ON THE HADWIGER NUMBERS OF CENTRALLY SYMMETRIC STARLIKE DISKS

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Abstract. The Hadwiger number $H(S)$ of a topological disk S in $\mathbb{R}^2$ is the maximal number of pairwise nonoverlapping translates of S that touch S. A conjecture of A. Bezdek., K. and W. Kuperberg [2] states that this number is at most eight for any starlike disk. A. Bezdek [1] proved that the Hadwiger number of a starlike disk is at most seventy five. In this note, we prove that the Hadwiger number of any centrally symmetric starlike disk is at most twelve.
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1. Introduction and Preliminaries

This paper deals with topological disks in the Euclidean plane $\mathbb{R}^2$. We make use of the linear structure of $\mathbb{R}^2$, and identify a point with its position vector. We denote the origin by o.

A *topological disk*, or shortly *disk*, is a compact subset of $\mathbb{R}^2$ with a simple, closed, continuous curve as its boundary. Two disks S_1 and S_2 are *nonoverlapping*, if their interiors are disjoint. If S_1 and S_2 are nonoverlapping and $S_1 \cap S_2 \neq \emptyset$, then S_1 and S_2 *touch*. A disk S is *starlike relative to a point* p, if, for every $q \in S$, S contains the closed segment with endpoints p and q. In particular, a convex disk C is starlike relative to any point $p \in C$. A disk S is *centrally symmetric*, if $-S$ is a translate of S. If $-S = S$, then S is *o-symmetric*.

The *Hadwiger number*, or *translative kissing number*, of a disk S is the maximal number of pairwise nonoverlapping translates of S that touch S. The Hadwiger number of S is denoted by $H(S)$. It is well known (cf. [8]) that the Hadwiger number of a parallelogram is eight, and the Hadwiger number of any other convex disk is six. In [9], the authors showed that the Hadwiger number of a disk is at least six. Recently, Cheong and Lee [4] constructed, for every $n > 0$, a disk with Hadwiger number at least n.

A. Bezdek, K. and W. Kuperberg [2] conjectured that the Hadwiger number of any starlike disk is at most eight (see also Conjecture 6, p. 95 in the book [3] of

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Brass, Moser and Pach). The only result regarding this conjecture is due to A. Bezdek, who proved in [1] that the Hadwiger number of a starlike disk is at most seventy five. Our goal is to prove the following theorem.

Theorem. *Let S be a centrally symmetric starlike disk. Then the Hadwiger number $H(S)$ of S is at most twelve.*

In the proof, Greek letters, small Latin letters and capital Latin letters denote real numbers, points and sets of points, respectively. For $u, v \in \mathbb{R}^2$, the symbol $\operatorname{dist}(u, v)$ denotes the *Euclidean distance* of u and v. For simplicity, we introduce a Cartesian coordinate system and, for a point $u \in \mathbb{R}^2$ with x-coordinate α and y-coordinate β, we may write $u = (\alpha, \beta)$. The *closed segment* (respectively, *open segment*) with endpoints u and v is denoted by $[u, v]$ (respectively, by (u, v)). For a subset A of $\mathbb{R}^2$, $\operatorname{int} A$, $\operatorname{bd} A$, $\operatorname{card} A$ and $\operatorname{conv} A$ denotes the *interior*, the *boundary*, the *cardinality* and the *convex hull* of A, respectively.

Consider a convex disk C and two points $p, q \in \mathbb{R}^2$. Let $[t, s]$ be a chord of C, parallel to $[p, q]$, such that $\operatorname{dist}(s, t) \geq \operatorname{dist}(s', t')$ for any chord $[s', t']$ of C parallel to $[p, q]$. The *C-distance* $\operatorname{dist}_C(p, q)$ of p and q is defined as

$$\operatorname{dist}_C(p, q) = \frac{2 \operatorname{dist}(p, q)}{\operatorname{dist}(s, t)}.$$

For the definition of C-distance, see also [10]. It is well known that the C-distance of p and q is equal to the distance of p and q in the normed plane with unit disk $\frac{1}{2}(C - C)$. The o-symmetric convex disk $\frac{1}{2}(C - C)$ is called the *central symmetral* of C. We note that $C \subset C'$ yields $\operatorname{dist}_C(p, q) \geq \operatorname{dist}_{C'}(p, q)$ for any $p, q \in \mathbb{R}^2$.

