On the Relative Distances of Seven Points in a Plane Convex Body

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November 24, 2006

∗ Partially supported by the Hung. Nat. Sci. Found. (OTKA), grant no. T043556.
∗∗ Partially supported by the Hung. Nat. Sci. Found. (OTKA), grant no. T043556 and T037752 and by the Alberta Ingenuity Fund.

Abstract

Let C be a convex body in the Euclidean plane. The relative distance of points p and q is twice the Euclidean distance of p and q divided by the Euclidean length of a longest chord in C with the direction, say, from p to q. We prove that, among any seven points of a plane convex body, there are two points at relative distance at most one, and one cannot be replaced by a smaller value. We apply our result to determine the diameter of point sets in normed planes.

Mathematics Subject Classification (2000): 52A40, 52A38, 52A10
Key words: convex body, normed space, relative distance, packing

1 Introduction

The focus of the paper is systems of "far" points in normed spaces in general, and normed planes in particular. Given $k \geq 3$, we look for sets of k points in a convex body C with minimum pairwise distance as large as possible. This is equivalent to packing C by congruent homothetic copies of C.

Let p and q be points in n-dimensional Euclidean space $\mathbb{E}^n$. Let $[p, q]$, (p, q), $|pq|$ and $\overrightarrow{pq}$ denote, respectively, the closed and the open segment, the distance and the vector with initial point p and terminal point q. Furthermore, if P is a simple polygon with edges $[a_1, a_2]$, $[a_2, a_3]$, ..., $[a_n, a_1]$, we use the notations $P = [a_1, a_2, \ldots, a_n]$ and $\operatorname{int} P = (a_1, a_2, \ldots, a_n)$. We use the usual notations $\operatorname{card} A$, $\operatorname{conv} A$, $\operatorname{int} A$, $\operatorname{bd} A$ for the cardinality, the convex hull, the interior and the boundary of a set A, respectively. We denote the family of plane convex bodies by $\mathcal{C}$ and the family of centrally symmetric plane convex bodies by $\mathcal{M}$. For simplicity, we call a plane convex body an *oval*, and identify a point and its position vector.

Let $C \subset \mathbb{E}^n$ be a convex body, and denote by r and s points in C such that $\overrightarrow{rs} || \overrightarrow{pq}$ and $|rs| \geq |r's'|$ where $\{r', s'\} \subset C$ and $\overrightarrow{r's'} || \overrightarrow{pq}$. The *$C$-length of* $[p, q]$, or equivalently, the *C-distance of p and q* is $2|pq|/|rs|$, and we denote it by $d_C(p, q)$ (see also [11]). If the convex body C is obvious, we may use the terms *relative distance of p and q* or *relative length of* $[p, q]$. Observe that for any convex bodies $D \subset C$ and points p, q we have $d_C(p, q) \leq d_D(p, q)$. It is a well-known fact that the unit ball of the normed space with norm $d_C(0, x)$ is $\frac{1}{2}(C - C)$.

Let $C \in \mathcal{C}$ and $k \geq 2$. Then compactness arguments yield that there is a greatest value $f_k(C)$ such that C contains k points in pairwise C-distances at least $f_k(C)$. Let $f_k = \min\limits_{C \in \mathcal{C}} \{f_k(C)\}$ and $F_k = \max\limits_{C \in \mathcal{C}} \{f_k(C)\}$. By Blaschke's Selection Theorem these values exist.

Numerous results appeared about the values $f_k(C)$, f_k and F_k. Here we list a few. Doliwka and Lassak [4] proved that, among any five boundary points of an oval, there is two at relative distance at most $\sqrt{5} - 1 \approx 1.236$ and that the value $\sqrt{5} - 1$ cannot be replaced by a smaller one. Böröczky and Lángi [2] showed that the result of Doliwka and Lassak remains true if we consider arbitrary points of the oval. In other words, $F_5 = \sqrt{5} - 1$. They also proved that $F_6 = 2 - \frac{2\sqrt{5}}{5} \approx 1.106$, and conjectured that $F_7 = 1$. We verify their conjecture.

Theorem 1 *Let $C \in \mathcal{C}$ and let $a_1, \ldots, a_7$ be points in C. Then $d_C(a_i, a_j) \leq 1$ for some $i \neq j$.*

Let us call an oval C *optimal* if it contains seven points at the minimum pairwise relative distance equal to one. In this case we say that the points *fit* C. The problem arises naturally to determine the optimal ovals and the set of points fitting them. We present the following three examples.

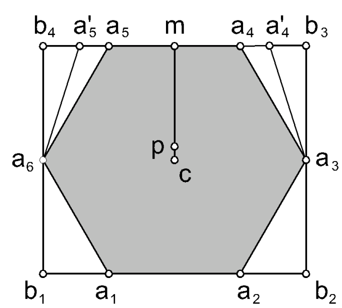

Figure 1

A result of Gołąb [6] states that there is an affine regular hexagon H inscribed in C for every $C \in \mathcal{M}$. The vertices and the centre of H fit C, and hence, C is optimal. Another example: any parallelogram P contains many sets of seven points at pairwise P-distances at least 1. Any oval $C \subset P$ containing such a set is optimal.

