Stochastic Differential Equations

Problem set No 3 — March 8, 2012

▷ **Exercise 1.** Applying Itô's formula to a suitable function $g(t, B_t)$, show that

(a) $\int_0^t s\, dB_s = tB_t - \int_0^t B_s\, ds$;

(b) $\int_0^t B_s\, dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$;

(c) $\int_0^t B_s^2\, dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s\, ds$.

▷ **Exercise 2.** Using Itô's formula, write X_t in the standard form of an Itô process, $dX_t = u(t,\omega)\, dt + v(t,\omega)\, dB_t(\omega)$:

(a) $X_t = 2 + t + e^{B_t}$;

(b) $X_t = (t_0 + t, B_t)$;

(c) $X_t = \big(B_1(t) + B_2(t) + B_3(t), B_2^2(t) - B_1(t)B_3(t)\big)$, where $\big(B_1(t), B_2(t), B_3(t)\big)$ is 3-dimensional Brownian motion.

▷ **Exercise 3.** Let X_t, Y_t be Itô processes in $\mathbb{R}$. Prove that

$$d(X_tY_t) = X_t\, dY_t + Y_t\, dX_t + dX_t \cdot dY_t\,.$$

Deduce the following *integration by parts* formula:

$$\int_0^t X_s\, dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s\, dX_s - \int_0^t dX_s \cdot dY_s\,.$$

In particular, if $X_t(\omega) = f(t,\omega)$, a function of bounded variation for a.a. ω, or in other words, $dX_t(\omega) = f'(t,\omega)\, dt$, and $dY_t = dB_t$, then

$$\int_0^t f(s)\, dB_s = f(t)\, B_t - \int_0^t B_s\, f'(s)\, ds\,.$$

▷ **Exercise 4.** Using Itô's formula (in particular, the previous exercise), show that following processes are martingales w.r.t. $\mathcal{F}_t := \sigma\{B_s : s \le t\}$:

(a) $X_t = e^{t/2} \cos B_t$;

(b) $X_t = e^{t/2} \sin B_t$;

(c) $X_t = (B_t + t) \exp(-B_t - t/2)$.

▷ **Exercise 5.** Let X_t be a 1-dimensional Itô process $dX_t(\omega) = v(t,\omega)^\mathsf{T} \, dB_t(\omega)$, with $v(t,\omega), B_t \in \mathbb{R}^n$ and $v \in \mathcal{V}^n[0,T]$ bounded. Prove that

$$M_t := X_t^2 - \int_0^t \|v_s\|^2 \, ds$$

is a martingale. The process $\langle X, X\rangle_t := \int_0^t \|v_s\|^2 \, ds$ is the *quadratic variation process* of X_t or M_t.

▷ **Exercise 6.** Let B_t be n-dimensional BM and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^2. Use Itô's formula to prove

$$f(B_t) - f(B_0) = \int_0^t \nabla f(B_s)^\mathsf{T} \, dB_s + \frac{1}{2} \int_0^t \Delta f(B_s) \, ds \, .$$

▷ **Exercise 7** (Exponential martingales)**.** Let $\theta(t,\omega) = (\theta_1(t,\omega), \ldots, \theta_n(t,\omega)) \in \mathbb{R}^n$ with $\theta_k(t,\omega) \in \mathcal{V}[0,T]$ for each k and some $T \le \infty$, and let $B(s)$ be Brownian motion in $\mathbb{R}^n$. Define

$$Z_t := \exp\left\{ \int_0^t \theta(s,\omega)^\mathsf{T} \, dB(s) - \frac{1}{2} \int_0^t \|\theta(s,\omega)\|^2 \, ds \right\} ; \qquad 0 \le t \le T \, .$$

(a) Use Itô's formula for a suitable $g(t, Y_t)$ to prove that $dZ_t = Z_t \theta(t,\omega) dB(t)$.

(b) Deduce that Z_t is a martingale for $t \le T$, provided that $Z_t \theta_k(t,\omega) \in \mathcal{V}[0,T]$ for each k.

