KIRCHBERGER–TYPE THEOREMS FOR SEPARATION BY CONVEX DOMAINS

ZSOLT LÁNGI* AND MÁRTON NASZÓDI**

Abstract. We say that a convex set K in $\mathbb{R}^d$ *strictly separates* the set A from the set B if $A \subset \operatorname{int}(K)$ and $B \cap \operatorname{cl} K = \emptyset$. The well–known Theorem of Kirchberger states the following. If A and B are finite sets in $\mathbb{R}^d$ with the property that for every $T \subset A \cup B$ of cardinality at most $d+2$, there is a half space strictly separating $T \cap A$ and $T \cap B$, then there is a half space strictly separating A and B. In short, we say that the *strict separation number* of the family of half spaces in $\mathbb{R}^d$ is $d+2$.

In this note we investigate the problem of strict separation of two finite sets by the family of positive homothetic (resp., similar) copies of a closed, convex set. We prove Kirchberger–type theorems for the family of positive homothets of planar convex sets and for the family of homothets of certain polyhedral sets. Moreover, we provide examples that show that, for certain convex sets, the family of positive homothets (resp., the family of similar copies) has a large strict separation number, in some cases, infinity. Finally, we examine how our results translate to the setting of non–strict separation.

1. Introduction and Preliminaries

A fundamental theorem in the study of separation properties of sets in Euclidean d–space $\mathbb{R}^d$ is the Theorem of Kirchberger [7]. It states that, for any two finite sets A and B in $\mathbb{R}^d$, A and B are strictly separable by a half space, if for any $T \subset A \cup B$ of cardinality at most $d+2$, $T \cap A$ and $T \cap B$ are strictly separable by a half space.

Let $\mathbf{B}^d[x,r]$ denote the closed Euclidean ball of radius r with x as centre in $\mathbb{R}^d$. Houle [6] proved the following. If A and B are finite sets in $\mathbb{R}^d$, and, for any $T \subset A \cup B$ of cardinality at most $d+2$, there is a ball $\mathbf{B}^d[x_T, r_T]$ such that $T \cap A \subset \operatorname{int} \mathbf{B}^d[x_T, r_T]$ and $T \cap B \subset \mathbb{R}^d \setminus \mathbf{B}^d[x_T, r_T]$, then there is a ball $\mathbf{B}^d[x,r]$ such that $A \subset \operatorname{int} \mathbf{B}^d[x,r]$ and $B \subset \mathbb{R}^d \setminus \mathbf{B}^d[x,r]$.

In [1] the following strengthening of this result is proved: If for every T, we have that $r_T \leq 1$ in the preceding statement, then there is a ball $\mathbf{B}^d[x,r]$ strictly separating A from B with $r \leq 1$.

For simplicity, we call a (possibly unbounded) closed, convex set with non–empty interior a *convex domain*. A compact convex domain is a *convex body*. We define separation of sets by convex domains as follows.

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Definition 1.1. Let $A, B \subset \mathbb{R}^d$, and $K \subset \mathbb{R}^d$ be a convex domain. We say that K *strictly separates* (resp., *separates*) A from B if $A \subset \operatorname{int} K$ and $B \cap K = \emptyset$ (resp., if $A \subset K$ and $B \cap \operatorname{int} K = \emptyset$).

Note the order of A and B in the preceding definition.

Definition 1.2. Let $\mathcal{F}$ be a family of convex domains in $\mathbb{R}^d$. Let n be the smallest positive integer (if it exists) such that the following holds for every two finite sets $A, B \subset \mathbb{R}^d$: If for every $T \subset A \cup B$ with $\operatorname{card} T \leq n$, there is a member of $\mathcal{F}$ that strictly separates $T \cap A$ from $T \cap B$, then there is a member of $\mathcal{F}$ that strictly separates A from B. We call n the *strict separation number* of $\mathcal{F}$, and denote it by $\operatorname{sep} \mathcal{F}$. If there is no such n, then we set $\operatorname{sep} \mathcal{F} := \infty$.