The third example is the following. Let $H = [a_1, a_2, \ldots, a_6]$ be a regular hexagon and $S = [b_1, b_2, b_3, b_4]$ be a rectangle circumscribed about H such that $[a_1, a_2] \subset [b_1, b_2]$ and $a_1 \in [b_1, a_2]$. Let c be the centre of H and $m = (b_3 + b_4)/2$. Let $a_4' \in (b_3, a_4)$ and $a_5' \in (a_5, b_4)$ such that $|a_4 a_4'| = |a_5 a_5'|$

and let $p \in (c, m)$; cf. Figure 1. Finally, let $C = [a_1, a_2, a_3, a'_4, a'_5, a_6]$. If p is close enough to c, all the pairwise C-distances of the vertices of C and p are at least one. We collect our results about optimal ovals and fitting sets of points in Theorem 2.

Theorem 2 *Let $C \in \mathcal{C}$ such that $Q = \operatorname{conv}\{a_1, a_2, \dots, a_7\} \subset C$ and $d_C(a_i, a_j) \geq 1$ for all $i \neq j$.*
2.1 *If C is strictly convex then Q is an affine regular hexagon with some a_i as centre.*
2.2 *If $\operatorname{card}(\operatorname{bd} Q \cap \{a_1, a_2, \dots, a_7\}) \neq 6$ then there is a parallelogram P such that $C \subset P$ and $d_P(a_i, a_j) \geq 1$ for all $i \neq j$.*

Using the idea of [5] (see also [11] and Theorem of [10]), we reformulate our theorems.

Corollary 3 *No oval is packed by seven homothetic copies of ratio greater than 1/3.*

Corollary 4 *Let $C \in \mathcal{C}$ be packed by seven homothetic copies of ratio 1/3 with points $a_1, a_2, \dots, a_7$ as centres. Let $Q = \operatorname{conv}\{a_1, a_2, \dots, a_7\}$.*
4.1 *If C is strictly convex then Q is an affine regular hexagon with some a_i as centre.*
4.2 *If $\operatorname{card}(\operatorname{bd} Q \cap \{a_1, a_2, \dots, a_7\}) \neq 6$ then there is a parallelogram P containing C such that P is packed by seven homothetic copies of ratio 1/3 with $a_1, a_2, \dots, a_7$ as centres.*

The following lemma is applied in the proof of Theorems 1 and 2 in Sections 2 and 3. We present applications of our theorems in Section 4. We note that analogous form of 5.1 has been verified in [9]. Theorem 1 when $Q = \operatorname{conv}\{a_1, a_2 \dots, a_7\}$ is not a hexagon is a consequence of [9] and Lemma 3 of [2]. In that case we prove Theorem 1 for the sake of Theorem 2.

Lemma 5 *Let $C \in \mathcal{C}$, $n \geq 6$, $D = [a_1, a_2, \dots, a_n] \subset C$ be a (possibly degenerate) convex n-gon and $T \subset D$ be an inscribed triangle of largest area with a side coinciding with a side of D.*
5.1 *D has a side of C-length at most one.*
5.2 *If the C-lengths of the sides of D are at least one then C is not strictly convex, and there is a parallelogram P such that $C \subset P$ and the sides of D are of P-length at least one.*

Proof. Without loss of generality, we assume that $T = [a_1, a_2, a_i]$ for a suitable i. Observe that $D \setminus T$ has a component W with at least three edges. We assume that $\{a_2, a_i\} \subset \operatorname{bd} W$; that is, $i \geq 5$. As relative distance and area ratio do not change under an affine transformation, we assume that T is an isosceles triangle and with right angle at a_1. Let b be the point such that $S = [a_1, a_2, b, a_i]$ is a square. Since T is a triangle of maximal area inscribed in D, we have $a_j \in [a_2, b, a_i]$ for $j = 3, \ldots, i-1$.

Let $m_1 = (a_2 + b)/2$, $m_2 = (b + a_i)/2$ and $m = (a_i + a_2)/2$. If $a_3 \in [a_2, m_1, m] \setminus [m, m_1]$ then $d_C(a_2, a_3) \leq d_T(a_2, a_3) < 1$ and we are done. If $a_{i-1} \in [a_i, m, m_2] \setminus [m, m_2]$ then $d_C(a_{i-1}, a_i) < 1$. We are left with the case $a_j \in S_0 = [m, m_2, b, m_3]$ for $3 \leq j \leq i-1$.

In this case $d_C(a_j, a_{j+1}) \leq d_T(a_j, a_{j+1}) \leq 1$ for $3 \leq j \leq i-1$. This proves 5.1. Moreover, if, for some $3 \leq j \leq i-2$, the points a_j and a_{j+1} are not on parallel sides of S_0 then $d_C(a_j, a_{j+1}) \leq d_T(a_j, a_{j+1}) < 1$. Let us examine the opposite case. Then $i = 5$ or $i = 6$, and, in the latter case, $a_3 = m_1$, $a_4 = b$ and $a_5 = m_2$, which implies that $S \subset C$. If $S \neq C$ then among the points there is two at C-distance less than one. If $S = C$ then the pairwise S-distances of the points are at least one.

Let us assume that $i = 5$ and that, say, $a_3 \in [m_1, m]$ and $a_4 \in [m_2, b]$. Let M denote the closed infinite strip containing S and bounded by the line through a_1 and a_2 and the line through b and a_5; cf. Figure 2. From $d_C(a_2, a_3) \geq 1$, we obtain that $C \subset M$. Let u, v be the endpoints of a maximal chord of C parallel to $\overrightarrow{a_3a_4}$, and N be the closed strip bounded by parallel supporting lines of C through u and v.

