Stochastic Differential Equations

Problem set No 7 — May 3, 2012

▷ **Exercise 1** (Change of conditional expectation)**.** Let $\mathbf{Q}$ and $\mathbf{P}$ be two probability measures on $(\Omega, \mathcal{F})$, with $\mathbf{Q} \ll \mathbf{P}$, and Radon-Nikodym derivative $\rho(\omega) = \frac{d\mathbf{Q}}{d\mathbf{P}}(\omega)$. Let $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra. Show that, for any $\mathcal{F}$-measurable variable X, we have

$$\mathbf{E}_{\mathbf{Q}}[\, X \mid \mathcal{G}\,]\, \mathbf{E}_{\mathbf{P}}[\, \rho \mid \mathcal{G}\,] = \mathbf{E}_{\mathbf{P}}[\, \rho X \mid \mathcal{G}\,]\,. \tag{1}$$

▷ **Exercise 2.** Let $\mathbf{P}$ be the measure on $\{\mathsf{H}, \mathsf{T}\}^n$ given by tossing a biased coin n times independently, giving probability 2/3 to H. Let $\tilde{\mathbf{P}}$ be the measure given by a fair coin. Let $Z_n(\omega) := \frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}(\omega)$, and consider the martingale $Z_m := \mathbf{E}[\, Z_n \mid \mathcal{F}_m\,]$ for $m \leq n$.

(a) Give the distribution of Z_{n+1} given $Z_n, \ldots, Z_1$ explicitly. Note the similarity with the martingale of Problem set No. 3, Exercise 7, Remark 2.

(b) Note that (1) of the previous exercise translates to $\tilde{\mathbf{E}}[X \mid \mathcal{F}_m] = \frac{1}{Z_m}\mathbf{E}[\, X Z_n \mid \mathcal{F}_m\,]$. Check this numerically for $n = 3$, $m = 2$, $X = \#\{\text{heads in } \omega\}$, and $\mathsf{HH} \in \mathcal{F}_2$.

(c) Interpret this exercise as a discrete version of Girsanov's theorem.

▷ **Exercise 3** (Cameron-Martin theorem)**.**

(a) Consider $F(t) = \int_0^t f(s)\, ds$ with $f \in L^2[0,1]$, a deterministic function. Show that if B_t is standard Brownian motion, then $\{F(t) + B_t : t \in [0,1]\}$ and $\{B_t : t \in [0,1]\}$ are mutually absolutely continuous w.r.t. each other.

(b) If $F(t)$ is such that the above $f(t)$ does not exist, then $\{F(t)+B_t : t \in [0,1]\}$ and $\{B_t : t \in [0,1]\}$ are singular w.r.t. each other.

▷ **Exercise 4.** Let $B(t) = (B_1(t), B_2(t))$, $t \leq T$, be a 2-dimensional Brownian motion w.r.t. $(\Omega, \mathcal{F}_T, \mathbf{P})$. Find a probability measure $\mathbf{Q}$ on $\mathcal{F}_T$ that is mutually abs. continuous w.r.t. $\mathbf{P}$, and under which the following $Y(t)$ becomes a martingale:

(a) $$dY(t) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} dt + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}; \qquad t \leq T\,.$$

(b) $$dY(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}; \qquad t \le T.$$

▷ **Exercise 5.** Let $a : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be bounded measurable function. Construct a weak solution X_t of the SDE

$$dX_t = a(X_t)\, dt + dB_t\,; \qquad X_0 = x \in \mathbb{R}^n\,.$$

▷ **Exercise 6.** Let $Y(t) = t + B(t)$, where $B(t)$ is a BM under $\mathbf{P}$. For each $T > 0$, find $\mathbf{Q}_T \sim \mathbf{P}$ on $\mathcal{F}_T$ such that $\{Y(t)\}_{t \le T}$ becomes a BM under $\mathbf{Q}_T$.

(a) Show that there exists a probability measure $\mathbf{Q}$ on $\mathcal{F}_\infty$ such that $\mathbf{Q}|_{\mathcal{F}_T} = \mathbf{Q}_T$ for all $T > 0$.

(b) Show that $\mathbf{P}\big[\lim_{t\to\infty} Y(t) = \infty\big] = 1$, while $\mathbf{Q}\big[\lim_{t\to\infty} Y(t) = \infty\big] = 0$. Why does not this contradict Girsanov's theorem?

▷ **Exercise 7.** Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be Lipschitz, and define $X_t = X_t^x$ by

$$dX_t = b(X_t)\, dt + dB_t\,; \qquad X_0 = x \in \mathbb{R}\,.$$

(a) Use Girsanov to prove that for all $M < \infty$, $x \in \mathbb{R}$, and $t > 0$, we have $\mathbf{P}\big[\, X_t^x > M \,\big] > 0$.

(b) Choose $b(x) = -r$, where $r > 0$ is a constant. Prove that, for all x, we have $X_t^x \to -\infty$ as $t \to \infty$, a.s.

▷ **Exercise 8.** Let B_t denote BM in $\mathbb{R}^n$, and consider the Itô diffusion

$$dX_t = \nabla h(X_t)\, dt + dB_t\,; \qquad X_0 = x \in \mathbb{R}^n\,,$$

where $h \in C_0^1(\mathbb{R}^n)$. We are going to relate this BM with drift to a BM killed at a certain rate V.

(a) Let

$$V(x) = \frac{1}{2}|\nabla h(x)|^2 + \frac{1}{2}\Delta h(x)\,.$$

Prove that, for any $f \in C_0(\mathbb{R}^n)$, we have

$$\mathbf{E}_x[\, f(X_t)\,] = \mathbf{E}_x\Big[\exp\Big(-\int_0^t V(B_s)\, ds\Big) \exp\big(h(B_t) - h(x)\big)\, f(B_t)\Big]\,. \tag{2}$$

(Hint: use Girsanov to express the LHS in terms of B_t, then use Itô's formula on $h(B_t)$.)

(b) Assume $V \ge 0$, and use Feynman-Kac with killing rate V to get a process Y_t and to reinterpret (2) as

$$T_t^X(f, x) = \exp(-h(x))\, T_t^Y(f \exp(h),\, x)\,,$$

where $T_t^X(f,x) = \mathbf{E}_x[\, f(X_t)\,]$ and similarly for Y.