Critical versus near-critical dynamics in the planar FK Ising model

Gábor Pete
http://www.math.toronto.edu/∼gabor

Joint work with Christophe Garban (ENS Lyon, CNRS)

Summary of talk

Together with Oded Schramm, we proved the existence and conformal covariance of the scaling limits of dynamical and near-critical percolation on the triangular lattice. These are closely related; both mechanisms (and hence the right space-time scalings) are governed by macroscopic pivotals.

Trying to generalize this to critical Ising and FK(2) on $\mathbb{Z}^2$, we found:

1. Dynamical FK(2) scaling limit works fine.

2. Near-critical FK(2) is **very different** from dynamical, with a different scaling. We can't prove a scaling limit.

3. Spin cluster evolution in dynamical Ising is completely mysterious.

4. Some results and conjectures on exceptional times with infinite clusters during the dynamics. E.g., we proved that, as opposed to percolation, there are no exceptional times in dynamical Ising.

The Ising and q-Potts models

Spin configuration $\sigma : V \longrightarrow \{1, \ldots, q\}$. For $q = 2$, usually $\{-1, +1\}$.

Hamiltonian: $H(\sigma) := \sum_{(x,y)\in E(\Gamma)} \mathbb{1}_{\{\sigma(x)\neq\sigma(y)\}}$.

For $\beta = 1/T \geqslant 0$ inverse temperature, Gibbs measure on configurations agreeing with some given boundary configuration ξ on $\partial V \subset V$:

$$\mathbf{P}_\beta^\xi[\sigma] := \frac{\exp(-\beta H(\sigma))}{Z_\beta^\xi}, \quad \text{where} \quad Z_\beta^\xi := \sum_{\sigma:\sigma|_{\partial V}=\xi} \exp(-\beta H(\sigma)).$$

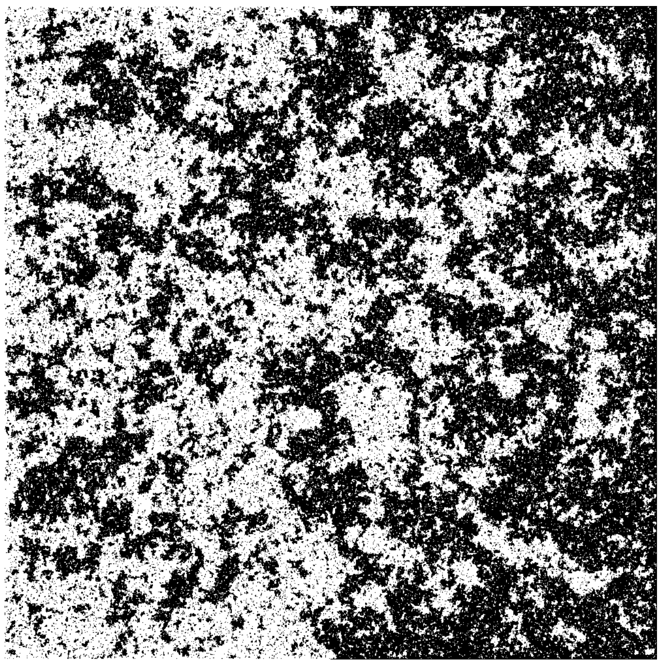
This Z_β is called the partition function.

Sometimes external field, favoring one kind of spin.