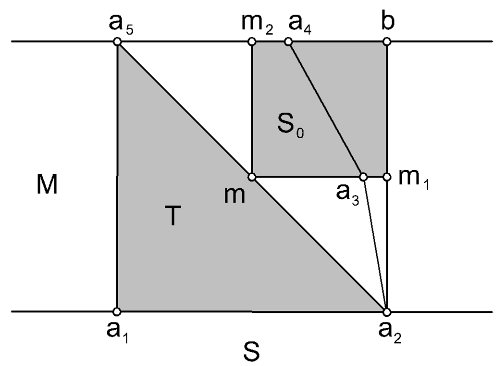

Figure 2

Then $C \subset P = M \cap N$, and the P-lengths of the sides of D are at least one. We observe also that C is not strictly convex.

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2 Proof of Theorems 1 and 2 when $Q = \text{conv}\{a_1, a_2, \ldots, a_7\}$ is a hexagon

Let us assume that $Q = [a_1, a_2, \ldots, a_6]$ and $a_7 \in \text{int}\, Q$. Let $a_i = q_i$ for $i = 1, 2, \ldots, 6$, $q_7 = q_1$ and $q_0 = q_6$.

We use the following terms and notations. For any i, j, k, l, where $1 \leq i, j, k, l \leq 6$ and $\{i, j\} \neq \{k, l\}$, α_i denotes the angle of Q at q_i, q_{ij} denotes the midpoint of the segment $[q_i, q_j]$, and $L_{ij,kl}$ denotes the straight line containing q_{ij} and q_{kl}. We note that $q_i = q_{ii}$ and set $L_{i,kl} = L_{ii,kl}$ and $L_{i,k} = L_{ii,kk}$. In addition, $S_i = [q_i, q_{i+1}]$ for

$i = 1, 2, \ldots, 6$ and M_i denotes the maximal chord of Q parallel to S_i with minimal Euclidean distance from S_i.

If $\alpha_{i-1} + \alpha_i + \alpha_{i+1}$ is greater than 2π, equal to 2π or less than 2π, where $i = 1, 2, \ldots, 6$, we say that q_i is a *large, normal* or *small vertex of* Q, respectively. Observe that q_i and q_{i+3} are either both normal, or one of them is large and the other one is small.

Note that $\alpha_i + \alpha_{i+1} \leq \pi$ implies that Q is contained in a parallelogram with S_i as side. From this it readily follows that there is a triangle T_i inscribed in Q with the following property: S_i is a side of T_i and T_i has maximum area of all triangles inscribed in Q. In this case the theorems follow from Lemma 3, and so we assume that the sum of every two consecutive angles of Q is greater than π.

Next, it is a simple matter to check that:
Case 1, every second vertex of Q is large, or
Case 2, Q has three consecutive vertices such that the second one is large and the two other ones are not small, or
Case 3, Q has three consecutive vertices such that the second one is normal and the two other ones are not small.

Case 1. Let the large vertices be q_1, q_3 and q_5, and $b_i = q_1+q_3+q_5-2q_i$ for $i = 1, 3, 5$; cf. Figure 3. Then $Q \subset [b_1, b_3, b_5]$ and every maximal chord of Q passes through q_1, q_3 or q_5. Let Q_i denote the homothetic copy of $\operatorname{int} Q$ with ratio $1/2$ and with q_i as centre. Let $P_i = [q_i, q_{(i-1)i}, q_{(i-1)(i+1)}, q_{(i+1)i}]$ for $i = 2, 4, 6$, $T_2 = [q_{13}, q_{14}, q_{36}]$, $T_4 = [q_{35}, q_{36}, q_{25}]$, $T_6 = [q_{15}, q_{25}, q_{14}]$ and $T = (q_{14}, q_{25}, q_{36})$.

We assume that $d_(q_i, q_{i+1}) \geq 1$ for each i. Then we need only to show that for any $p \in \operatorname{int} Q$,

$(*)_i \qquad d_Q(p, q_i) < 1$

for some i. Let $p \in \operatorname{int} Q$. We consider the position of p with respect to certain polygons. By symmetry, we assume that $p \in Q_1 \cup P_2 \cup T_2 \cup T$.

We have:
(1) $(*)_1$ *for* $p \in Q_1$;
(2) $(*)_2$ *for* $p \in P_2$;
(3) $(*)_2$ *or* $(*)_4$ *or* $(*)_6$ *for* $p \in T_2$;
(4) $(*)_2$ *or* $(*)_4$ *or* $(*)_6$ *for* $p \in T$.

The statements in (1) and (2) are easy to show, and also (3) with the condition that $d_Q(q_2, q_{14}) < 1$ and $d_Q(q_2, q_{36}) < 1$. We show (3) for $d_Q(q_2, q_{14}) \geq 1$ and $d_Q(q_2, q_{36}) \geq 1$. When exactly one of $d_Q(q_2, q_{14})$ and $d_Q(q_2, q_{36})$ is at least one, the proof is similar.

Let $\{s_1\} = L_{35,25} \cap [q_{13}, q_{15}]$ and $\{s_2\} = L_{15,25} \cap [q_{13}, q_{35}]$. From $d_Q(q_1, q_2) \geq 1$ and $d_Q(q_2, q_3) \geq 1$, we have that q_2 is in the parallelogram with vertices q_{13}, $(q_1 + b_5)/2$, b_5 and $(q_3 + b_5)/2$. Thus the set of points in $[q_{13}, q_{35}, q_{15}]$, at Q-distance less than one from q_2, is $[q_{13}, s_1, q_{25}, s_2] \setminus ([s_1, q_{25}] \cup [q_{25}, s_2])$. Similar statements are obtained for q_4 and q_6. Let $\{w_1\} = L_{35,36} \cap [q_{13}, q_{14}]$, $\{w_2\} = L_{15,14} \cap [q_{13}, q_{36}]$

