The near-critical planar FK-Ising model

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arXiv:1111.0144

The Ising and q-Potts models

Spin configuration $\sigma : V \longrightarrow \{1, \ldots, q\}$. For $q = 2$, usually $\{-1, +1\}$.

Hamiltonian: $H(\sigma) := \sum_{(x,y)\in E(G)} \mathbb{1}_{\{\sigma(x)\neq\sigma(y)\}}$.

For $\beta = 1/T \geqslant 0$ inverse temperature, Gibbs measure on configurations agreeing with some given boundary configuration ξ on $\partial V \subset V$:

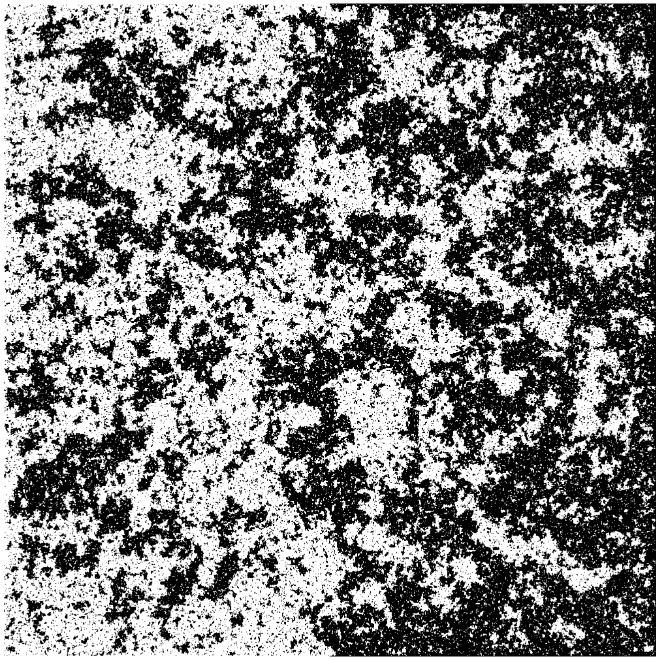
$$\mathbf{P}^{\xi}_{\beta}[\sigma] := \frac{\exp(-\beta H(\sigma))}{Z^{\xi}_{\beta}}, \quad \text{where} \quad Z^{\xi}_{\beta} := \sum_{\sigma:\sigma|_{\partial V}=\xi} \exp(-\beta H(\sigma)) .$$

This Z_β is called the partition function.

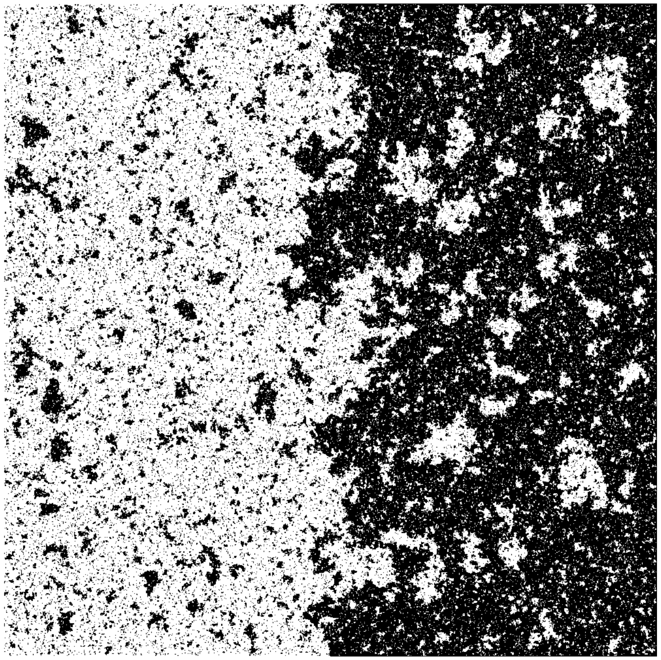
Sometimes external field, favoring one kind of spin.

But it's more interesting to vary β: decay of correlations? Effect of ξ?

The critical temperature of Ising



$\beta = 0.881374$



$\beta = 0.9$

Theorem (Onsager 1944, Aizenman-Barsky-Fernández 1987, Beffara-Duminil-Copin 2010). $\beta_c(\mathbb{Z}^2) = \ln(1+\sqrt{2}) \approx 0.881374.$

Onsager also showed that $\mathbf{E}^{\xi}_{\beta_c}[\sigma(0)] = n^{-1/8+o(1)}$ for $\xi = +1_{\partial B_n(0)}$.