We prove the theorem in Section 2. During the proof we present two remarks, showing that as we broaden our knowledge of S, we are able to prove better and better upper bounds on its Hadwiger number.

2. Proof of the theorem

Let S be an o-symmetric starlike disk. Let $\mathfrak{F} = \{S_i : i = 1, 2, \ldots, n\}$ be a family of translates of S such that $n = H(S)$ and, for $i = 1, 2, \ldots, n$, $S_i = c_i + S$ touches S and does not overlap with any other element of $\mathfrak{F}$. Let $K = \operatorname{conv} S$, $X = \{c_i : i = 1, 2, \ldots, n\}$, $C = \operatorname{conv} X$ and $\bar{C} = \operatorname{conv}\big(X \cup (-X)\big)$. Furthermore, let R_i denote the closed ray $R_i = \{\lambda c_i : \lambda \in \mathbb{R} \text{ and } \lambda \geq 0\}$.

First, we prove a few lemmas.

Lemma 1. *The disk S is starlike relative to the origin o. Furthermore, $o \in \operatorname{int} S$.*

Proof. Let S be starlike relative to $p \in S$, and assume that $p \neq o$. By symmetry, S is starlike relative to $-p$. Consider a point $q \in S$. Since S is starlike relative to p and $-p$, the segments $[p, q]$ and $[-p, q]$ are contained in S. Thus, any segment $[p, r]$, where $r \in [-p, q]$, is contained in S. In other words, we have $\operatorname{conv}\{p, -p, q\} \subset S$, which yields that $[o, q] \subset S$. The second assertion follows from the first and the symmetry of S. □

Lemma 2. *If $x + S$ and $y + S$ are nonoverlapping translates of S, then we have $\operatorname{dist}_K(x, y) \geq 1$.*

Proof. Without loss of generality, we may assume that $x = o$. Suppose that $y \in \operatorname{int} K$. Note that there are points $p, q \in S$ such that $y \in \operatorname{int}\operatorname{conv}\{o, p, q\}$. By the symmetry of S, $[y - p, y]$ and $[y - q, y]$ are contained in $y + S$. Since $y \in \operatorname{int}\operatorname{conv}\{o, p, q\}$, the segments $[y - p, y]$ and $[o, q]$ cross, which yields that S and $y + S$ overlap; a contradiction. Hence, $y \notin \operatorname{int} K$. Since $\operatorname{int} K$ is the set of points in the plane whose distance from o, in the norm with unit ball K, is less than one, we have $\operatorname{dist}_K(o, y) \geq 1$. □

Remark 1. The Hadwiger number $H(S)$ of S is at most twenty four.

Proof. Note that, for every value of i, K and $c_i + K$ either overlap or touch. Since K is o-symmetric, it follows that $c_i \in 2K$, and $c_i + \frac{1}{2}K$ is contained in $\frac{5}{2}K$. By Lemma 2, $\{c_i + \frac{1}{2}K : i = 1, 2, \ldots, n\} \cup \{\frac{1}{2}K\}$ is a family of pairwise nonoverlapping translates of $\frac{1}{2}K$. Thus, $n \leq 24$ follows from an area estimate. □

Lemma 3. *If $j \neq i$, then $R_i \cap \operatorname{int} S_j = \emptyset$. Furthermore, $R_i \cap S_j \subset (o, c_i)$.*

Proof. Since S and S_i touch, there is a (possibly degenerate) parallelogram P such that $\operatorname{bd} P \subset (S \cup S_i)$ and $[o, c_i] \subset P$ (cf. Figure 1). Note that if $\operatorname{int}(x+S)$ intersects neither S nor S_i, then $x \notin P$ and $\operatorname{int}(x + S) \cap (o, c_i) = \emptyset$.