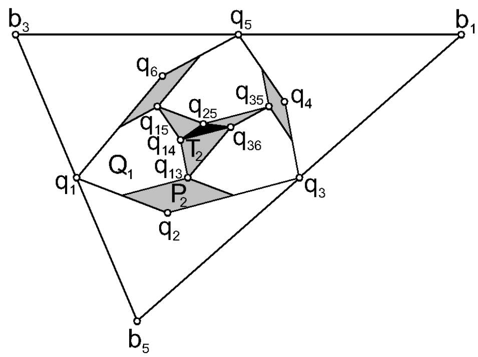

Figure 3

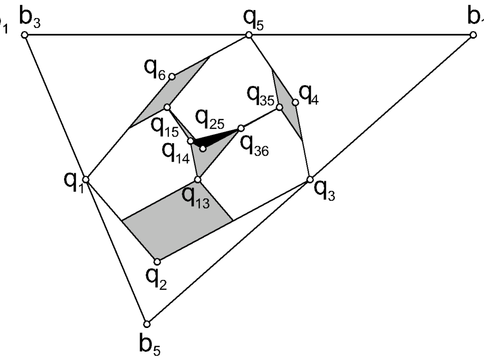

Figure 4

and $\{w\} = [q_{14}, w_2] \cap [q_{36}, w_1]$. As $d_Q(q_2, q_{14}) \geq 1$ and $d_Q(q_2, q_{36}) \geq 1$, it follows that w_1, w_2 and w exist. Note that $p \in [q_{13}, w_2, w, w_1]$, $p \in [q_{14}, w_2, q_{36}] \setminus [w_2, q_{14}]$ and $p \in [q_{14}, w_1, q_{36}] \setminus [w_1, q_{36}]$ imply $(*)_2$, $(*)_4$ and $(*)_6$, respectively.

Next, we show (4). If $T \cap (T_2 \cup T_4 \cup T_6) \neq \emptyset$ then $T \subset T_2 \cup T_4 \cup T_6$, and our theorems follow from (3), and so we suppose that $T \cap (T_2 \cup T_4 \cup T_6) = \emptyset$; cf. Figures 3 and 4. We need to distinguish between positions of lines that contain a vertex of T and a side of some T_i. If $L_{15,25} \cap T = L_{35,25} \cap T = \emptyset$, we have $(*)_2$. If $L_{15,25} \cap T \neq \emptyset \neq L_{35,25} \cap T$ then $L_{13,14} \cap T = L_{15,14} \cap T = \emptyset$ and so we have $(*)_4$. Finally, we show that our hypotheses allow only these two possibilities.

Assume that $L_{15,25} \cap T \neq \emptyset$ and $L_{35,25} \cap T = \emptyset$. Let us take $v = \frac{1}{2}(\overrightarrow{q_1q_6} + \overrightarrow{q_3q_2} + \overrightarrow{q_5q_4})$. As $\alpha_i + \alpha_{i+1} > \pi$ for $i = 1, 2, \ldots, 6$, Q has a vertex u such that the half line V with endpoint u and with tangential vector v intersects $\operatorname{int} Q$.

Assume that u is a large vertex, say, $u = q_1$. Observe that $\overrightarrow{q_{53}q_{52}} = \frac{1}{2}\overrightarrow{q_3q_2} = v - \frac{1}{2}(\overrightarrow{q_1q_6} + \overrightarrow{q_5q_4}) = v - \overrightarrow{q_{15}q_{46}}$. Let $n = q_{15} + v + \overrightarrow{q_{52}q_{36}}$. Then $\overrightarrow{q_{46}n} = \overrightarrow{q_{35}q_{36}} = \overrightarrow{q_5q_{56}}$, which implies $\overrightarrow{q_{45}n} = \overrightarrow{q_5q_6}$. For $i = 1, 2, \ldots, 6$, let H_i denote the open half plane bounded by $L_{i5,(i+1)5}$ and containing Q_5. As $\overrightarrow{q_{45}n} = \overrightarrow{q_5q_6}$, $n \in H_3 \cap H_4 \cap H_5$. From $V \cap \operatorname{int} Q \neq \emptyset$, we have $q_{15} + v \in H_6 \cap H_1$. From $L_{15,25} \cap T \neq \emptyset$, we obtain $q_{36} \in H_6 \cap H_1$. This implies $n \in H_6 \cap H_1$. Let $D = \operatorname{cl}(H_3 \cap H_4 \cap H_5 \cap H_6 \cap H_1)$. Then $n \in D$, and we have $d_D(q_5, q_6) < 1$. As the maximal chords of Q and D in the direction of $\overrightarrow{q_5q_6}$ coincide, we have also $d_Q(q_5, q_6) < 1$.

If u is a small vertex, a similar consideration yields the contradiction.

Case 2. Let q_2 be large and q_1 and q_3 be not small. We show that $d_C(q_i, q_j) \leq 1$, and if C is strictly convex then $d_C(q_i, q_j) < 1$ for some $i \neq j$.

Recall that $S_i = [q_i, q_{i+1}]$ and M_i is the maximal chord of Q parallel to S_i with the minimal Euclidean distance from S_i. Note that, as the sum of any two consecutive angles of Q is greater than π, every maximal chord of Q intersects S_j and S_{j+3}

for some $j \in \{1,2,3\}$. If M_i intersects S_j and S_{j+3}, we say that M_i is *a j-type maximal chord*. Observe that M_j is not j-type and M_6 is not 3-type. If M_6 and M_3 are 1-type and 2-type, respectively, then we observe that q_2 is not a small vertex; a contradiction. Hence, we have twelve possibilities depending on the types of M_1, M_2, M_3 and M_6. Let $\{d_1\} = L_{5,6} \cap L_{1,2}$, $\{d_2\} = L_{6,1} \cap L_{2,3}$, $\{d_3\} = L_{1,2} \cap L_{3,4}$ and $\{d_4\} = L_{2,3} \cap L_{4,5}$.

i) M_3 and M_6 are 1-type, M_1 is 2-type and M_2 is 3-type.