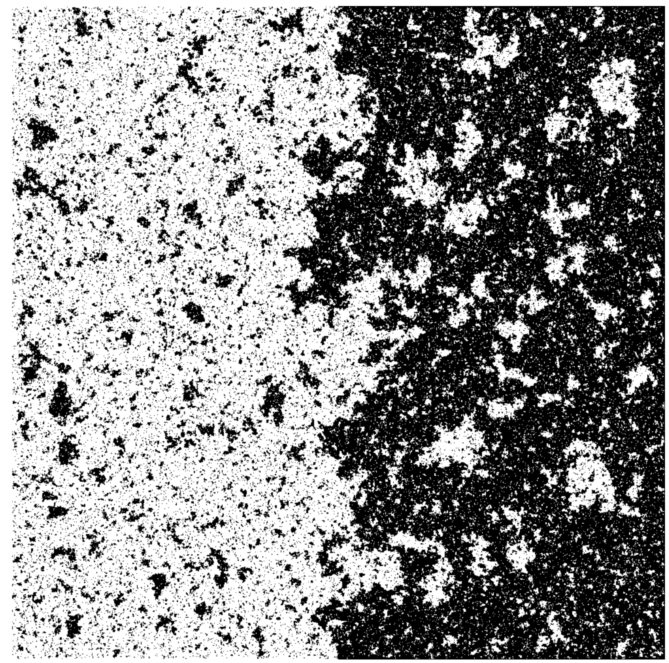
Instead, vary β now and look for change in decay of spin correlations.

Above critical β_c: non-uniqueness of infinite volume measures. Can be produced via different boundary conditions ξ.

The critical temperature of Ising



$\beta = 0.881374$



$\beta = 0.9$

Theorem (Onsager 1944, Aizenman-Barsky-Fernández 1987, Beffara-Duminil-Copin 2010). $\beta_c(\mathbb{Z}^2) = \frac{1}{2}\ln(1+\sqrt{2}) \approx 0.881374$.

Onsager also showed that $\mathbf{E}^{\xi}_{\beta_c}[\sigma(0)] \asymp n^{-1/8}$ for $\xi = +1_{\partial B_n(0)}$.

The random cluster model FK(p, q)

Fortuin-Kasteleyn (1969): for $\omega \in \{0,1\}^{E(G)}$ and $\xi \in \{0,1\}^{\partial E(G)}$ for $\partial E(G) \subset E(G)$,

$$\mathbf{P}^{\xi}_{\mathrm{FK}(p,q)}[\omega] = \frac{p^{|\omega|}\,(1-p)^{|E(G)\setminus\omega|}\,q^{|\mathrm{cl}(\omega)|}}{Z^{\xi}_{p,q}}.$$

$q = 1$: Bernoulli(p) bond percolation.
$q \to 0$, then $p \to 0$: Uniform Spanning Tree

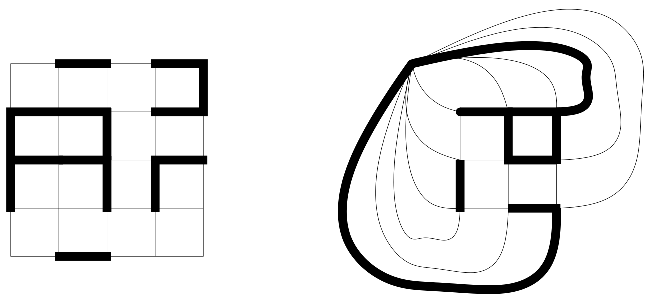
For $q \in \{2, 3, \dots\}$: color each cluster independently with one of q colors, then forget ω: get q-Potts, with $\beta = \beta(p) = -\frac{1}{2}\ln(1-p)$.

Therefore, Correl$^{\xi}_{\beta,q}[\sigma(x), \sigma(y)] = \mathbf{P}^{\xi}_{\mathrm{FK}(p,q)}[x \longleftrightarrow y]$!

If $q \geqslant 1$, then increasing events are positively correlated: FKG-inequality.

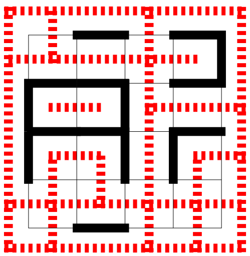
For $q < 1$, there should be negative correlations, proved only for UST, which is a determinantal process.

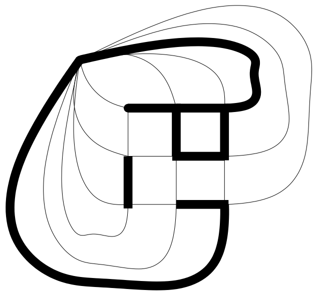
FK(p, q) on $\mathbb{Z}^2$



Exhaustion of $\mathbb{Z}^2$ by finite boxes, with free of wired boundaries. Limit measures exist, $\text{FK}^{\text{free}}(p, q)$ and $\text{FK}^{\text{wired}}(p, q)$.