Definition 1.3. Let $K \subset \mathbb{R}^d$ be a convex set with non–empty interior. Let $\mathcal{H}(K)$ and $\mathcal{S}(K)$ denote, respectively, the family of *positive homothetic images* of K, and the family of the images of K under *orientation–preserving similarities*.

In the present note, we find $\operatorname{sep}(\mathcal{H}(K))$ and $\operatorname{sep}(\mathcal{S}(K))$ for various convex domains in $\mathbb{R}^d$. First, in Section 2, we show that $\operatorname{sep}(\mathcal{H}(K)) = 4$ for any plane convex body K. In Section 3, we study strict separation by certain types of polyhedral sets. Next, in Section 4, we construct convex domains with large separation numbers. Finally, in Section 5, we state the analogues of our results for non–strict separation.

We denote the origin by o, and the standard basis vectors of $\mathbb{R}^d$ by $e_1, \dots, e_d$. For $x, y \in \mathbb{R}^d$, we denote the closed segment with endpoints x and y by $[x, y]$.

2. Strict separation in the plane

Theorem 2.1. *Let K be a convex body in $\mathbb{R}^2$. Then for every two finite sets $A, B \subset \mathbb{R}^2$ the following holds: If for every subset T of $A \cup B$ of at most four points, $T \cap A$ is strictly separated from $T \cap B$ by a positive homothetic copy of K of homothety ratio less than one then A is strictly separated from B by a positive homothetic copy of K of homothety ratio less than one.*

Combining this Theorem with Remark 4.7, we obtain the following.

Corollary 2.2. *We have* $\operatorname{sep}(\mathcal{H}(K)) = 4$.

The main tool used in the proof of Theorem 2.1 is a topological version of Helly's Theorem (cf. [3] and [2]).

Definition 2.3. A *homology cell* is a topological space whose singular homology groups are isomorphic to those of a point. Note that homology cells are non–empty, and that all non–empty contractible spaces are homology cells.

Theorem 2.4 (Topological Helly Theorem)**.** *Let $\mathcal{F}$ be a finite family of open subsets of $\mathbb{R}^d$ such that the intersection of each n elements of $\mathcal{F}$ is a homology cell for all $n \leq d+1$. Then the intersection of all elements of $\mathcal{F}$ is a homology cell.*

Proof of Theorem 2.1. First, we prove the theorem in the case that K is strictly convex. Let $A, B \subset \mathbb{R}^2$ be two finite sets such that for every four–point subset T of $A \cup B$, $T \cap A$ is strictly separated from $T \cap B$ by a positive homothetic copy of K of homothety ratio less than one. We may assume that A is not empty.

For every $a \in A$ let $C_a := \{(p,r) \in \mathbb{R}^2 \times (0,1) : a \in \operatorname{int}(rK+p)\}$.

For every $b \in B$ let $C_b := \mathbb{R}^2 \times (-1,1) \setminus \{(p,r) \in \mathbb{R}^2 \times [0,1) : b \in rK+p\}$.

Let $\mathcal{F} := \{C_a : a \in A\} \cup \{C_b : b \in B\}$. We show that the intersection of any at most four members of $\mathcal{F}$ is contractible, and hence, a homology cell. Clearly, members of $\mathcal{F}$ are open homology cells in $\mathbb{R}^2 \times (-1,1)$. By the assumption, every four members of $\mathcal{F}$ intersect. Let T be a set of at most four points from $A \cup B$. Let $A' := T \cap A, B' := T \cap B$. Let (q,ρ) be an arbitrary element of $\cap\mathcal{F}'$, where $\mathcal{F}' := \{C_s : s \in T\}$. We describe a contraction of $\cap\mathcal{F}'$ onto (q,ρ), which, by Theorem 2.4, implies our statement (Theorem 2.4 applies as $\mathbb{R}^2 \times (-1,1)$ is homeomorphic to $\mathbb{R}^3$). We may assume that A' and B' are non–empty, otherwise $\cap\mathcal{F}'$ is clearly contractible to a point. Let (p,r) be in $\cap\mathcal{F}'$. Now,

$$A' \subset (\operatorname{int}(rK)+p) \cap (\operatorname{int}(\rho K)+q) \quad \text{and} \quad B' \cap [(rK+p) \cup (\rho K+q)] = \emptyset. \tag{2.1}$$