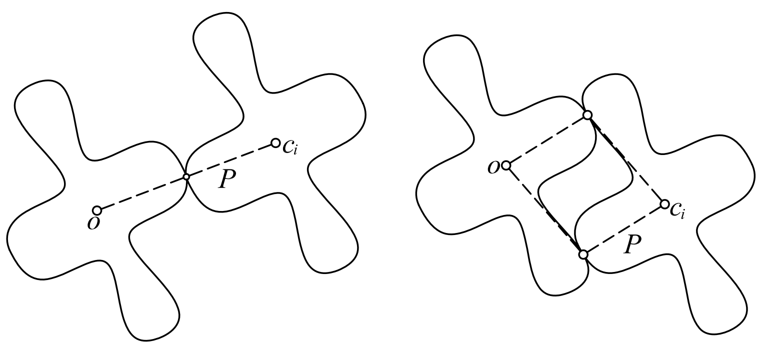

FIGURE 1

If $S_j \cap R_i = \emptyset$, we have nothing to prove. Let $S_j \cap R_i \neq \emptyset$ and consider a point $c_j + p \in S_j \cap R_i$. Since $o \in \operatorname{int} S$, $c_j + p \neq o$ and $c_j + p \neq c_i$. By the previous paragraph, if $c_j + p \in (o, c_i)$, then $c_j + p \notin \operatorname{int} S_j$. Thus, we are left with the case that $c_j + p \in R_i \setminus [o, c_i]$. By symmetry, $c_i - p \in S_i$. Note that $(c_i, c_i - p) \cap (o, c_j) \neq \emptyset$, which yields that $\operatorname{int} S_i$ intersects (o, c_j); a contradiction. □

Lemma 4. *We have $o \in \operatorname{int} C$, and $X \subset \operatorname{bd} C$.*

Proof. Assume that $o \notin \operatorname{int} C$. Note that there is a closed half plane H, containing o in its boundary, such that $C \subset H$. Let p be a boundary point of S satisfying $S \subset p + H$. Then, for $i = 1, 2, \ldots, n$, we have $S_i \subset p + H$. Observe that, for any value of i, $2p + S$ touches S and does not overlap S_i. Thus, $\mathfrak{F} \cup \{2p + S\}$ is a family of pairwise nonoverlapping translates of S in which every element touches S, which contradicts our assumption that $\operatorname{card} \mathfrak{F} = n = H(S)$.

Assume that $c_i \notin \operatorname{bd} C$ for some i, and note that there are values j and k such that $c_i \in \operatorname{int}\operatorname{conv}\{o, c_j, c_k\}$. Since S_j and S_k touch S, $\frac{1}{2}c_j$ and $\frac{1}{2}c_k$ are contained in K. Observe that at least one of $d_j = c_i - \frac{1}{2}c_j$ and $d_k = c_i - \frac{1}{2}c_k$ is in the exterior of the closed, convex angular domain D bounded by $R_j \cup R_k$ (cf. Figure 2). Since d_j and d_k are points of $c_i + K$, we obtain $(c_i + K) \setminus D \neq \emptyset$. On the other hand, Lemma 3 yields that $S_i \subset D$, hence, $c_i + K = \operatorname{conv} S_i \subset D$; a contradiction. □

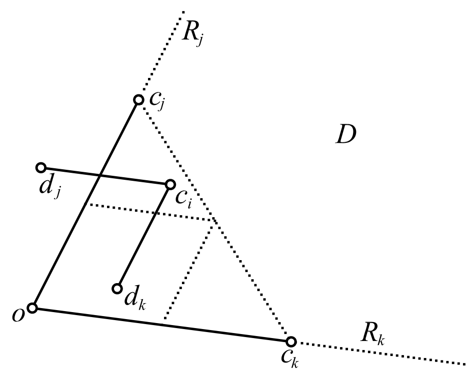

FIGURE 2

Remark 2. The Hadwiger number $H(S)$ of S is at most sixteen.