FK(p,q) on $\mathbb{Z}^2$





Exhaustion of $\mathbb{Z}^2$ by finite boxes, with free of wired boundaries. Limit measures exist, FK$^{\text{free}}(p,q)$ and FK$^{\text{wired}}(p,q)$.

Self-dual point $p_{sd}(q) = \sqrt{q}/(1+\sqrt{q})$.

FK$^{\text{free}}(p,q)$ =FK$^{\text{wired}}(p,q)$ for all $p \neq p_{sd}(q)$. (Welsh '93, Grimmett '95)

Critical point $p_c(q)$: threshold for existence of infinite cluster.

Theorem (Beffara & Duminil-Copin 2010). $p_c(q) = p_{sd}(q)$ for $q \geqslant 1$.

At $p_{sd}(q=2)$: FK$^{\text{free}}$ =FK$^{\text{wired}}$, no percolation. (Conjectured for all $q \leqslant 4$.) (Simplest proof by W. Werner '09.)

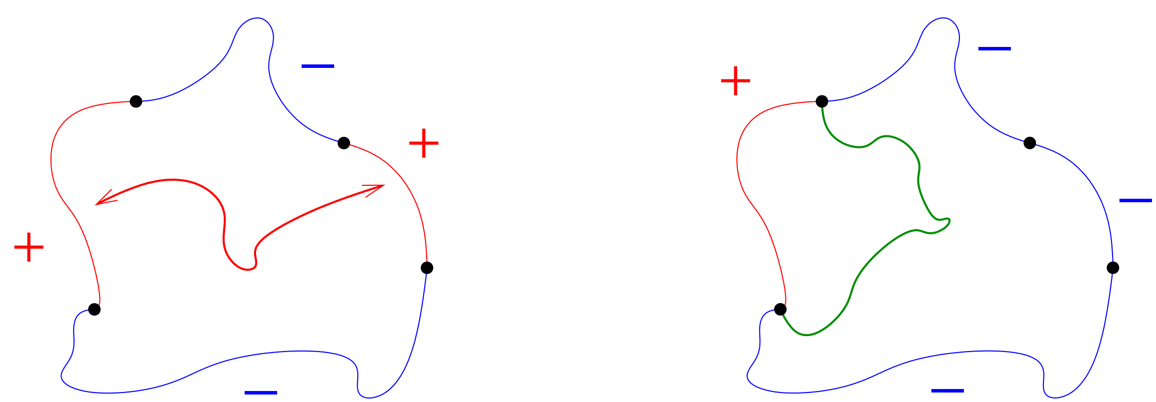
Russo-Seymour-Welsh theory

Proving criticality relies on proving box-crossing and RSW estimates.
For general $q \geqslant 1$, tricky gluing argument by Beffara & Duminil-Copin.
For $q = 2$, stronger results using Smirnov's conformal invariant observable:

Theorem (Duminil-Copin, Hongler & Nolin '09). At $p = p_c(2)$, with *any* boundary condition ξ around a piecewise smooth quad (D, a, b, c, d) with four marked boundary points, for any mesh $\eta > 0$,

$$0 < c_1 < \mathbf{P}^{\xi}_{\mathrm{FK}}\left[ab \longleftrightarrow cd \text{ in } D \cap \mathbb{Z}^2_\eta\right] < c_2 < 1\,.$$

This implies quad-crossing bounds in Ising with *certain* boundary conditions:

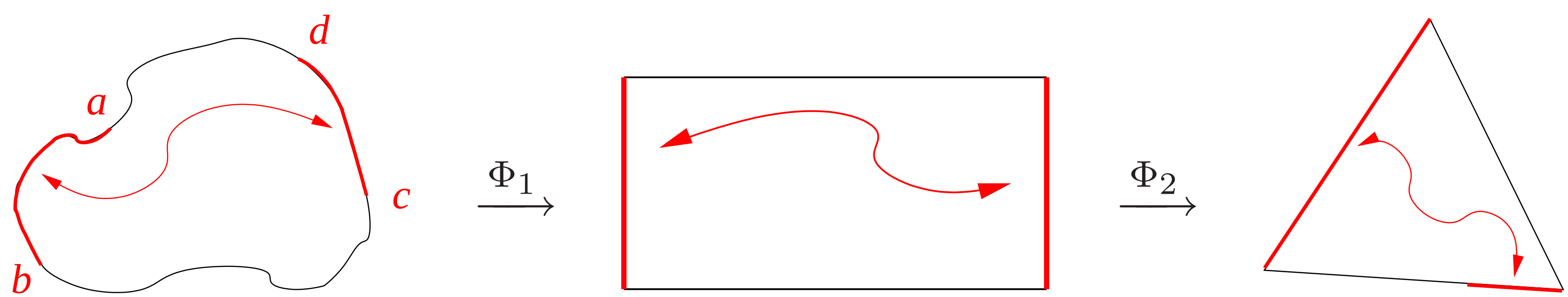


Conformal invariance at criticality

Theorem (Smirnov '01, Smirnov '07, Chelkak-Smirnov '10). For critical site percolation on Δ_η, and for FK$(p_c(2), 2)$ and β_c-Ising on a large class of graphs G_η, if $\mathcal{Q} \subset \mathbb{C}$ is a piecewise smooth quad, then

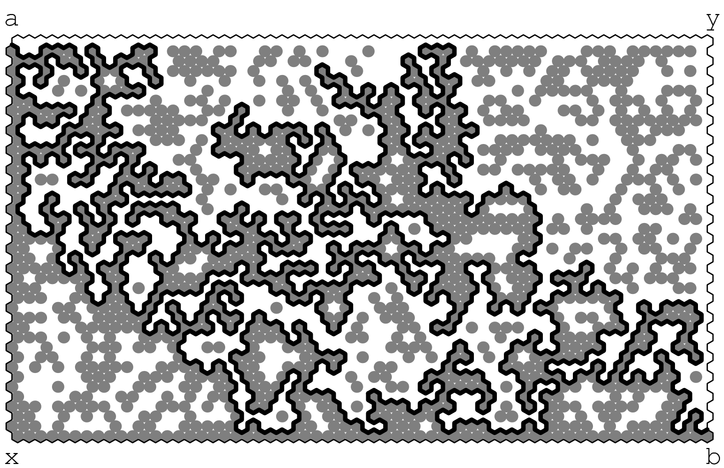
$$\lim_{\eta \to 0} \mathbf{P}\Big[ab \longleftrightarrow cd \text{ inside } \mathcal{Q} \cap G_\eta \Big]$$

exists, is strictly between 0 and 1, and conformally invariant.

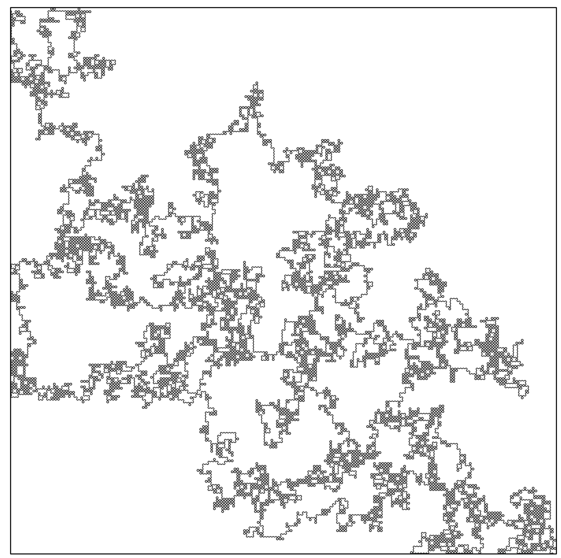


Moreover, there is a continuum scaling limit, encoding macroscopic connectivity structure, cluster boundaries, etc., Aizenman '95, Schramm '00, Camia-Newman '06, Sheffield '09. In physics, only correlation functions.