If $|q_1 d_2| < |q_1 q_6|$ then it follows from the type of M_1 that $d_Q(q_1, q_2) < 1$. Similarly, $|q_2 d_3| < |q_1 q_2|$ implies $d_Q(q_2, q_3) < 1$, and $|q_3 d_4| < |q_2 q_3|$ implies $d_Q(q_3, q_4) < 1$. Assume that $|q_1 d_2| \geq |q_1 q_6|$, $|q_2 d_3| \geq |q_1 q_2|$ and $|q_3 d_4| \geq |q_2 q_3|$. Let f_1 be the intersection of $L_{1,2}$ and the line through q_6 parallel to $L_{3,4}$, f_2 be the intersection of $L_{2,3}$ and the line through q_1 parallel to $L_{3,4}$, and g be the intersection of $L_{1,2}$ and the line through q_4 parallel to $L_{1,6}$; cf. Figure 5. Since q_3 is not small and $|q_3 d_4| \geq |q_2 q_3|$, we have $|q_3 q_4| \geq |q_3 d_3|$. From $|q_2 d_3| \geq |q_1 q_2|$, we obtain that $|q_3 d_3| \geq |q_1 f_2|$. As $|q_1 d_2| \geq |q_1 q_6|$ and Q is nondegenerate, we have also $|q_1 f_2| > |q_6 f_1|$, and so $2|q_6 f_1| < |q_4 d_3|$. Since $2|q_6 q_1| < |q_4 g|$ and M_6 is 1-type, we obtain that $d_Q(q_1, q_6) < 1$.

If M_3 and M_6 are 2-type, M_1 is 3-type and M_2 is 1-type then a similar argument yields $d_Q(q_{i-1}, q_i) < 1$ for some $i \in \{1,2,3,4\}$.

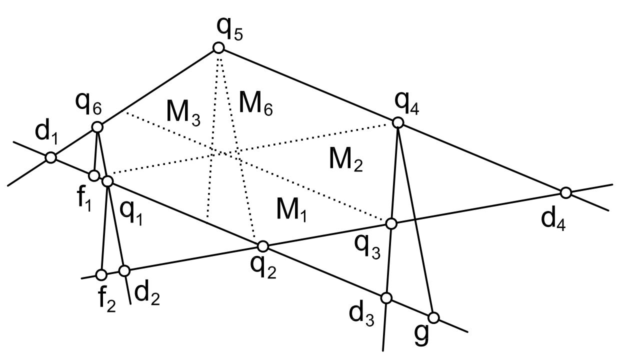

Figure 5

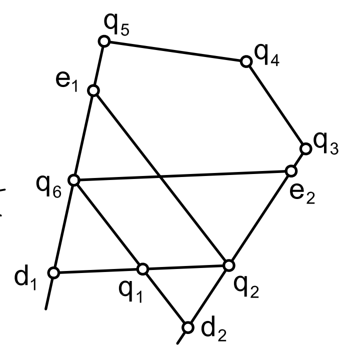

Figure 6

ii) M_6 and M_1 are 2-type.

Let e_1 denote the intersection of S_5 and the line through q_2 parallel to S_6, and e_2 denote the intersection of S_2 and the line through q_6 parallel to S_1; cf. Figure 6. Since M_6 and M_1 are 2-type, e_1 and e_2 exist. Observe that $|d_1 q_1| \leq |q_1 q_2|$ or $|d_2 q_1| \leq |q_1 q_6|$. If $|d_1 q_1| \leq |q_1 q_2|$, we have $2|q_1 q_6| \leq |q_2 e_1|$ and thus $d_C(q_1, q_6) \leq d_Q(q_1, q_6) \leq 1$. Similarly, from $|d_2 q_1| \leq |q_1 q_6|$, we have $d_C(q_1, q_2) \leq 1$. A detailed analysis shows that, under the assumption that C is strictly convex and $d_C(q_i, q_j) \geq 1$ for every $i \neq j$, Q is an affine regular hexagon. Note that in this case all the vertices of Q

are normal ($\alpha_{i-1}+\alpha_i+\alpha_{i+1}=2\pi$ for $i=1,2,\ldots,6$), but we assumed that Q has a small vertex, a contradiction.

If M_1 and M_2 are 3-type, or M_2 and M_3 are 1-type then a similar argument shows our theorems. Hence, we have examined all the possibilities for the types of M_1, M_2, M_3 and M_6.

Case 3. Let q_2 be normal and q_1, q_3 be not small. As $\alpha_1+\alpha_2+\alpha_3=2\pi$, $L_{6,1}$ and $L_{3,4}$ are parallel. Since relative distance is affine invariant, we assume that the triangle $[q_1,q_3,q_5]$ is regular and also that $\angle q_6q_1q_3 \leq 90°$. Let $b_i = q_1+q_3+q_5-2q_i$ for $i=1,3,5$. Since Q is convex and q_1 and q_5 are not small, $\{q_2,q_4,q_6\} \subset [b_1,b_3,b_5]$. Let $f=(q_3+b_1)/2$ and L be the line containing q_{13} and parallel to $L_{1,6}$.