Proof. Gołąb [7] proved that the circumference of every centrally symmetric convex disk measured in its norm is at least six and at most eight. Fáry and Makai [6] proved that, in any norm, the circumferences of any convex disk C and its central symmetral $\frac{1}{2}(C-C)$ are equal. Thus, the circumference of C measured in the norm with unit ball $\frac{1}{2}(C-C)$ is at most eight.

Since $C \subset 2K$, we have $\operatorname{dist}_C(p,q) \geq \operatorname{dist}_{2K}(p,q) = \frac{1}{2}\operatorname{dist}_K(p,q)$ for any points $p, q \in \mathbb{R}^2$. By Lemma 2, $\operatorname{dist}_K(c_i, c_j) \geq 1$ for every $i \neq j$. Thus, $X = \{c_i : i = 1, 2, \ldots, n\}$ is a set of n points in the boundary of C at pairwise C-distances at least $\frac{1}{2}$. Hence, $n \leq 16$. □

Now we are ready to prove our theorem. By [5], there is a parallelogram P, circumscribed about $\bar{C}$, such that the midpoints of the edges of P belong to $\bar{C}$. Since the Hadwiger number of any affine image of S is equal to $H(S)$, we may assume that $P = \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq 1 \text{ and } |\beta| \leq 1\}$. Note that the points $e_x = (1,0)$ and $e_y = (0,1)$ are in the boundary of $\bar{C}$.

First, we show that there are two points r_x and s_x in S, with x-coordinates ρ_x and σ_x, respectively, such that $e_x \in \operatorname{conv}\{o, 2r_x, 2s_x\}$ and $\rho_x + \sigma_x \geq 1$.

Assume that $e_x = c_i$ for some value of i. Since S and S_i touch, there is a (possibly degenerate) parallelogram $P_i = \operatorname{conv}\{o, r_x, s_x, c_i\}$ such that $c_i = r_x + s_x$, $([o, r_x] \cup [o, s_x]) \subset S$ and $([c_i, r_x] \cup [c_i, s_x]) \subset S_i$ (cf. Figure 1). Observe that $c_i \in \operatorname{conv}\{o, 2r_x, 2s_x\}$ and $\rho_x + \sigma_x = 1$. If $e_x = -c_i$, we may choose r_x and s_x similarly.

Assume that $e_x \in (c_i, c_j)$ for some values of i and j. Consider a parallelogram $P_i = \mathrm{conv}\{o, r_i, s_i, c_i\}$ such that $c_i = r_i + s_i$, $([o, r_i] \cup [o, s_i]) \subset S$ and $([c_i, r_i] \cup [c_i, s_i]) \subset S_i$. Let L denote the line with equation $x = \frac{1}{2}$. We may assume that L separates s_i from o. We define r_j and s_j similarly. If the x-axis separates the points s_i and s_j, we may choose s_i and s_j as r_x and s_x. If both s_i and s_j are contained in the open half plane, bounded by the x-axis and containing c_i or c_j, say c_i, we may choose r_j and s_j as r_x and s_x (cf. Figure 3). If e_x is in $(-c_i, c_j)$ or $(-c_i, -c_j)$, we may apply a similar argument.

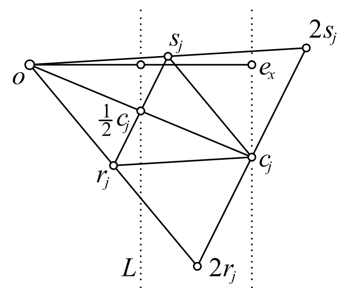

FIGURE 3

Analogously, we may choose points r_y and s_y in S, with y-coordinates ρ_y and σ_y, respectively, such that $e_y \in \mathrm{conv}\{o, 2r_y, 2s_y\}$ and $\rho_y + \sigma_y \geq 1$. We may assume that $\rho_x \leq \sigma_x$ and that $\rho_y \leq \sigma_y$.