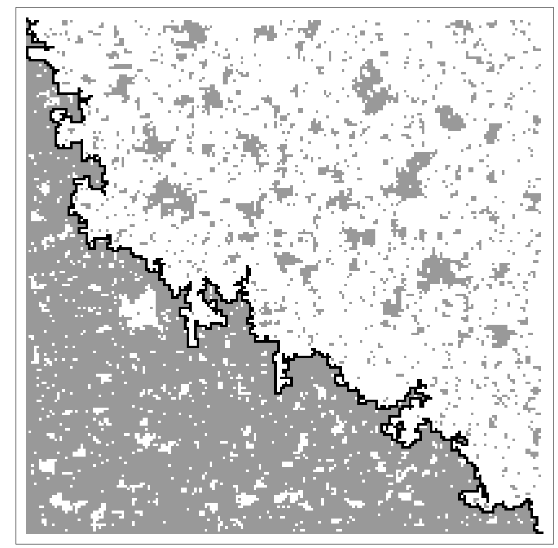
Schramm-Loewner Evolution SLE_κ



$q = 1,\ \kappa = 6$



$q = 2,\ \kappa = 16/3$



$\kappa = 3$

Conjecture. In FK$(p_c(q), q)$ for $0 \leqslant q \leqslant 4$, the scaling limit of the exploration path is SLE$_\kappa$, with $\kappa(q) = 4\pi / \arccos(-\sqrt{q}/2) \in [4, 8]$.

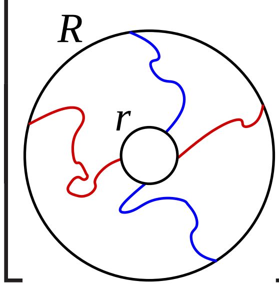
For the corresponding "outer-boundary type" models, we have $16/\kappa$.

Known for $q = 0, 1, 2$.

SLEs and critical exponents

Using the SLE_6 curve, several critical exponents can be computed (Lawler-Schramm-Werner, Smirnov-Werner 2001), e.g., $\alpha_1(r,R) = (r/R)^{5/48+o(1)}$,

$$\alpha_4(r,R) := \mathbf{P}\Big[$$



$$\Big] = (r/R)^{5/4+o(1)}. \quad (\rho_4 = 5/4)$$

Also implies off-critical exponent $\theta(p_c+\epsilon) = \epsilon^{5/36+o(1)}$, by Kesten (1987).

For discrete results from SLE: RSW crossing estimates $\Longrightarrow$ Separation of interfaces phenomenon $\Longrightarrow$ quasi-multiplicativity of arm probabilities. For percolation, done by Kesten (1987).

Garban (2011): $\alpha_4^{\mathrm{FK}(2)}(n) = n^{-35/24+o(1)}$ and $\alpha_4^{\mathrm{Ising}}(n) = n^{-21/8+o(1)}$.

$21/8 > 2 \implies$ no macroscopic pivotals in Ising $\leftrightarrow$ no self-touches of SLE_3.

Dynamical versions

Ising Glauber dynamics: i.i.d. Poisson clocks on vertices. When clock rings, update spin in a local manner: the conditional law of new spin depends only on the old spins in a bounded neighbourhood. Keep Ising stationary.

Example 1: Gibbs sampler = heat-bath dynamics. **Example 2**: Metropolis.

FK(p, q) heat-bath dynamics: i.i.d. Poisson clocks on edges. Not quite local stationary dynamics:

$$\mathbf{P}^{G}_{p,q}\big[e \text{ is on} \bigm| \omega \text{ on } G \setminus \{e\}\big] = \begin{cases} p & \text{if } \{x \overset{\omega}{\longleftrightarrow} y\} \text{ in } G \setminus \{e\} \\ \frac{p}{p+(1-p)q} & \text{otherwise.} \end{cases}$$

Open problem. Does this make sense on infinite $\mathbb{Z}^2$? (Information leaking from infinity?) Limits of dynamics on finite boxes do exist (using monotonicity, Grimmett 1995), but they are non-Fellerian processes. Are they given by these local transition rules?