We define a path

$$\ell_{(p,r)} : [0,1] \longrightarrow \cap\mathcal{F}',\ t \mapsto (p(t), r(t))$$

such that $\ell_{(p,r)}(0) = (p,r)$ and $\ell_{(p,r)}(1) = (q,\rho)$. It is well known that the intersection of a convex planar curve with a positive homothetic copy of itself has at most two connected components (cf. p. 16 of [4]). Since K is strictly convex, $\operatorname{bd}(rK+p) \cap \operatorname{bd}(\rho K+q)$ contains exactly two points, say u and v. We have two cases.

Case 1, the longest chord of $(rK+p)$ parallel to the line $\overline{uv}$ is in the same closed half plane bounded by $\overline{uv}$ as the longest chord of $\rho K+q$ parallel to $\overline{uv}$. Then, it is easy to see that there is a unique path $p : [0,1] \longrightarrow \mathbb{R}^2$ starting at $p(0) := p$ and ending at $p(1) := q$ such that for all $t \in [0,1]$

$$(rK+p) \cap (\rho K+q) \subseteq ((1-t)r_p + t\rho)K + p(t) \subseteq (rK+p) \cup (\rho K+q). \tag{2.2}$$

We note that at any time, $u, v \in [((1-t)r_p + tr_q)K + p(t)]$. We define $\ell_{(p,r)}(t) := (p(t), (1-t)r_p + t\rho)$.

Case 2, the longest chord of $(rK+p)$ parallel to the line $\overline{uv}$ is not in the same closed half plane bounded by $\overline{uv}$ as the longest chord of $\rho K+q$ parallel to $\overline{uv}$. Then, we define $\ell_{(p,r)}$ piecewise. First, let $0 < \bar{r}$ be such that the longest chord of $\bar{r}K$ parallel to $\overline{uv}$ is of length $\operatorname{dist}(u,v)$. Clearly, $\bar{r} < r$. Now, there is a unique path $p(t) : [0,\frac{1}{2}] \longrightarrow \mathbb{R}^2$ starting at $p(0) := p$ such that for all $t \in [0,\frac{1}{2}]$

$$(rK+p) \cap (\rho K+q) \subseteq ((1-2t)r_p + 2t\bar{r})K + p(t) \subseteq (rK+p) \cup (\rho K+q). \tag{2.3}$$

For $t \in [0,\frac{1}{2}]$, we define $\ell_{(p,r)}(t) := (p(t), (1-2t)r_p + 2t\bar{r})$. Next, we can continue this path $p(t) : [\frac{1}{2},1] \longrightarrow \mathbb{R}^2$ in a unique way such that

$$(rK+p) \cap (\rho K+q) \subseteq ((2-2t)\bar{r} + (2t-1)\rho)K + p(t) \subseteq (rK+p) \cup (\rho K+q). \tag{2.4}$$

For $t \in \left[\frac{1}{2}, 1\right]$, we define $\ell_{(p,r)}(t) := (p(t), (2-2t)\bar{r} + (2t-1)\rho)$. By Equations (2.1-2.4), $\ell_{(p,r)}(t)$ is in $\cap\mathcal{F}'$. Clearly, $\ell : \cap\mathcal{F}' \times [0,1] \longrightarrow \cap\mathcal{F}'; ((p,r),t) \mapsto \ell_{(p,r)}(t)$ is a contraction of $\mathcal{F}'$ onto (q, ρ).

Next, we drop the assumption that K is strictly convex. Let $A, B \subset \mathbb{R}^2$ be finite sets such that every four of their points are strictly separated by a positive homothetic copy of K of homothety ratio less than one.

Let $\varepsilon > 0$ be such that for every 4–point subset T of $A \cup B$ there is an $(x_T, r_T) \in \mathbb{R}^2 \times (0,1)$ such that $(T \cap A) + \mathbf{B}^2[o, \varepsilon] \subset (1-\varepsilon) \operatorname{int} K$ and $((T \cap B) + \mathbf{B}^2[o, \varepsilon]) \cap (1+\varepsilon)K = \emptyset$.