Let Q_1, Q_2, Q_3 and Q_4 denote the four closed quadrants of the coordinate system in counterclockwise cyclic order. We may assume that $X \cap Q_1 \neq \emptyset$, and that Q_1 contains the points with nonnegative x- and y-coordinates. We relabel the indices of the elements of $\mathfrak{F}$ in a way that $R_1, R_2, \ldots, R_n$ are in counterclockwise cyclic order, and the angle between R_1 and the positive half of the x-axis, measured in the counterclockwise direction, is the smallest amongst all rays in $\{R_i : i = 1, 2, \ldots, n\}$.

If $\mathrm{card}(Q_i \cap X) \leq 3$ for each value of i, the assertion holds. Thus, we may assume that, say, $j = \mathrm{card}(Q_1 \cap X) > 3$. By Lemma 3, $[c_i, c_i - s_y]$ does not cross the rays R_1 and R_j for $i = 2, 3, \ldots, j-1$. Thus, the y-coordinate of c_i is at least σ_y (cf. Figure 4, note that c_i is not contained in the dotted region). Similarly, the x-coordinate of c_i is at least σ_x for $i = 2, \ldots, j-1$. Thus, $\sigma_x \leq 1$ and $\sigma_y \leq 1$, which yield that $\rho_x \geq 0$ and $\rho_y \geq 0$. Since $\sigma_x \geq 1 - \rho_x$ and $\sigma_y \geq 1 - \rho_y$, each c_i, with $2 \leq i \leq j-1$, is contained in the rectangle $T = \{(\alpha, \beta) \in \mathbb{R}^2 : 1 - \rho_x \leq \alpha \leq 1 \text{ and } 1 - \rho_y \leq \beta \leq 1\}$.

Let $B = \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq \rho_x \text{ and } |\beta| \leq \rho_y\}$. Note that if S and $p + S$ are nonoverlapping and $u, v \in S$, then the parallelogram $\mathrm{conv}\{o, u, v, u+v\}$ does not contain p in its interior. Thus, applying this observation with $\{u, v\} \subset \{\pm r_x, \pm\frac{\rho_x}{\sigma_x} s_x, \pm r_y, \pm\frac{\rho_y}{\sigma_y} s_x\}$, we obtain that $p \notin \mathrm{int}\, B$ (cf. Figure 5, the dotted parallelograms show the region "forbidden" for p).

Furthermore, if r_x and s_x do not lie on the x-axis, and r_y and s_y do not lie on the y-axis, then the interiors of these parallelograms cover B, apart from some points of S, and thus, we have $p \notin B$. If p is on a vertical side of B, then r_y or s_y lies on the y-axis (cf. Figure 6). Note that if r_y lies on the y-axis, then $e_y \in \mathrm{conv}\{o, 2r_y, 2s_y\}$ yields $\rho_y \geq \frac{1}{2}$, or that also s_y lies on the y-axis. Thus, it follows in this case that