Main dynamical questions

Question 1: How much time does it take to change macroscopic crossing events? (How noise sensitive are the crossing events? Complexity theory: "primitive" Boolean functions under iid measure are quite stable.)

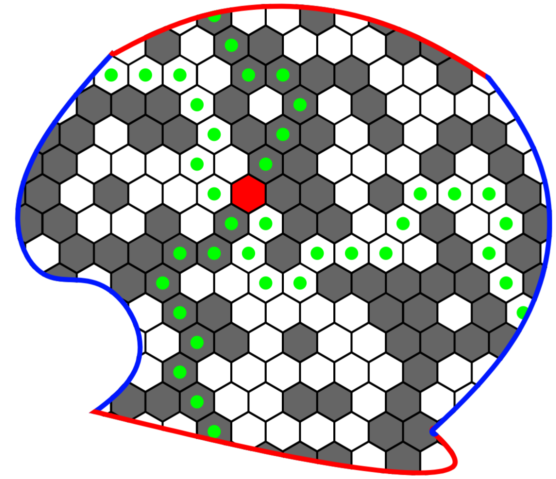
Question 2: On an infinite lattice, are there random times with exceptional behavior, e.g., an infinite cluster? (Dynamical sensitivity?)

The more noise-sensitive the system is, the more chances there are to see exceptional events.

Question 3: With a well-chosen rate $r(\eta)$ for the clocks, is there a scaling limit of the process, giving a Markov process on continuum configurations?

RW $\to$ BM: shrinking steps to η, need to speed up time: $r(\eta) = \eta^{-2}$.

Good time scale: pivotals



A site (or bond) is pivotal in ω if flipping it changes the existence of a left-right crossing. Equivalent to having alternating 4 arms. For nice quads, there are not many pivotals close to $\partial\mathcal{Q}$, hence $\mathbf{E}|\mathrm{Piv}_\eta| \asymp \eta^{-2}\,\alpha_4(\eta,1)$.

Taking $r(\eta) = 1/\mathbf{E}|\mathrm{Piv}_\eta|$, the expected number of pivotal switches in unit time will be about 1, so let's fix that.

Short time: small expectation $\implies$ $\mathbf{P}[\exists$ pivotal switch$]$ is small, so things don't change.

Long time: large expectation $\not\Longrightarrow$ probability. But with a second moment argument ($\rho_4 < 2$), at this scale things do start changing, great.

But do crossing events completely decorrelate after long time? YES, but it's hard, and needs Fourier analysis w.r.t. product measure on hypercube: Benjamini-Kalai-Scramm '98, Schramm-Steif '05, Garban-P.-Schramm '08.

Dynamical percolation and FK(2) scaling limits

Theorem (GPS 2010-11). On Δ_η, with $r(\eta) = 1/\mathbf{E}|\mathrm{Piv}_\eta| = \eta^{3/4+o(1)}$ clocks, the scaling limit of dynamical percolation exists, is Markov, and conformally covariant: if domain changes by $\phi(z)$, then time scales locally by $|\phi'(z)|^{3/4}$. By GPS '08, the process is ergodic ($t^{-2/3}$ correlation decay).

Proof. Step 0. Work in quad-crossing description of the full scaling limit (Schramm-Smirnov '10), uniqueness following from Camia-Newman '06.

Step 1. Scaling limit of counting measure on macroscopic ρ-important pivotals exists: pivotal measures $\mu^\rho(\omega)$, measurable w.r.t. continuum percolation, conformally covariant, with exponent 3/4.

So, can hope that scaling limit of dynamics is given by $\omega_{t=0}$ plus a "filtered" Poisson point process $(\mathscr{P}^\rho)_{\rho>0}$ of flips from μ^ρ(domain) $\times$ Lebesgue(time). This was suggested by Camia-Fontes-Newman '06.

Step 2. Stability: no new macroscopic info from invisible scales: originally unimportant points do not become important *and* then switch.

