The Free Uniform Spanning Forest is disconnected in some virtually free groups, depending on the generating set

Gábor Pete
Alfréd Rényi Institute of Mathematics & Technical University, Budapest
http://www.math.bme.hu/∼gabor

Joint work with Ádám Timár
Rényi Institute
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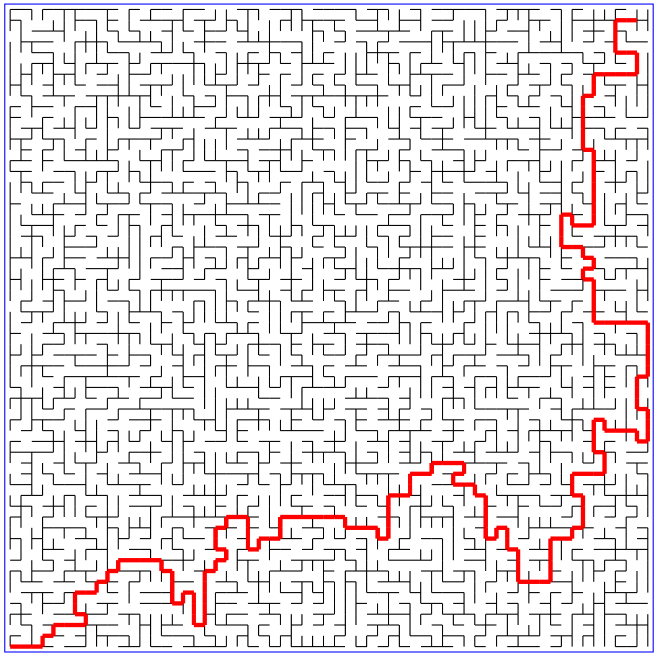
Outline of talk

- The Uniform Spanning Tree UST on finite graphs, and its connections to random walks and electric networks.
- Infinite volume limits: the Free and Wired Uniform Spanning Forests, FUSF and WUSF. Why are they natural and interesting?
- Our result: the FUSF behaves very unexpectedly on some tree-like graphs.
- Many open questions.

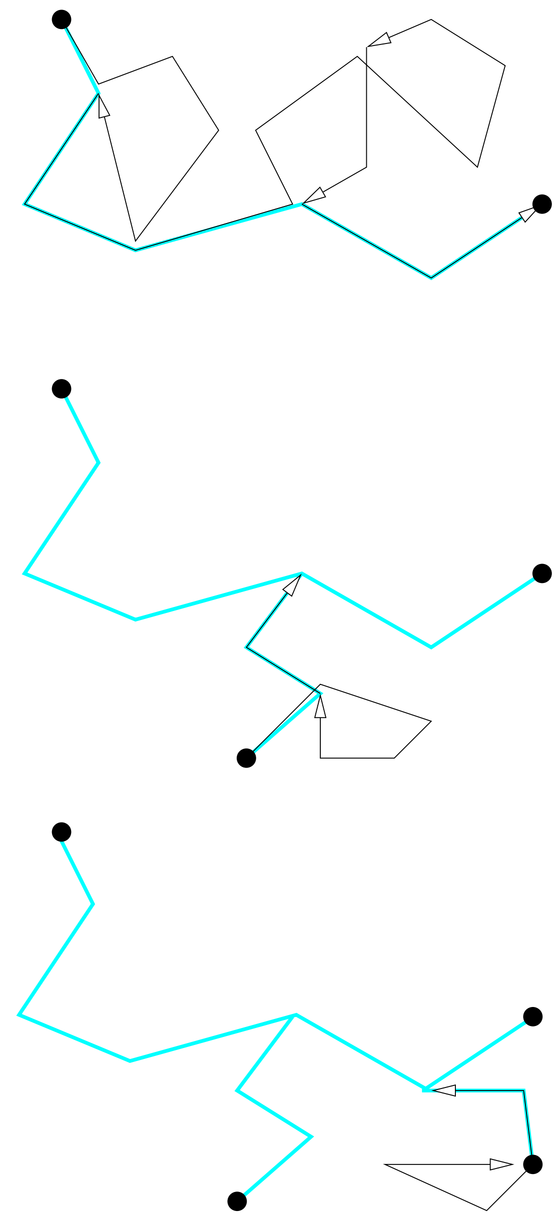
Uniform Spanning Tree (UST)



On a finite graph $G(V, E)$, from all spanning trees, take one uniformly at random.