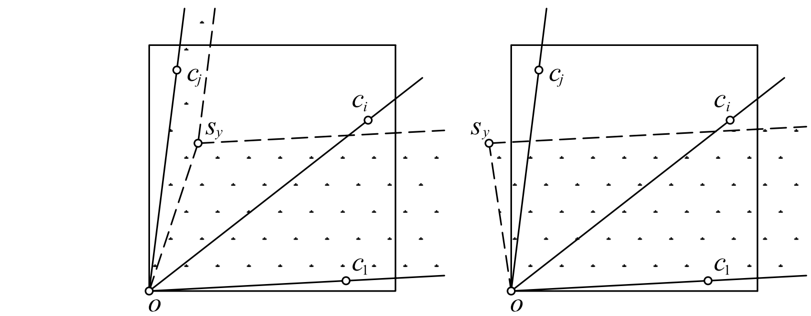

FIGURE 4

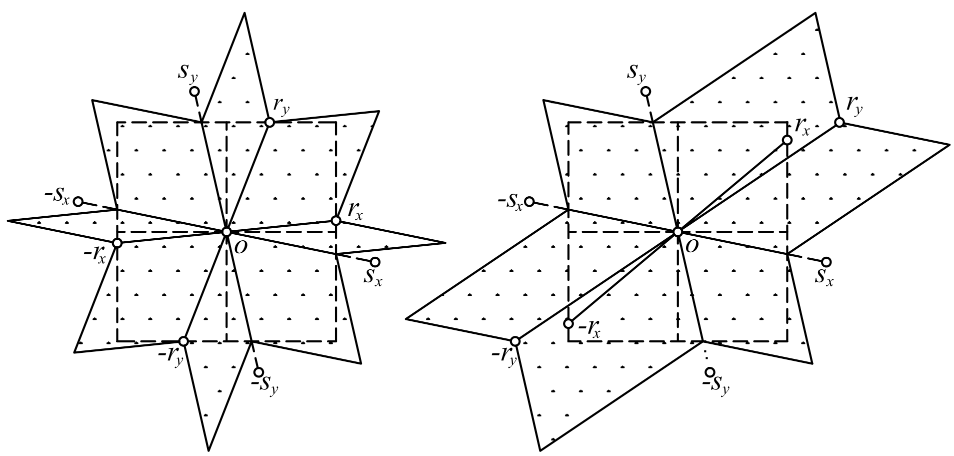

FIGURE 5

$\frac{1}{2}e_y \in S$. Similarly, if p is on a horizontal side of B, then $\frac{1}{2}e_x \in S$. We use this observation several times in the next three paragraphs.

Note that $T = \left(1 - \frac{\rho_x}{2}, 1 - \frac{\rho_y}{2}\right) + \frac{1}{2}B$. Since for any $2 \leq i < k \leq j-1$, $c_i + \frac{1}{2}B$ and $c_k + \frac{1}{2}B$ do not overlap, it follows that c_i and c_k lie on opposite sides of T. By Lemma 4, we immediately obtain that $j \leq 5$.

Assume that $j = 5$. Then, we have $\operatorname{card}(X \cap T) = 3$, which implies that two points of $X \cap T$ are consecutive vertices of T. Without loss of generality, we may assume that $c_4 = (1-\rho_x, 1)$, $c_3 = (1,1)$ and $c_2 = (\tau, 1-\rho_y)$ for some $\tau \in [1-\rho_y, 1]$. Since $c_3 - c_4$ lies on a vertical side of B, we obtain that $\frac{1}{2}e_y \in S$. From the position of $c_3 - c_2$, we obtain similarly that $\frac{1}{2}e_x \in S$. Thus, if c_1 is not on the x-axis or c_5 is not on the y-axis, then $R_1 \cap \operatorname{int} S_2 \neq \emptyset$ or $R_5 \cap \operatorname{int} S_4 \neq \emptyset$, respectively; a contradiction. Hence, from $\frac{1}{2}e_x, \frac{1}{2}e_y \in S$, it follows that $c_1 = e_x$ and $c_5 = e_y$. By Lemma 4, we have that $c_2 = (1, 1-\rho_y)$, which yields that, for example, S_1 and S_2 overlap; a contradiction.

We are left with the case $j = 4$. We may assume that c_2 and c_3 lie, say, on the vertical sides of T. Then we immediately have $\frac{1}{2}e_y \in S$. If c_4 is not on the y-axis,

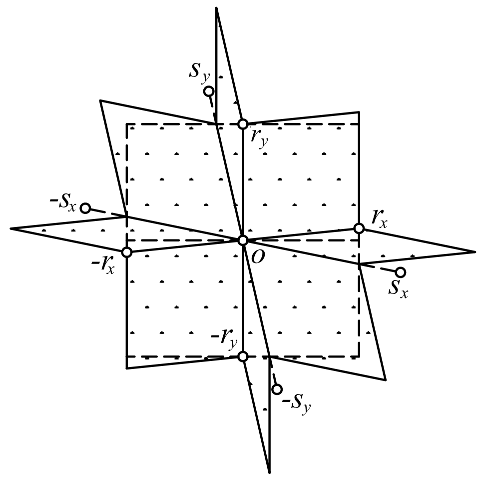

FIGURE 6

then $R_4 \cap \operatorname{int} S_3 \neq \emptyset$, and thus, it follows that $c_4 = e_y$. We show, by contradiction, that $\operatorname{card}((Q_1 \cup Q_2) \cap X) \leq 6$.

