methods of *Theorem 1.3 (b), (d)* and *(f)*. However, the construction for $G_{k,2}(n)$, when $n$ is even, is not so straightforward as was for $k = 3$. First we construct it for $n = 2$:

Our $k$-dimensional cube board is nothing else in this case, but the partially ordered set $\left(2^{\{1,\ldots,k\}}, \subseteq\right)$, with edges between subsets differing by exactly one element in addition or deletion. (In fact, this is the *Hasse-diagram* of our poset.) We can divide our poset into $k+1$ levels according to the number of elements in the subsets: $0^{\text{th}}, 1^{\text{st}}, \ldots, k^{\text{th}}$. Now we choose the following $\lceil k/2 \rceil + 1$ subsets (i.e. cubes of the board) to be black in the initial configuration: two arbitrary subsets from the $1^{\text{st}}$ level, and $\lceil (k-2)/2 \rceil$ subsets from the $2^{\text{nd}}$ level, such that their union covers exactly those $k-2$ points which are not in the two subsets chosen from the $1^{\text{st}}$ level. These subsets paint black the entire $1^{\text{st}}$ level at once, and all the other subsets afterwards, so we are ready.

For $n > 2$ ($n$ is even) we only have to add $k(n-2)/2$ initial black cubes to the previous construction, i.e. $(n-2)/2$ for each direction. This can be carried out in the natural way, similarly to *Figure 1.4.a*. ∎

Finally, based on our results above, we state the following conjecture:

**Conjecture 1.1.** *For $k+1 \leq l \leq 2k$ the Perimeter Lemma is sharp, i.e.* $G_{k,l}(n) = \frac{l-k}{l} n^k + O(n^{k-1})$.

The case $l < k$ seems to be hopeless at this time.