Intimately related to electric networks (Kirchhoff 1847) and random walks (Aldous-Broder 1989 and Wilson 1996).

UST **and** LERW



David Wilson's algorithm (1996) to generate the UST via loop-erased random walks.

- Pick $x_0 \in V$.
- Pick $x_1 \in V$. Run random walk from x_1 until hitting x_0, and erase all the loops as they are created. Get a simple path T_1.
- Pick $x_2 \in V$. Run random walk from x_2 until hitting T_1. Get tree T_2.
- And so on, until all vertices are included.

Amazingly, we get the UST, regardless of how $x_0, x_1, x_2, \ldots$ are chosen.

UST **and electric networks**

Kirchhoff's Effective Resistance Formula (1847):

$$\mathbf{P}\big[\,(x,y)\in \mathsf{UST}\,\big] = i^{x,y}(x,y) = \mathcal{R}(x\leftrightarrow y),$$

where $\mathcal{R}(x\leftrightarrow y)$ is the effective resistance between x and y,
and $i = i^{x,y} : \overleftrightarrow{E} \longrightarrow \mathbb{R}$ is the current flow from x to y, total flow 1:

- **antisymmetric** for every $(u,v)\in\overleftrightarrow{E}$: $i(u,v) = -i(v,u)$,
- **node law** at every $u\in V\setminus\{x,y\}$: $\sum_{v\sim u} i(u,v) = 0$,
- **cycle law** for every cycle $C=(u_i)_{i=0}^t$: $\sum_{i=0}^t i(u_i,u_{i+1}) = 0$,
- **total flow**: $\sum_{v\sim x} i(x,v) = \sum_{v\sim y} i(v,y) = 1$.

Kirchhoff proved this with linear algebra: his Matrix-Tree Theorem.

Easy with Wilson's algorithm:
$\mathbf{P}\big[\,(x,y)\in \mathsf{UST}\,\big] = \mathbf{P}_x[\,\text{1st hit } y \text{ via } (x,y)\,] = \mathbf{E}_x[\,N_{x,y} - N_{y,x}\,] = i^{x,y}(x,y)$.

Generalization by Burton & Pemantle (1993), the Transfer Current Theorem:

$$\mathbf{P}\big[\, e_1, \ldots, e_k \in \mathsf{UST} \,\big] = \det \left[Y(e_i, e_j)_{i,j=1}^{k} \right],$$

where $Y(e, f) = i^e(f)$. So, the UST is a determinantal process.

Maybe surprisingly, $Y(e, f) = Y(f, e)$. Why? What is this matrix?

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Consider $\mathcal{H} := \ell^2_{\text{antisymm}}(\overleftrightarrow{E}, \mathbb{R})$ with standard inner product.
A basis: $\chi^{x,y} := \mathbf{1}_{(x,y)} - \mathbf{1}_{(y,x)}$, for all $(x,y) \in E$. Two subspaces:

$$\text{star space} \quad \bigstar := \text{span}\Big\{ \sum_{y \sim x} \chi^{x,y} : x \in V \Big\},$$

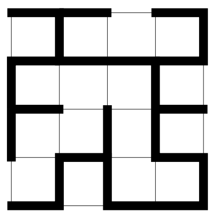
$$\text{cycle space} \quad \diamondsuit := \text{span}\Big\{ \sum_{i=0}^{t} \chi^{x_i, x_{i+1}} : \text{all oriented cycles } (x_i)_{i=0}^{t} \Big\}.$$

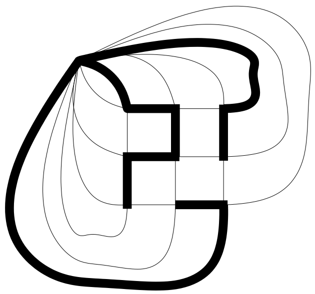
Node law: $\bigstar^\perp$. Cycle law: $\diamondsuit^\perp$. Easy: $\bigstar \oplus \diamondsuit = \mathcal{H}$ and $i^{x,y} = P_\bigstar \chi^{x,y}$.