Subcase 3.1, $q_2 \notin L$. Let $d = (q_3+b_5)/2$. We show that if $d_Q(q_i,q_{i+1}) \geq 1$ for every i then $q_2 \in [q_1,b_5,d] \setminus [q_1,d]$, $q_4 \in [q_5,f,b_1]$ and $q_6 \in [q_{15},q_5,b_3]$. This yields $\frac{\pi}{6} < \angle q_3q_1q_2$, $\frac{\pi}{6} \leq \angle q_3q_5q_4$ and that q_3 is a large vertex, a contradiction. In the case $|q_1q_6| \leq |q_3q_4|$, our arguments are deduced from Figure 7 with $k = q_{13}+\frac{1}{2}\overrightarrow{q_6q_1}$; $r = q_1+\frac{1}{2}\overrightarrow{q_5q_{13}}$; $s = r+\frac{1}{2}\overrightarrow{q_{13}k}$; $c=(q_1+q_5+b_3)/3$ and t being the intersection of the line containing $[q_{15},s]$ and the line containing $[q_1,c]$. We argue similarly if $|q_1q_6| \geq |q_3q_4|$.

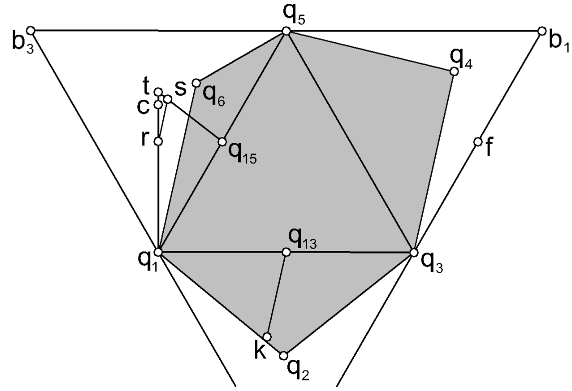

Figure 7

Subcase 3.2, $q_2 \in L$. Observe that $q_3 \in M_1$ and $M_1 \cap ((q_1,q_6) \cup S_5) \neq \emptyset$. If $M_1 \cap (q_1,q_6) \neq \emptyset$ then $d_C(q_1,q_2) \leq d_Q(q_1,q_2) = 1$. Moreover, if $d_C(q_1,q_2)=1$ then M_1 is maximal also in C, which implies that C is not strictly convex. Similarly, if $M_2 \cap (q_3,q_4) \neq \emptyset$ then $d_C(q_2,q_3) \leq 1$, and $d_C(q_2,q_3) < 1$ or C is not strictly convex.

Assume that $M_1 \cap S_5 \neq \emptyset \neq M_2 \cap S_3$. Let w be the intersection of $L_{1,6}$ and the line containing M_1. Observe that $[q_1,q_3,w]$ is a homothetic copy of $[q_1,q_2,q_{13}]$ of ratio -2, and $2|q_{13}q_2| \geq |q_1q_6|$. Similarly, we obtain that $2|q_{13}q_2| \geq |q_3q_4|$. As above, this and $d_Q(q_1,q_6) \geq 1$ imply that $q_6 \in [q_{15},q_5,b_3]$, $q_4 \in [q_5,f,b_1]$ and $\frac{\pi}{6} \leq \angle q_2q_1q_3$. Since q_1 is not a small vertex, it follows that $\angle q_2q_1q_3 = \frac{\pi}{6}$, $q_4 \in [q_5,f]$, $q_6 \in [b_3,q_{15}]$ and $M_1 = [q_3,q_6]$. Let $\{x\} = L_{1,3} \cap L_{4,5}$. Notice that $[q_3,q_4,x]$ is a homothetic copy of $[q_1,q_2,q_{13}]$ of ratio -2, $|q_3q_4| = 2|q_2q_{13}|$ and $|q_1q_6| = 2|q_2q_{13}|$. Observe that $q_1 \in M_5$, $d_Q(q_5,q_6)=1$ and $M_5 \cap S_4 \neq \emptyset$. Let $\{y\} = M_5 \cap S_4$. As $d_C(q_5,q_6) \leq d_Q(q_5,q_6)$, we may assume that $d_C(q_5,q_6)=1$. In this case, $[q_1,y]$ is a maximal chord of C. If $y \neq q_4$ then $y \in (q_4,q_5)$ and C is not strictly convex.

If $y = q_4$, Q is a regular hexagon. Let c be the centre of Q. If $p \neq c$ is a point of $[q_i, q_{i+1}, c]$ then $d_Q(q_i, p) < 1$ or $d_Q(q_{i+1}, p) < 1$. Hence, the only point of Q at Q-distance at least one from every vertex of Q is the centre of Q.

The last case is $a_7 \in \operatorname{bd} Q$. We regard Q as a degenerate heptagon and prove Theorems 1 and 2 in Section 3.

3 Proof of Theorems 1 and 2 when $Q = \operatorname{conv}\{a_1, a_2, \ldots, a_7\}$ is not a hexagon

We assume that no triangle, of the largest possible area inscribed in Q, has a side that coincides with a side of Q; otherwise, Theorems 1 and 2 follow from Lemma 5.

Case 1, $Q = [a_1, a_2, \ldots, a_7]$.

Let T be a triangle of the largest possible area inscribed in Q such that the vertices of T are also vertices of Q. Assume that $T = [a_1, a_3, a_6]$. Since relative distance is affine invariant, we assume that T is a regular triangle. Let $b_i = a_1 + a_3 + a_6 - 2a_i$ for $i = 1, 3, 6$. As T is a triangle of the largest area and Q is convex, we have $a_2 \in [a_1, b_6, a_3]$, $\{a_4, a_5\} \subset [a_3, b_1, a_6]$ and $a_7 \in [a_6, b_3, a_1]$.