Scaling limits for FK(2) heat-bath and Ising Glauber?

Theorem (Garban-P. 2011). Assuming uniqueness of the quad-crossing full scaling limit for FK$(p_c(2), 2)$, on any compact $D \cap \mathbb{Z}^2_\eta$, with $r(\eta) = 1/\mathbf{E}|\mathrm{Piv}_\eta| = \eta^{13/24+o(1)}$ clocks, the scaling limit of the heat bath dynamics exists, is Markov, and conformally covariant with exponent $13/24$.

For quad-crossings by spin clusters in the Ising Glauber model, situation is very unclear: because of $\rho_4 > 2$, dynamical second moment argument for pivotal switches doesn't work, hence even the right time scale is unclear, maybe need more than $n^2\alpha_4(n)$.

And even if the right time scale is given by $\alpha_4(n)$, more small pivotals are switching than big ones, hence stability (no cascade of information from microscopic scales) becomes unclear.

Maybe it's "physically irrelevant" anyway. . . Though mixing time is not given by magnetization, either. . . And once we have natural dynamics on CLE$_6$ and CLE$_{16/3}$, why not on CLE$_3$?

The near-critical ensemble in percolation

Standard coupling: to each site (or bond) $x \in G$, assign $V(x)$ i.i.d. Unif$[0, 1]$, and let x be **open at level** p if $V(x) \leqslant p$.

Dynamical version: starting from criticality, whenever a clock rings, switch to **open**. So, at time t, each site is open with probability $\sim 1/2 + t\, r(\eta)$, with our old $r(\eta)$. May also take $t < 0$, bias towards $closed$.

Super/sub-critical as $t \to \pm\infty$. But maybe changes are faster because of monotonicity? Could critical window be smaller than the dynamical?

Kesten (1987): Multi-arm probabilities stay comparable inside window! Above window, already supercritical. (Doesn't need Fourier analysis.)

Borgs-Chayes-Kesten-Spencer (2001): Finite size versions of previous.

Nolin-Werner (2008): Subsequential limits of the near-critical interface exist, and are singular w.r.t. the critical interface SLE_6.

GPS (2010-11): Scaling limit of near-critical ensemble exists, etc.

The correlation length in near-critical percolation

How did Kesten find the off-critical exponent $\theta(p_c+\epsilon) \approx \epsilon^\beta$, with $\beta = \frac{\rho_1}{2-\rho_4}$?

Stability: Fixed domain, mesh η, $p = p_c + \epsilon$ with $\epsilon \leqslant C\eta^{2-\rho_4} = \eta^{3/4+o(1)}$ — we are still basically critical. So, at $p_c + \epsilon$ and unit mesh, critical in domains of size $\epsilon^{-4/3+o(1)}$, and supercritical in larger domains.

This $\epsilon^{-1/(2-\rho_4)} = \epsilon^{-4/3+o(1)}$ is called the correlation length.

His proof of stability used Russo's formula for $\frac{d}{dp}\alpha_1^p(n)$ and $\frac{d}{dp}\alpha_4^p(n)$. Also follows from our dynamical stability argument (Step 2 above).

Therefore, to have $0 \longleftrightarrow \infty$ at $p_c + \epsilon$, need

1. $0 \longleftrightarrow \epsilon^{-4/3+o(1)}$, happening with the critical probability $\alpha_1(1, \epsilon^{-1/(2-\rho_4)}) = \epsilon^{\rho_1/(2-\rho_4)}$;
2. $\epsilon^{-4/3+o(1)} \longleftrightarrow \infty$, happening with the supercritical probability $\asymp 1$,

and done.

The near-critical ensemble in FK(p, q)

Want a monotone coupling as p varies, i.e., random $Z \in [0,1]^{E(G)}$ labeling such that $Z_{\leqslant p} \subset E(G)$ is FK(p,q), preferably Markov in p. Asymmetric heat-bath is not good. Instead, Grimmett '95: define a Markov chain Z_t on labelings with the right stationary measure.