Hence Y is the matrix of the projection $P_\bigstar$ in the basis $\{\chi^{x,y} : (x,y) \in E\}$.

Free and Wired Uniform Spanning Forests

On an infinite graph G, take exhaustion $G_n \uparrow G$ by finite subgraphs, maybe with some boundary conditions, and hope there is a weak limit.





On each G_n, free or wired UST.

Free: just take the subgraph.

Wired: collapse boundary vertices into one.

The limit would be a random spanning forest: cycles cannot appear in the limit, but connecting paths may get very long, disconnected in the limit.

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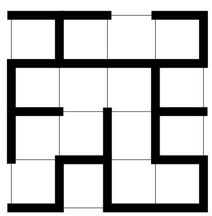

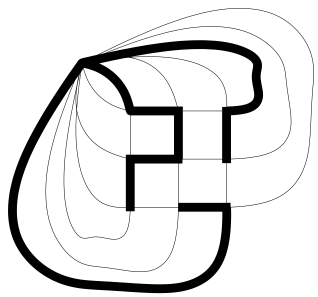

On each G_n, free or wired UST.

$\diamondsuit_n^{\mathsf{f}} \leqslant \diamondsuit_{n+1}^{\mathsf{f}} \leqslant \diamondsuit_G$.

$\bigstar_n^{\mathsf{w}} \leqslant \bigstar_{n+1}^{\mathsf{w}} \leqslant \bigstar_G$.

Thus $P_{\bigstar_n^{\mathsf{w}}} \uparrow P_{\bigstar}$ and $P_{\diamondsuit_n^{\mathsf{f}}}^{\perp} \downarrow P_{\diamondsuit}^{\perp}$, so $\mathsf{UST}_{G_n^{\mathsf{w}}} \uparrow \mathsf{WUSF}_G$ and $\mathsf{UST}_{G_n^{\mathsf{f}}} \downarrow \mathsf{FUSF}_G$. I.e., the limit forests exist, and are independent of the exhaustion.

But now $\mathcal{H}_G = \bigstar_G \oplus \diamondsuit_G \oplus \nabla \mathsf{HD}_G$, where HD_G is the set of harmonic functions $h : V \longrightarrow \mathbb{R}$ with finite Dirichlet energy: $\nabla h(x, y) := h(x) - h(y)$ lies in $\mathcal{H}_G$. Hence $\mathsf{HD}_G \neq \mathbb{R} \iff P_{\bigstar} < P_{\diamondsuit}^{\perp} \iff \mathsf{WUSF}_G \precneqq \mathsf{FUSF}_G$. This was observed by Benjamini, Lyons, Peres, Schramm (2001).

E.g., on $\mathbb{Z}^d$ (or any amenable transitive graph), WUSF = FUSF.
On transitive graphs with infinitely many ends, WUSF $\neq$ FUSF.

On any transient graph, WUSF can be generated by Wilson's algorithm rooted at infinity: "$x_0 = \infty$". This is immensely useful:

On $\mathbb{Z}^d$, the WUSF = FUSF is a single tree iff $d \leqslant 4$, by Pemantle '91.
For $d > 4$, two random walks avoid each other with positive probability.

But no good algorithm for FUSF. . . Need to take the limit.

FROM NOW ON, WE WILL ONLY TALK ABOUT TRANSITIVE GRAPHS.
In fact, Cayley graphs Cay(Γ, S) of finitely generated groups. Then both forests have translation-invariant laws, by independence on the exhaustion.