Let $s_1 = (a_3 + a_6)/2$, $s_2 = (a_3 + b_1)/2$, $s_3 = (b_1 + a_6)/2$, $t_1 = (a_6 + a_1)/2$, $t_2 = (a_6 + b_3)/2$ and $t_3 = (b_3 + a_1)/2$. If $d_Q(a_3, a_4) < 1$ or $d_Q(a_5, a_6) < 1$, we are done, and so, we have that $\{a_4, a_5\} \subset [s_1, s_2, b_1, s_3]$. Note that the convexity of Q implies $d_Q(a_4, a_5) \leq 1$ and thus, Theorem 1; cf. Figure 8. To prove Theorem 2, we assume that $d_Q(a_i, a_{i+1}) \geq d_C(a_i, a_{i+1}) \geq 1$ for every i. Then $d_C(a_4, a_5) = 1$ and a_4, a_5 are on parallel sides of the rhombus $[s_1, s_2, b_1, s_3]$. We assume that $a_4 \in [s_1, s_2]$ and $a_5 \in [b_1, s_3]$.

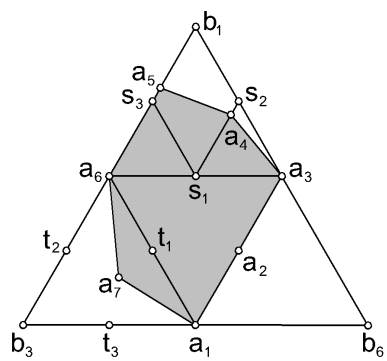

Figure 8

Let L_1 be the line through a_1 and a_3, and L_2 be the line through b_1 and b_3. Let H_1 and H_2 be the open half planes containing (a_2, a_6) and bounded by the lines L_1 and L_2, respectively. Observe that there are points $u \in (a_5, a_6)$ and $v \in (a_1, a_3)$ such that $\overrightarrow{uv} || \overrightarrow{a_3, a_4}$. As $d_C(a_3, a_4) \geq 1$, $[u, v]$ is a maximal chord of C, and so, $C \subset H_1 \cap H_2$.

Since $C \subset H_1$, $a_2 \in [a_1, a_3]$. Thus $a_2 = (a_1 + a_3)/2$ and $d_C(a_1, a_2) = d_C(a_2, a_3) = 1$. Due to $d_C(a_1, a_3) = 2$, there are parallel supporting lines L_3 and L_4 of C through a_1 and a_3. Let $a_1 \in L_3$ and $a_3 \in L_4$. Let P be the parallelogram bounded by L_1, L_2, L_3, L_4. Clearly, $C \subset P$.

We show that the P-length of every side of Q is at least one. We verify that $d_P(a_7, a_1) \geq 1$ and $d_P(a_6, a_7) \geq 1$, as the other inequalities are trivial.

From $d_Q(a_6,a_7) \geq 1$ and $d_Q(a_7,a_1) \geq 1$ we have $a_7 \in [t_1,t_2,b_3,t_3]$. This implies $d_P(a_7,a_1) \geq 1$. Let x be the vertex of P on $[a_6,b_3]$. As $s_3 \in Q \subset P$ and $[a_1,a_3]$ is a side of P, we have $x \in [a_6,t_2]$. Let $\{t\} = [a_1,t_2] \cap [t_1,t_3]$. Observe that $a_7 \in [t_1,t,t_2]$. If $a_7 \notin [t_1,t] \cup [t,t_2]$ then $d_Q(a_6,a_7) < 1$; a contradiction. If $a_7 \in [t_1,t] \cup [t,t_2]$ then $d_P(a_6,a_7) = 1$.

Case 2, Q is a triangle or Q contains a quadrangle R such that, with a suitable labelling of the a_i's, $R = [a_1,a_2,a_3,a_4]$ and $\operatorname{card}(\operatorname{int} R \cap \{a_5,a_6,a_7\}) \geq 2$.

The proof in Case 2 is a refined version of the proof of Lemma 3 in [2], hence we omit it.

4 Applications

Bateman and Erdős [1] asked what is the smallest diameter of a set of k points in $\mathbb{E}^n$ with pairwise distances at least one. In their paper they showed that, for seven points in the plane, the smallest diameter is two. The extension of this result for normed planes was, to our knowledge, first proposed by K. Bezdek (Problem Session, Workshop on Discrete Geometry and Convexity, Auburn University, Auburn, Alabama, USA, April, 2000). In 2005, Brass and Swanepoel conjectured that, in every normed plane, if S is a set of seven points with pairwise distances at least one, then the diameter of S is at least two (cf. Problem 10 on p. 71, [3]). In this section, we consider the question of Bateman and Erdős, and extend their result to normed planes. We prove also the conjecture of Brass and Swanepoel.

Let $C \in \mathcal{M}$ and $k \geq 2$. Compactness arguments show the existence of the smallest diameter $g_k(C)$ of a set of k points in the normed plane for a unit disk C such that the pairwise distances of the points are at least one. Let $g_k = \min_{C \in \mathcal{M}} \{g_k(C)\}$ and $G_k = \max_{C \in \mathcal{M}} \{g_k(C)\}$. By a Blaschke type theorem these values exist.