The random cluster model FK(p, q)

Fortuin-Kasteleyn (1969): for $\omega \in \{0,1\}^{E(G)}$ and $\xi \in \{0,1\}^{\partial E(G)}$ for $\partial E(G) \subset E(G)$,

$$\mathbf{P}^{\xi}_{\mathrm{FK}(p,q)}[\omega] = \frac{p^{|\omega|}\,(1-p)^{|E(G)\setminus\omega|}\,q^{|\mathrm{clusters}(\omega)|}}{Z^{\xi}_{\mathrm{FK}(p,q)}}.$$

$q = 1$: Bernoulli(p) bond percolation. $q \to 0$, then $p \to 0$: UST

For $q \in \{2, 3, \ldots\}$, Edwards-Sokal coupling: color each cluster independently with one of q colors, then forget ω: get q-Potts, with $\beta = \beta(p) = -\ln(1-p)$. Partition functions are equal: $Z_{\mathrm{FK}(p,q)} = Z_{\beta(p),q}$.

Therefore, Correl$^{\xi}_{\beta,q}[\sigma(x), \sigma(y)] = \mathbf{P}^{\xi}_{\mathrm{FK}(p,q)}[x \longleftrightarrow y]$!

If $q \geqslant 1$, then increasing events are positively correlated: FKG-inequality.

For $q < 1$, there should be negative correlations, proved only for UST, which is a determinantal process.

Critical spin-Ising and FK-Ising on $\mathbb{Z}^2$

Fermonic observables, conformal invariance, convergence to SLE_3, $\text{SLE}_{16/3}$: Smirnov '06, '10, Chelkak-Smirnov '10, Kemppainen-Smirnov '11, etc.

FK-Ising RSW estimates for rectangles by Duminil-Copin-Hongler-Nolin '10.

Separation of interfaces, quasi-multiplicativity of arm probabilities, pivotal exponents by Duminil-Copin & Garban '12?:

$$\alpha_4^{\mathrm{FK}(2)}(n) = n^{-35/24+o(1)} \text{ and } \alpha_4^{\mathrm{Ising}}(n) = n^{-21/8+o(1)}.$$

The FK(p,q) heat-bath dynamics

I.i.d. Poisson clocks on edges. Not quite local stationary dynamics:

$$\mathbf{P}^G_{p,q}\big[e \text{ is on } \big| \, \omega \text{ on } G \setminus \{e\}\big] = \begin{cases} p & \text{if } \{x \overset{\omega}{\longleftrightarrow} y\} \text{ in } G \setminus \{e\} \\ \frac{p}{p+(1-p)q} & \text{otherwise.} \end{cases}$$

Open problem. Does this make sense on infinite $\mathbb{Z}^2$? (Information leaking from infinity?) Limits of dynamics on finite boxes do exist (using monotonicity, Grimmett 1995), but they are non-Fellerian processes. Are they given by these local transition rules?

The near-critical ensemble in FK(p,q)

Want a monotone coupling as p varies, i.e., random $Z \in [0,1]^{E(G)}$ labeling such that $Z_{\leqslant p} \subset E(G)$ is FK(p,q), preferably Markov in p. Asymmetric heat-bath is not good. Instead, Grimmett '95: define a Markov chain Z_t on labelings with the right stationary measure.

Set $T_e(Z) := \inf \big\{ p : \text{endpoints of } e \text{ are connected in } Z_{\leqslant p} \setminus \{e\} \big\}$.

If e rings at time t, then, to get the right conditional distribution on e in $Z_{\leqslant p}$, need

$$\mathbf{P}[Z_t(e) \leqslant p] = \begin{cases} p & \text{if } p \geqslant T_e(Z_{t-}) \\ \frac{p}{p+(1-p)q} & \text{if } p < T_e(Z_{t-}). \end{cases}$$

We can get this simultaneously for all p by defining this update rule for $Z_t(e)$. Makes sense if $q \geqslant 1$. Note Dirac point mass at $T_e(Z_{t-})$.

First difference from asymmetric heat-bath: from specific heat (variance of energy) computation on $\mathbb{Z}^2$, density of edges in $Z_{\leqslant p_c+\epsilon} \setminus Z_{\leqslant p_c}$ is not $\asymp \epsilon$, but $\epsilon \log(1/\epsilon)$ for $q=2$, and polynomial blowup for $q>2$.