Set $T_e(Z) := \inf \big\{ p : \text{endpoints of } e \text{ are connected in } Z_{\leqslant p} \setminus \{e\} \big\}$.

If e rings at time t, then, to get the right conditional distribution on e in $Z_{\leqslant p}$, need

$$\mathbf{P}[Z_t(e) \leqslant p] = \begin{cases} p & \text{if } p \geqslant T_e(Z_{t-}) \\ \frac{p}{p+(1-p)q} & \text{if } p < T_e(Z_{t-}). \end{cases}$$

We can get this simultaneously for all p by defining this update rule for $Z_t(e)$. Makes sense if $q \geqslant 1$. Note Dirac point mass at $T_e(Z_{t-})$.

First difference from asymmetric heat-bath: from specific heat (variance of energy) computation on $\mathbb{Z}^2$, density of edges in $Z_{\leqslant p_c+\epsilon} \setminus Z_{\leqslant p_c}$ is not $\asymp \epsilon$, but $\epsilon \log(1/\epsilon)$ for $q = 2$, and polynomial blowup for $q > 2$.

Onsager vs pivotals

From Onsager '44 magnetization results: $\mathbf{P}^{\mathbb{Z}^2}_{p_c(2),2}\big[0 \longleftrightarrow R\big] = R^{-1/8+o(1)}$ and $\mathbf{P}^{\mathbb{Z}^2}_{p_c(2)+\epsilon,2}\big[0 \longleftrightarrow \infty\big] = \epsilon^{1/8+o(1)}$. This gives a correlation length $\epsilon^{1+o(1)}$. But Garban computed $1/(2-\rho_4) = 24/13$, which is much larger!

1. Correlation length is not given by amount of pivotals at criticality.

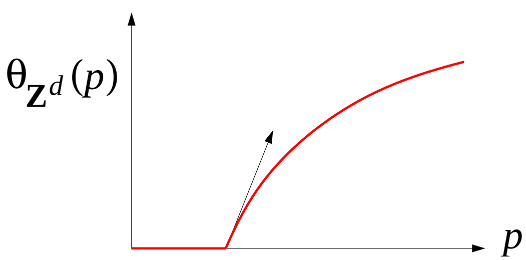
2. Near-critical window is much shorter than dynamical window.

3. Asymmetric heat bath is **very** different from the monotone coupling. When raising p in the monotone coupling, open bonds do not arrive in a uniform, Poissonian way, but with self-organization, to create more pivotals and build long connections. Would contradict Markov property in p, unless there are clouds of open bonds appearing together.

We don't understand geometry of clouds, but at least can see directly that they are happening, due to the Dirac mass in the update rule. Intuitively: good to open many edges together, without lowering number of clusters.

Exceptional times with infinite clusters

Häggström-Peres-Steif '97: No exceptional times for Bernoulli($p \neq p_c$). No exceptional times at $p = p_c$ for bond percolation on $\mathbb{Z}^d$, $d \geqslant 19$.



The latter is essentially due to Hara-Slade '90 on the off-critical exponent $\beta = 1$: even switching asymmetrically, $\mathbf{E}\big[$number of ϵ-subintervals of $[0,1]$ with exceptional times$\big] = O(1)$. But the exceptional set is closed without isolated points.

Garban-P.-Schramm '08: On Δ or $\mathbb{Z}^2$, Hausdorff dimension of exceptional times is a.s. $1 - \beta = 1 - \frac{\rho_1}{2-\rho_4}$, which is $31/36$ on Δ, and positive on $\mathbb{Z}^2$.

Garban-P. '11: For the Ising Glauber dynamics, no exceptional times for infinite spin clusters (even with *-connections), due to having few pivotals.

Moreover, assuming $\mathrm{SLE}_{\kappa(q)}$ conjectures, no exceptional times for $q \in (q^*, 4)$, (i.e., $\kappa \in (4, \kappa^*)$), despite having many macroscopic pivotals (meaning noise sensitivity?), since dimension $\leqslant 1 - \rho_1/(2-\rho_4) < 0$.