What properties depend only on the group?

(0) On any Cayley graph, $\mathbf{E}_{\mathsf{WUSF}}[\,\deg(o)\,] = 2$.

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(1) FUSF $>$ WUSF iff $\diamondsuit^{\perp} > \bigstar$, and the difference is given by ∇HD. Quantitatively: for $G = \mathsf{Cay}(\Gamma, S)$, $\mathbf{E}_{\mathsf{FUSF}(G)}[\deg(o)] = 2 + 2\dim_{\Gamma}(\mathsf{HD}_G)$, where $\dim_{\Gamma}$ is the Γ-invariant von Neumann dimension.

Just like in Hodge theory, $\dim_{\Gamma}(\mathsf{HD}_G)$ is the 1st ℓ^2-Betti number of the Cayley graph (Lyons 2009), which does not depend on the generating set: $\mathbf{E}_{\mathsf{FUSF}}[\deg(o)] = 2 + 2\beta_1^{(2)}(\Gamma)$.

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(4) Number of ends: 1 or 2 in the WUSF (Morris 2003).
If FUSF $\neq$ WUSF, then all the FUSF trees have continuum many ends (Hutchcroft-Nachmias 2017, Timár 2018).

Natural conjectures, and our answers

Lyons-Peres (2016): is the number of trees also in the FUSF independent of the generating set?

More generally, is it a quasi-isometry invariant for transitive graphs?

In the k-regular tree $\mathbb{T}^k$, the FUSF is obviously the entire tree.

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Theorem 1 (Timár-P.). For every d there is k_d such that if $k \geqslant k_d$, and H is a connected finite d-regular transitive graph on more than $k^{5/2}$ vertices, then the FUSF on $\mathbb{T}^k \times H$ has **infinitely many trees** a.s.

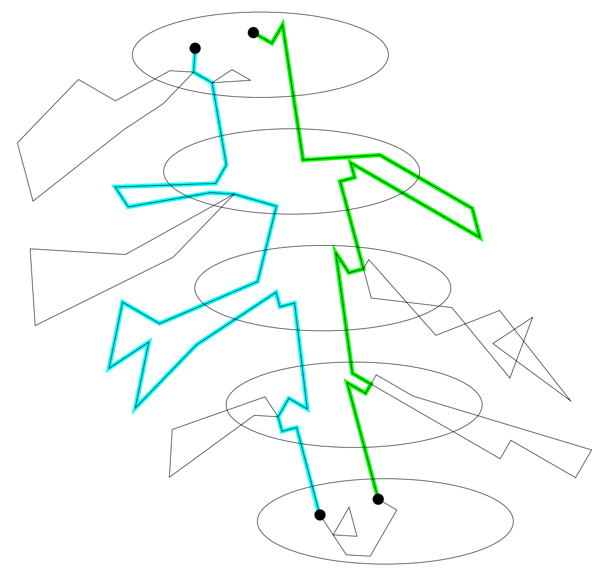
Theorem 2 (Timár-P.). For k large enough, the group $\mathbb{F}_k \times \mathbb{Z}_{k^8}$ has **a Cayley graph** (cycle in the 2nd coordinate) in which the FUSF has infinitely many components, and **another Cayley graph** (complete graph in the 2nd coordinate) in which the FUSF is connected.

Strategy of proof for Theorem 1

The ball and sphere of radius n around root $o \in \mathbb{T}^k$ will be T_n and S_n.

Generate UST in $T_n \times H$ by Wilson's algorithm, first taking LERW from $\mathfrak{a} = (o, h_{\mathfrak{a}})$ to $\mathfrak{b} = (o, h_{\mathfrak{b}})$. **GOAL:** the LERW contains some boundary vertex $(z, h_z) \in S_n \times H$ with a positive probability not depending on n.