To formulate our next theorem, we introduce for any $C \in \mathcal{M}$ and $k \geq 2$, $\overline{f}_k(C) = \max\{f_k(D) | D \in \mathcal{C} \text{ and } \frac{1}{2}(D-D) = C\}$. In other words, we consider the maximum over the ovals 'generating' the normed plane with unit disk C.

Theorem 6 *Let $k \geq 2$ and $C \in \mathcal{M}$. Then $g_k(C) \cdot \overline{f}_k(C) = 2$.*

Proof. We show only that $\overline{f}_k(C) \geq 2/g_k(C)$, since the opposite direction is simple. Let $a_1,a_2,\ldots,a_k$ be points in the normed plane with unit disk C at pairwise distances at least one such that $\operatorname{diam}(\{a_1,a_2,\ldots,a_k\}) = g_k(C)$. Let $b_i = \frac{2}{g_k(C)} a_i$ for $i = 1,2,\ldots,k$. The pairwise distances of the points b_i are at least $2/g_k(C)$ and $\operatorname{diam}(\{b_1,b_2,\ldots,b_k\}) = 2$. According to [8], there is a plane convex body D of constant width two containing b_i for $i = 1,2,\ldots,k$. As D is of constant width two, we have $\frac{1}{2}(D-D) = C$. Hence, D contains k points at pairwise D-distances at least $2/g_k(C)$.

■

Using Theorem 6, we determine g_k and G_k for small values of k.

Theorem 7 *Let $C \in \mathcal{M}$ and $g_k(C)$ be defined as above, $g_k = \min_{C\in\mathcal{M}}\{g_k(C)\}$ and $G_k = \max_{C\in\mathcal{M}}\{g_k(C)\}$.*
7.1 *If $k \le 4$ then $g_k = 1$.*
7.2 $g_5 = \frac{\sqrt{5}+1}{2}$.
7.3 $g_6 = \frac{5+\sqrt{5}}{4}$, *and* $g_6(C) = \frac{5+\sqrt{5}}{4}$ *if, and only if, C is an affine regular ten-gon.*
7.4 $g_7 = g_8 = g_9 = 2$, *and* $g_9(C) = 2$ *if, and only if, C is a parallelogram.*
7.5 $G_2 = G_3 = 1$.
7.6 $G_4 = \sqrt{2}$.
7.7 $G_5 = G_6 = G_7 = 2$ *and* $g_7(C) = 2$.

Proof. The statements in 7.1 to 7.6 follow from Theorem 6 and from results in [1], [2], [5], [7] and [10]. We prove only 7.7.
First, we show that $\overline{f}_5(P) = 1$ for any parallelogram P. Observe that $f_5(P) = 1$. Hence, it is enough to show that if C is an oval of constant width two in the normed plane with unit disk P then C is a translate of P. Let $P = [a_1, a_2, a_3, a_4]$. Let L_1 and L_2 be the supporting lines of C parallel to $\overrightarrow{a_1a_2}$ such that the translate of L_1 by $\overrightarrow{a_1a_4}$ is L_2. Let $[b_1, b_2] = C \cap L_1$ and $[b_3, b_4] = C \cap L_2$ such that $\overrightarrow{b_1b_2}$ and $\overrightarrow{b_4b_3}$ are positive multiples of $\overrightarrow{a_1a_2}$. Let $c_3 = b_1 + \overrightarrow{a_1a_3}$ and $c_4 = b_2 + \overrightarrow{a_2a_4}$. Since C is of constant width two, we have $[c_3, c_4] \subset [b_3, b_4]$. Observe that $|c_3c_4| = 2|a_1a_2| - |b_1b_2|$. As $|b_3b_4| \le |a_1a_2|$, this implies that $|b_1b_2| = |b_3b_4| = |a_1a_2|$, $c_3 = b_3$ and $c_4 = b_4$. Hence, $D = [b_1, b_2, b_3, b_4]$ is a translate of P. As $D \subset C$ and C is an oval of constant width two, we have $C = D$.
Due to [11], every oval contains five points at pairwise relative distances at least one. Hence, $g_5(C) = 2/\overline{f}_5(C) \le 2$ for any $C \in \mathcal{M}$. We have shown that $g_5(P) = 2$ for any parallelogram P. Consequently, we have $G_5 = 2$. In [5], the authors show that every centrally symmetric oval contains seven points at pairwise relative distances at least one. Thus $\overline{f}_7(C) \ge 1$ for any $C \in \mathcal{M}$. This implies that $G_7 \le 2$. Now $G_5 \le G_6 \le G_7$ yields that $G_5 = G_6 = G_7 = 2$. Since $G_7 = g_7 = 2$, we obtain that $g_7(C) = 2$ for every $C \in \mathcal{M}$.

■

The next theorem is based on Theorem 2. As its proof is similar to that of Theorem 7, we omit it.

Theorem 8 *Let $a_1, a_2, \ldots, a_7$ be points, at pairwise distances at least one and with diameter two, in the normed plane with unit disk C. Let $Q = \operatorname{conv}\{a_1, a_2, \ldots, a_7\}$.*
8.1 *If C is strictly convex then $a_1, a_2, \ldots, a_7$ are the vertices and the centre of an affine regular hexagon.*
8.2 *If* $\operatorname{card}(\operatorname{bd} Q \cap \{a_1, a_2, \ldots, a_7\}) \ne 6$ *then there is a parallelogram P such that*

$C \subset P$ and the pairwise distances of $a_1, a_2, \ldots, a_7$ are at least one in the normed plane with unit disk P.

Acknowledgements. The authors are indebted to K. Bezdek for raising a question that led to Theorems 6, 7 and 8, and to T. Bisztriczky for his helpful remarks.

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