Next, let K_n be a sequence of strictly convex bodies in $\mathbb{R}^2$ converging to K such that $(1-\varepsilon)K \subset K_n \subset (1+\varepsilon)K$ for all n. Let $\bar{A} := A + \mathbf{B}^2[o, \varepsilon]$ and $\bar{B} := B + \mathbf{B}^2[o, \varepsilon]$. By the choice of ε, for every n, every four points of $\bar{A} \cup \bar{B}$ are strictly separated by a positive homothetic copy of K_n of homothety ratio less than one. Hence, by the previous paragraph, for any finite subset S of $\bar{A} \cup \bar{B}$, $\bar{A} \cap S$ is strictly separated from $\bar{B} \cap S$ by a positive homothetic copy of K_n of homothety ratio less than one. Thus, Blaschke's Selection Theorem yields that there is a positive homothetic copy $r_n K_n + x_n$ of K_n $(0 < r_n < 1)$ that (not necessarily strictly) separates $\bar{A}$ from $\bar{B}$.

We may assume that $A \neq \emptyset$. Then, $\{x_n\}$ is a bounded sequence in $\mathbb{R}^2$ and $\{r_n\}$ is bounded in $\mathbb{R}$. By taking a suitable subsequence of $\{K_n\}$, we may assume that both $\{x_n\}$ and $\{r_n\}$ converge, say $x := \lim x_n$ and $r := \lim r_n$. Clearly, if $\operatorname{card} A > 1$, then $0 < r$, and $rK + x$ separates $\bar{A}$ from $\bar{B}$. Hence, it strictly separates A from B. □

We note that a slight modification of the above proof yields a new proof of the result in [1] cited in the Introduction.

3. Polyhedral Sets

Theorem 3.1. *Let K be the intersection of m closed half spaces, where $1 \leq m \leq d$, such that their outer normal vectors are linearly independent. Then* $\operatorname{sep}(\mathcal{H}(K)) = m + 1$*. Furthermore, if K is a d–simplex, then* $\operatorname{sep}(\mathcal{H}(K)) = d + 2$*.*

Proof. Let K be a cone as in the theorem or a simplex in $\mathbb{R}^d$, and let m be the number of its facets. Let $A, B \subset \mathbb{R}^d$ be finite sets, and assume that for every $T \subset A \cup B$ with $\operatorname{card} T \leq m+1$, a member of $\mathcal{H}(K)$ strictly separates $T \cap A$ from $T \cap B$.

Let $N(K) := \{u_i : i = 1, 2, \ldots, m\}$ be the family of the outer normal vectors of the facet hyperplanes K. Note that if K' is the intersection of m closed half spaces with outer normal vectors $u_1, u_2, \ldots, u_m$, then $K' \in \mathcal{H}(K)$. Vica versa, $K' \in \mathcal{H}(K)$ yields $N(K') = N(K)$.

Let $\alpha_i := \max\{\langle a, u_i \rangle : a \in A\}$, and let $a_i \in A$ be a point such that $\langle a_i, u_i \rangle = \alpha_i$. Observe that $M := \bigcap_{i=1}^m \{x \in \mathbb{R}^d : \langle x, u_i \rangle \leq \alpha_i\}$ is a member of $\mathcal{H}(K)$ containing A, and that a positive homothet of K contains A if, and only if, it contains M. Similarly, a positive homothet of K contains $\{a_i : 1, 2, \ldots, m\}$ if, and only if, it contains M. By our assumption, $B \cap M = \emptyset$, since otherwise $\{a_i : i = 1, 2, \ldots, m\}$ is not strictly separable from $\{b\}$ for any $b \in B \cap M$. Thus, the finiteness of B and

$N(K)$ yields the existence of a positive homothet of K strictly separating A from B. It follows that $\operatorname{sep}(\mathcal{H}(K)) \leq m+1$.

Now, assume that K is a simplex. Let A be the set of centres of the $d+1$ facets of K and let B be the singleton containing the centroid of K. This example shows that $\operatorname{sep}(\mathcal{H}(K)) \geq d+2$.