Assume that $\operatorname{card}((Q_1 \cup Q_2) \cap X) > 6$. Note that in this case $\operatorname{card}(Q_2 \cap X) = 4$, and both c_5 and c_6 are either on the horizontal sides, or on the vertical sides of $T' = (-2+\rho_x, 0)+T$. If they are on the horizontal sides, then $\frac{1}{2}e_x \in S$, $c_5 = (-1, 1)$, $c_7 = -e_x$, and, by Lemma 4, $c_6 = (-1, 1-\rho_y)$. Thus, S_6 overlaps both S_5 and S_7; a contradiction, and we may assume that c_5 and c_6 are on the vertical sides of T'.

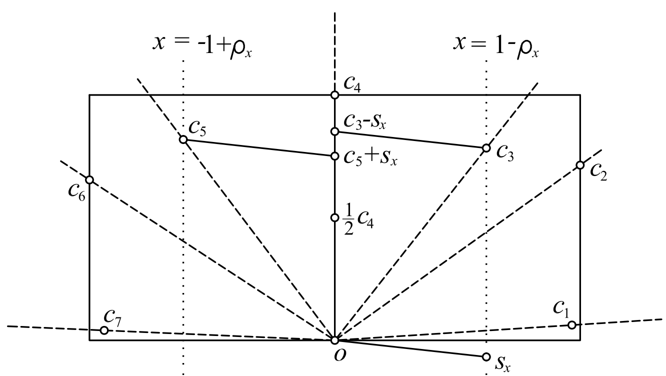

FIGURE 7

Since the y-coordinate of c_2 is at least $\frac{1}{2}$, and since $\left(c_3, c_3 - \frac{1}{2}e_y\right)$ does not intersect the ray R_2, we obtain that the y-coordinate of c_3 is at least $\frac{3}{4}$. Similarly, the y-coordinate of c_5 is at least $\frac{3}{4}$. Note that $c_3 - s_x$ and $c_5 + s_x$ are on the positive half of the y-axis. Then it follows from Lemma 3 that $c_3 - s_x$ and $c_5 + s_x$ lie on the open segment (o, c_4). If $c_3 - s_x \notin \left(\frac{1}{2}c_4, c_4\right)$ or $c_5 + s_x \notin \left(\frac{1}{2}c_4, c_4\right)$, then we have $c_5 + s_x \notin (o, c_4)$ or $c_3 - s_x \notin (o, c_4)$, respectively. Thus, both $c_5 + s_x$ and $c_3 - s_x$ belong to $\left(\frac{1}{2}c_4, c_4\right)$, and a neighborhood of $\frac{1}{2}c_4$ intersects S_4 in a segment, which yields that S_4 is not a disk; a contradiction.

Assume that $\operatorname{card}(Q_4 \cap X) > 3$. Then $\operatorname{card}((Q_1 \cup Q_4) \cap X) > 6$ yields that $\operatorname{card}((Q_3 \cup Q_4) \cap X) \leq 6$, and the assertion follows. Thus, we may assume that $\operatorname{card}(Q_4 \cap X) \leq 3$.

Finally, assume that $\operatorname{card}(Q_3 \cap X) > 3$. Then we have $\operatorname{card}((Q_3 \cup Q_4) \cap X) \leq 6$ or $\operatorname{card}((Q_2 \cup Q_3) \cap X) \leq 6$. In the first case we clearly have $\operatorname{card} X \leq 12$. In the second case, by the argument used for $Q_1 \cap X$, we obtain that $-e_x \in X$ and $\operatorname{card}(Q_2 \cap X) \leq 3$, from which it follows that $\operatorname{card}((Q_1 \cup Q_2 \cup Q_3) \cap X) \leq 9$. Since $\operatorname{card}(Q_4 \cap X) \leq 3$, the assertion holds.

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