Onsager vs pivotals

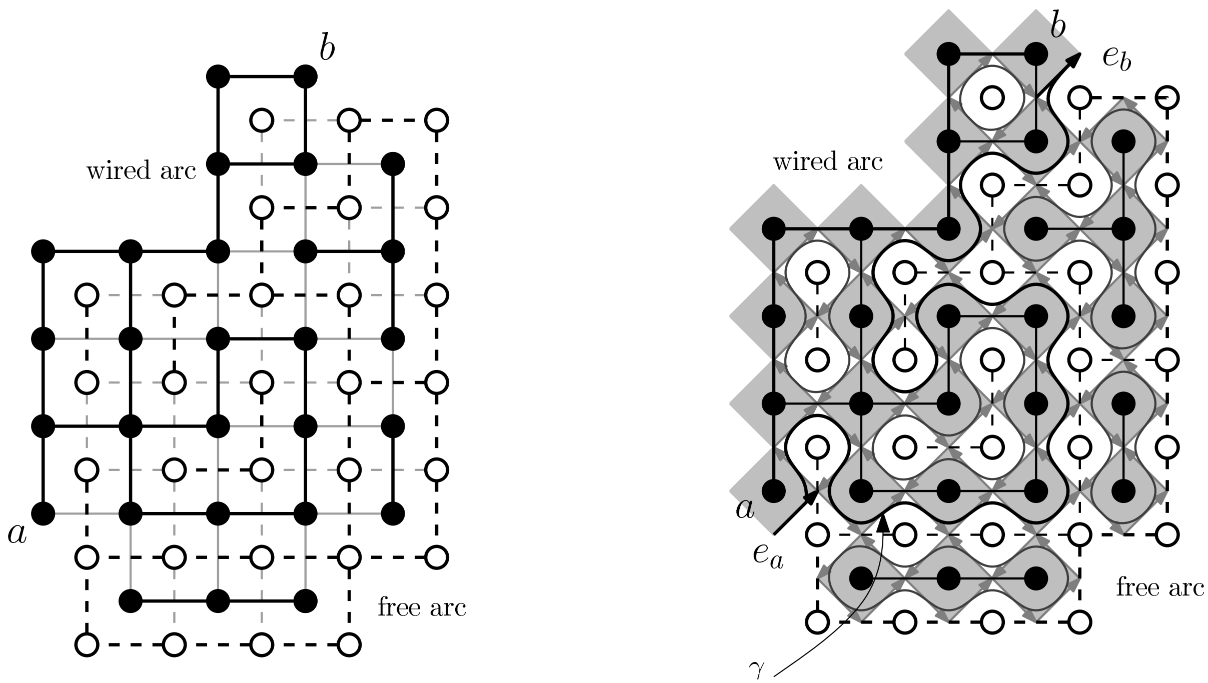
From Onsager '44 magnetization results: $\mathbf{P}^{\mathbb{Z}^2}_{p_c(2),2}\big[0 \longleftrightarrow R\big] = R^{-1/8+o(1)}$ and $\mathbf{P}^{\mathbb{Z}^2}_{p_c(2)+\epsilon,2}\big[0 \longleftrightarrow \infty\big] = \epsilon^{1/8+o(1)}$. This gives a correlation length $\epsilon^{1+o(1)}$. But DC & G computed $1/(2-\xi_4) = 24/13$, which is much larger!

Hence, correlation length is not given by amount of pivotals at criticality. Stability in near-critical window fails, the changes are faster. How come?

Conclusion: *Any* monotone coupling is **very** different from asymmetric heat bath. When raising p in the monotone coupling, open bonds do not arrive in a uniform, Poissonian way, but with self-organization, to create more pivotals and build long connections. Would contradict Markov property in p, unless there are clouds of open bonds appearing together.

We don't understand geometry of clouds, but at least can see directly that they are happening, due to the Dirac mass in the update rule. Intuitively: good to open many edges together, without lowering number of clusters.

Computing the correlation length



Smirnov's fermonic observable $F = F_p$ for any medial edge $e \in E_\diamond$:

$$F(e) := \mathbf{E}_{p,2}^{G,a,b}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}W_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}\right),$$

where γ is the exploration interface from a to b, and W_γ is the winding.

Relation to connectivity: if $u \in G$ is a site next to the free arc, and e is the appropriate medial edge next to it, then $|F(e)| = \mathbf{P}_{p,2}^{G,a,b}(u \leftrightarrow \text{wired arc})$.

Massive harmonicity (Beffara-Duminil-Copin): if X has four neighbors in $G \setminus \partial G$, then $\Delta_p F(e_X) = 0$, where the operator Δ_p is

$$\Delta_p g(e_X) := \frac{\cos[2\alpha]}{4} \left(\sum_{Y \sim X} g(e_Y) \right) - g(e_X),$$

with some $\alpha = \alpha(p)$, equalling 0 iff $p = p_c$.

Complicated boundary conditions. But, at p_c, $H(e^+) - H(e^-) := |F(e)|^2$, this H approximately solves a discrete Dirichlet boundary problem, hence $\mathbf{P}_{p_c,2}^{G,a,b}(u \leftrightarrow \text{wired arc}) \simeq (\text{harmonic measure of wired arc seen from } u)^{1/2}$, and can compute that crossing probabilities are between 0 and 1.

At $p \neq p_c$, need harmonic measure w.r.t. massive random walk, killing particle at each step with probability depending on $\cos(2\alpha)$, roughly $|p - p_c|^2$. $|p - p_c| < \frac{c}{n}$: during the roughly n^2 steps to boundary, particles dies with probability bounded away from 1, so everything is roughly the same.