Next, let K be a cone with $m \leq d$ facets as in the theorem. In order to show that $\operatorname{sep}(\mathcal{H}(K)) \geq m+1$ we consider an $m-1$ dimensional affine subspace H of $\mathbb{R}^d$ such that $K \cap H$ is a simplex of dimension $m-1$ in H, and choose A and B in H as in the paragraph above. □

Our next theorem shows that no similar statement holds for $\operatorname{sep}(\mathcal{S}(K))$ of the same objects. For simplicity, we call the intersection of two nonparallel closed half spaces a *wedge*, and say that the *angle of a wedge* is π minus the angle between the outer normal vectors of these two half spaces.

Theorem 3.2. *Let $k \geq 5$ be odd, and let $W \subset \mathbb{R}^d$ be a wedge with an angle strictly less than $\frac{\pi}{k}$. Then* $\operatorname{sep}(\mathcal{S}(W)) \geq \frac{3k+1}{2}$.

Proof. First, we prove the statement for $d=2$. Let B be the vertex set of a regular k–gon with centre at the origin. It follows from the Inscribed Angle Theorem that the angle between two diameters of B, sharing a point of B, is equal to $\frac{\pi}{k}$. Let $A := tB$, where t is chosen in a way that every point of A lies on a diameter of B.

For $p \notin \operatorname{conv} A$, let W_p denote the intersection of the closed half planes H^+ such that $A \subset H^+$ and $p \in \operatorname{bd} H^+$. Note that W_p is a wedge with apex p such that the union of the two supporting lines of $\operatorname{conv} A$ contains the boundary of W_p. Furthermore, there is a wedge with apex p strictly separating A from B if, and only if, $B \cap W_p = \emptyset$. Let Y_p denote the unbounded component of $W_p \setminus \operatorname{conv} A$ (cf. Figure 1).

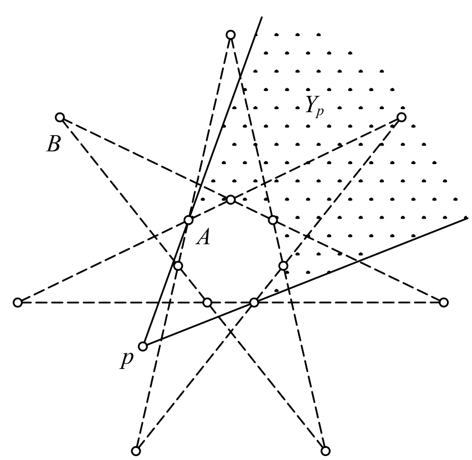

FIGURE 1

Assume that W is a wedge, with apex p, that strictly separates A from B. Then $B \cap W_p = \emptyset$, which yields that $p \notin Y_b$ for any $b \in B$. Observe that $\bigcup_{b\in B} Y_b = \mathbb{R}^2$, which shows that there is no wedge of any angle that strictly separates A from B. On the other hand, W_b separates A from B for any $b \in B$, and the angle of W_b is $\frac{\pi}{k}$.

Let $T \subset A \cup B$ with $\operatorname{card} T \leq \frac{3k-1}{2}$ and let $0 \leq \alpha \leq \frac{\pi}{k}$. Note that if there is a point $b \in B \setminus T$, then W_b may be modified to a wedge W of angle α with $T \cap A \subset \operatorname{int} W$ and $T \cap B \subset \mathbb{R}^2 \setminus W$. On the other hand, if $B \subset T$, then $A \cap T \leq \frac{k-1}{2}$, and $A \setminus T$ contains two consecutive vertices of A, which yields the existence of a wedge of angle α with the same separation property.

Assume that $d \geq 3$. Let us embed the example in the planar case into an affine plane P of $\mathbb{R}^d$, and note that there is no strip in P bounded by two parallel lines that strictly separates A from B. Since the intersection of a wedge in $\mathbb{R}^d$ with P is a wedge or a strip in P, the assertion follows from the planar case. □

Problem 3.3. *Determine* $\operatorname{sep}(\mathcal{S}(W))$ *for a wedge* W *in* $\mathbb{R}^2$ *with angle* α*, for every* $\alpha \in (0, \pi)$*.*

We mention one more result about strictly separating by polyhedral sets. Let $a_i, b_i \in \mathbb{R}$, with $a_i < b_i$, for $i = 1, 2, \ldots, d$. The set $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$ is called an *axis-parallel brick*. We denote the family of axis-parallel bricks in $\mathbb{R}^d$ by $\mathcal{B}^d$. The problem of strictly separating two sets by axis-parallel bricks has been examined by Lay in [8]. In particular, he proved the following theorem (cf. Theorem 2 in [8]).