Remark 1. A sufficient condition that Z_t be a martingale is the *Kazamaki condition*

$$\mathbf{E}\Big[\exp\Big(\frac{1}{2} \int_0^t \theta(s,\omega)^\mathsf{T} \, dB(s) \Big) \Big] < \infty \qquad \text{for all } t \le T \, .$$

This, in turn, is implied by the stronger *Novikov condition*

$$\mathbf{E}\Big[\exp\Big(\frac{1}{2} \int_0^T \|\theta(s,\omega)\|^2 \, ds \Big) \Big] < \infty \, .$$

Remark 2. The simplest discrete analogue of these exponential martingales is

$$M_n := 1 - \prod_{i=1}^n (1 - X_i) \, , \qquad M_0 = 0 \, ,$$

where the $X_i \in \{-1, 1\}$ are i.i.d. fair coin flips. This describes the *double the stake until you win* strategy, which almost certainly earns you money in a fair game, provided you have an unbounded credit.

▷ **Exercise 8.** In each of the cases below, find the process $f(t,\omega) \in \mathcal{V}[0,T]$ for Itô's representation, i.e., such that

$$F(\omega) = \mathbf{E}[\,F\,] + \int_0^T f(t,\omega)\,dB_t(\omega)\,.$$

(a) $F(\omega) = B_T(\omega)\,;$

(b) $F(\omega) = B_T(\omega)^2\,;$

(c) $F(\omega) = \int_0^T B_t(\omega)\,dt\,;$

(d) $F(\omega) = \sin B_T(\omega)\,.$

▷ **Exercise 9.** Given an $\mathcal{F}_T$-measurable variable Y with $\mathbf{E}[\,Y^2\,] < \infty$, consider the martingale $M_t = \mathbf{E}[\,Y \mid \mathcal{F}_t\,]$ for $t \in [0,T]$. According to the martingale representation theorem, there exists a unique process $g(t,\omega) \in \mathcal{V}[0,T]$ such that

$$M_t = M_0 + \int_0^t g(s,\omega)\,dB(s)\,, \qquad t \in [0,T]\,.$$

Find g in the following cases:
(a) $Y = B_T^2\,;$ **(b)** $Y = B_T^3\,;$ **(c)** $Y = e^{\sigma B_T}$ (hint: use that $e^{\sigma B_t - \sigma^2 t/2}$ is a martingale).

▷ **Exercise 10** (Bessel(d) processes)**.**

(a) Show that if X_t is an Itô process $dX_t(\omega) = \sum_{i=1}^d v_i(t,\omega)\,dB_i(t,\omega)$, then, in distribution, $dX_t = \big(\sum_{i=1}^d v_i(t)^2\big)^{1/2} dB(t)$.

(b) The previous was used in one of the lectures to show that the d-dimensional Bessel-squared process $Z(t) = \sum_{i=1}^d B_i(t)^2/2$ satisfies the equation $dZ(t) = \frac{d}{2}dt + \sqrt{2Z(t)}\,dB(t)$. From this, conclude that the Bessel(d) process $Y(t) = \sqrt{2Z(t)}$ satisfies $dY(t) = \frac{d-1}{2Y(t)} + dB(t)$.

(c) Show that d-dimensional Brownian motion is not recurrent for $d \geq 2$, while neighbourhood-recurrent for $d = 2$ and transient for $d \geq 3$. (Therefore, at least for $d = 2, 3, \ldots$, our equation makes sense for $Y(t)$ started at $Y(0) = 0$.)

▷ **Exercise 11** (Bonus on Bessel(3))**.**

(a) Consider simple random walk $\{X_t\}_{t\in\mathbb{N}}$ on $\{0, 1, \ldots, n\}$, started at $X_0 = i > 0$, conditioned to hit n before 0, stopped at n. Show that this is a Markov chain. Denote it by Y_t^n.

(b) Show that Y_t^n has a limit process Y_t as $n \to \infty$, a Markov chain on the state space $\mathbb{N}$. Show that this Y_t is transient. It may be called SRW on $\mathbb{N}$ conditioned on never hitting zero.

(c) What does Y_t have to do with the 3-dimensional Bessel process?