Want a "leaf bag" $\{z\} \times H$ such that if π_{there} is the SRW path from $\mathfrak{a}$ to that bag, and π_{back} is the path back to $\mathfrak{b}$, then:
(1) No backtracking on ray of bags from o to z in π_{there} or π_{back}; **(2)** no intersections between π_{there} or π_{back} outside this ray; **(3)** nor inside.



To guarantee (1) and (2), will take k large, so that the set of bags that π_{there} enters outside the ray is *disjoint* from those of π_{back}.

To guarantee (3), we will take H large, so that there is enough space in each bag for π_{there} and π_{back} to avoid each other.

Strategy of proof for Theorem 1

Requirements (1) and (2) are only about SRW on $\mathbb{T}^k$. Closely related to:

Proposition (Sparsely visited rays). For the SRW excursion on T_n away from o, with uniformly positive probability, there exists a ray that is fully visited, but the local times are all bounded by $O(k)$.

For any $z \in S_n$, we show that $\mathbf{P}[\,\text{ray to } z \text{ is good}\,] \geqslant \left(\frac{c \log k}{k}\right)^n$.
Hence the expected number of good leafs is $> (c \log k)^n$.
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Get positive probability via the 2nd Moment Method.

Once we have disconnectedness with positive probability, also get infinitely many trees almost surely:

Proposition ($1 - \infty$ law). On any transitive tree-like graph, any invariant spanning forest that has only infinite trees and satisfies a weak insertion tolerance, has either 1 or ∞ many trees a.s.

Strategy of proof for Theorem 2 (dependence on the generating set)

1. When H is a cycle, long compared to k, then Theorem 1 applies: we get a disconnected FUSF.

2. Now take a generating set so that H is the complete graph.

This makes the SRW that generates the LERW spend a lot of time in each bag $\{v\} \times H$ before moving in the tree-coordinate, leaving long pieces of the LERW in the bag:

Lemma. Running SRW in K_n for time $\gg \sqrt{n}$, resulting LERW is $\asymp \sqrt{n}$.

This makes it very likely that the π_{there} and π_{back} trajectory pieces meet in that bag. Hence the loop-erasure erases every long excursion away from the root bag $\{o\} \times H$.

Open questions

Problem 1. If Γ is a finitely generated treeable group with WUSF $\neq$ FUSF, does it always have two generating sets such that the FUSF is disconnected in the first Cayley graph, while it is connected in the second?

Problem 2. Is it true that if the FUSF in some $\mathbb{T}^k \times H$ is disconnected, then any two components touch each other only at finitely many places?

The measurable cost of a group is the infimum of all the average degrees of *connected* invariant random graphs on the group. Gaboriau (2002) has shown that

$$\text{cost}(\Gamma) \geqslant \beta_1^{(2)}(\Gamma) + 1,$$

and asked if there is always equality.

When the FUSF can be made connected by adding a small density invariant bond percolation, then YES.

E.g., for infinite Kazhdan groups, $\beta_1^{(2)}(\Gamma) = 0$ has been known for long, but Hutchcroft & P (2020) showed only recently that $\text{cost}(\Gamma) = 1$.

Now that the FUSF can be disconnected when it "shouldn't be", maybe it is also difficult to make it connected:

Problem 3. For the FUSF in any $\mathbb{T}^k \times H$, is the union of the FUSF with an independent Bernoulli(ϵ) bond percolation connected, for any $\epsilon > 0$? If not, is there any invariant way to make the FUSF connected by adding an arbitrarily small density edge percolation?

$$\mathsf{disco}(H) := \inf \left\{ k : \mathsf{FUSF}(\mathbb{T}^k \times H) \text{ is disconnected} \right\} \in \{3, 4, \ldots, \infty\}.$$

We know that disco(P_2) $= \infty$ from Tang (2019), while Theorem 1 implies that if ℓ is large enough, then the cycle C_ℓ of length ℓ has disco(C_ℓ) $< \infty$.

Problem 4. What is the smallest ℓ for which disco(C_ℓ) $< \infty$? In particular, what is disco(C_3)?

Problem 5. Are there infinitely many finite graphs H with disco(H) $= \infty$?