Theorem 3.4 (Lay)**.** *We have* $\operatorname{sep}(\mathcal{B}^d) = d + 1$*.*

4. Convex Domains with Large Strict Separation Numbers

In this section, we find convex domains with large strict separation numbers. For $1 \leq p < \infty$, we denote the unit ball of the ℓ_p^d–space by $\mathbf{B}_p^d := \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} |x_i|^p \leq 1\}$, and the cube by $\mathbf{B}_\infty^d := \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \max_{i=1}^{d} |x_i| \leq 1\}$. Since Euclidean balls appear often, we keep our old notation $\mathbf{B}^d[o, 1]$ for $\mathbf{B}_2^d$.

Theorem 4.1. *We have* $\operatorname{sep}(\mathcal{H}(\mathbf{B}_\infty^d)) = \infty$*, where* $d \geq 3$*.*

Proof. Let k be given. First, we set $A_0 := \{o\} \subset \mathbb{R}^2$, and construct a set $B \subset \mathbb{R}^2$ of cardinality at least k such that there is no translate of $\mathbf{B}_\infty^2$ that strictly separates A_0 from B whereas, for every $b \in B$, there is a translate of $\mathbf{B}_\infty^2$ that strictly separates A_0 from $B \setminus \{b\}$.

Note that $x + \mathbf{B}_\infty^2$ strictly separates A_0 from B if, and only if, $x \in \operatorname{int} \mathbf{B}_\infty^2$ and $x \notin b + \mathbf{B}_\infty^2$ for any $b \in B$. Thus, it is sufficient to show that for any k, there is a set B, of cardinality k, such that

$$\mathbf{B}_\infty^2 \subset \bigcup_{b \in B} \left(b + \mathbf{B}_\infty^2\right), \quad \text{and} \quad \mathbf{B}_\infty^2 \not\subset \bigcup_{b' \in B, b' \neq b} \left(b' + \mathbf{B}_\infty^2\right)$$

for every $b \in B$. The way to construct such a set for $k = 14$ is shown in Figure 2. We note that, in our construction, there is no homothetic copy of $\mathbf{B}_\infty^2$ of at least unit homothety ratio that strictly separates A_0 from B.

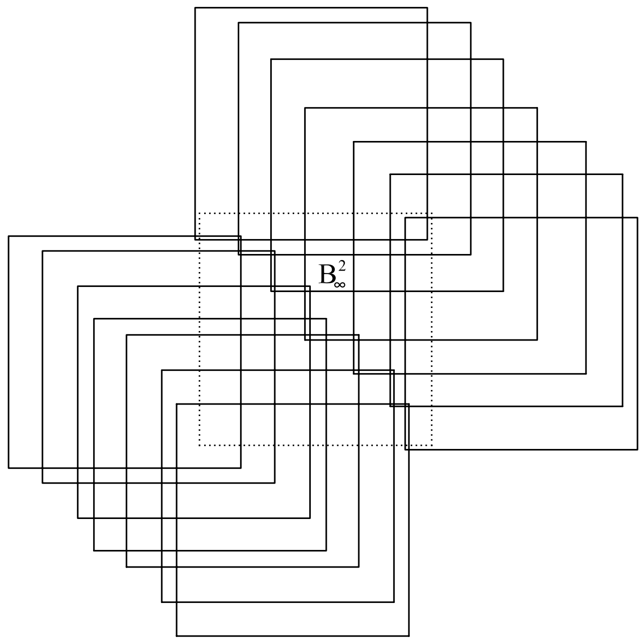

FIGURE 2

Now we embed $\mathbb{R}^2$ into $\mathbb{R}^d$ as the plane $P := \operatorname{span}\{e_1, e_2\}$. Let $A := \{-e_3, e_3\}$. Note that any homothetic copy of $\mathbf{B}^d_\infty$ containing A is of homothety ratio at least one, and contains o. Furthermore, its intersection Z with P is a square of side length at least two. Thus, Z contains a point of B, which yields that there is no homothetic copy of $\mathbf{B}^d_\infty$ strictly separating A from B. On the other hand, removing any point from $A \cup B$, the remaining points of $A \cup B$ are strictly separable. $\square$

It is easy to modify the proof of Theorem 4.1 to prove the following.

Corollary 4.2. *For every dimension $d \geq 3$, and for every positive integer k, there is a value $p(k)$ such that $\operatorname{sep}(\mathcal{H}(\mathbf{B}^d_p)) \geq k$ with $p > p(k)$. In particular, for every k, there is a strictly convex, smooth o–symmetric convex body K with $\operatorname{sep}(\mathcal{H}(K)) \geq k$.*

Problem 4.3. *Prove or disprove that $\operatorname{sep}(\mathcal{H}(\mathbf{B}^d_p))$ is finite for any $1 < p < \infty$.*

Problem 4.4. *Determine $\operatorname{sep}(\mathcal{H}(\mathbf{B}^d_1))$ for the cross–polytope $\mathbf{B}^d_1$.*

The following two problems are from K. Bezdek (oral communication).

Problem 4.5. *Prove or disprove that $\operatorname{sep}(\mathcal{H}(K))$ is finite for any (centrally symmetric) smooth and strictly convex body K.*

Problem 4.6. *Prove or disprove that there is a neighborhood of the d–dimensional Euclidean ball of unit radius (in the sense of the Hausdorff metric) such that $\operatorname{sep}(\mathcal{H}(K))$ is finite for any convex body in that neighborhood.*

A point b on the boundary of a convex set K is called *regular* if K has a unique support hyperplane at b.

Remark 4.7. For every convex body $K \subset \mathbb{R}^d$, we have $\operatorname{sep}(\mathcal{H}(K)) \geq d+2$. To prove this assertion, we consider $d+1$ affinely independent directions $u_1, \ldots, u_{d+1} \in \mathbb{S}^{d-1}$ such that the support hyperplanes of K that are orthogonal to u_i support K at a regular boundary point for all i. Now, let A be the set of vertices of the simplex $S := \{x \in \mathbb{R}^d : \langle x, u_i \rangle \leq 1, i = 1, \ldots, d+1\}$ and B be a singleton set containing an arbitrary interior point of S. Clearly, any $d+1$ points of $A \cup B$ are strictly separated by a sufficiently large positive homothetic copy of K but there is no $K' \in \mathcal{H}(K)$ that separates A from B.

Theorem 4.8. *Let C_0 be a strictly convex body in $\mathbb{R}^{d-1}$ and let K be the cylinder over C_0; that is, $K := C_0 \times [-1, 1] \subset \mathbb{R}^d$. Then $\operatorname{sep}(\mathcal{H}(K)) = \infty$.*

Proof. For any $k > 0$, we construct two sets A and B with the property that for any k–point subset T of $A \cup B$, there is a positive homothetic copy K' of K such that $T \cap A \subset \operatorname{int}(K')$ and $T \cap B \cap K' = \emptyset$, but there is no positive homothet of K that strictly separates A and B.

Let $A := \{-e_d, e_d\}$ and let $H := \{x \in \mathbb{R}^d : \langle x, e_d \rangle = 0\}$. By Lemma 4.9, there is a minimal covering $b_1 - C_0, b_2 - C_0, \ldots, b_l - C_0$ of $-C_0$ by at least $k+1$ translates of $-C_0$.

By the minimality of the covering, we have that for every $i = 1, \ldots, l$ there is a point p_i in $-C_0$ that is contained in $b_i - C_0$ and is not contained in the other members of the covering. Let $B := \{b_1, \ldots, b_l\}$.

We show that A and B are as promised. First, let T be a subset of $A \cup B$ of k points. Then, by the construction, there is a positive homothet of K that contains $T \cap A$ in the interior and does not intersect $T \cap B$, namely $(1+\varepsilon)K + p_i$, if $b_i \notin T$ and $\varepsilon > 0$ is sufficiently small. On the other hand, it is not difficult to see that no positive homothet separates A from B. $\square$

Lemma 4.9. *Let C be a strictly convex body in $\mathbb{R}^d$, and let $k > 0$. Then there is a covering of C with at least k translates of C which is minimal; that is, omitting any of the translates yields a family of translates that does not cover C.*

Proof. Let $u \in \mathbb{S}^{d-1}$ be an arbitrary direction. Let $A := \{x \in \operatorname{bd} C : \text{ there is a } t > 0 \text{ such that } x + tu \in \operatorname{int} C\}$. Next, choose translation vectors $b_1, \ldots, b_l$ such that for all $i = 1, \ldots, l$ we have

$$\operatorname{Vol}_{d-1}(\operatorname{bd} C \cap (b_i + C)) < \frac{\operatorname{Vol}_{d-1}(A)}{k} \tag{4.1}$$

and

$$\bigcup_{i=1}^{l} (b_i + \operatorname{int} C) \supseteq \operatorname{bd} C.$$

This is possible, since C is strictly convex. Now, for a sufficiently small $\varepsilon > 0$ we have that the family $\{b_i + C : i = 1, \ldots, l\} \cup \{\varepsilon u + C\}$ covers C. We choose a minimal covering of C from this family of $l+1$ sets. By (4.1) and since $(\varepsilon u + C) \cap A = \emptyset$, it follows that this minimal covering family has at least k members. $\square$

We provide the following example to show that, for every $\varepsilon > 0$, there is a convex body K at a Hausdorff distance less than ε from the Euclidean unit ball B^d in $\mathbb{R}^d$, where $d \geq 3$, such that $\operatorname{sep}(\mathcal{H}(K)) \geq d+3$. This example shows that the result of Houle about the strict separation number of the family of Euclidean balls is not stable.

Example 4.10. Let $H_\alpha := \{x \in \mathbb{R}^d : \langle x, e_d\rangle = \alpha\}$, $H' := \{x \in \mathbb{R}^d : \langle x, e_{d-1}\rangle = 0\}$ and $L := H_0 \cap H'$. It is not hard to see that there is an o–symmetric convex body K within Hausdorff distance ε from $\mathbf{B}^d$ with the following properties. Denoting the sections $H_\alpha \cap K$ of K by K_α (where $0 \leq \alpha$), we have

(1) K_α is a homothetic image of a $(d-1)$–dimensional ellipsoid,

(2) the centre of K_α is αe_d,

(3) $K_1 = \{e_d\}$,

(4) K_0 is not a Euclidean ball, and K_α is a Euclidean $(d-1)$–ball for exactly one $\alpha > 0$, and

(5) $K \cap H'$ is a Euclidean $(d-1)$–ball.

Let A_0 be the vertex set of a regular simplex inscribed in $\mathbf{B}^d[o,1] \cap L$ and let $B := \{e_{d-1}, -e_{d-1}\}$. First, we show that the unique homothet of the Euclidean ball $\mathbf{B}^d[o,1] \cap H_0$ in the hyperplane H_0 that (not necessarily strictly) separates A_0 from B is itself.

Consider a point $x \in H_0$ with $x \neq o$. Clearly, if $x \in \{\lambda e_{d-1} : \lambda \in \mathbb{R}\}$, then no ball $\mathbf{B}^d[x,r]$ separates A_0 from B. Hence, we may assume that $x \notin \{\lambda e_{d-1} : \lambda \in \mathbb{R}\}$, and that L separates x and $-e_{d-1}$. This yields that $\angle xoe_{d-1} \leq \frac{\pi}{2}$. Let $P_a := \{y \in H_0 : \langle y, a\rangle \geq 0\}$ for any $a \in A_0$, and note that $\bigcap_{a\in A_0} P_a = \{\lambda e_{d-1} : \lambda \in \mathbb{R}\}$. Thus, there is a point a of A_0 such that $\angle aox > \frac{\pi}{2}$. Since $||a|| = ||e_{d-1}|| = 1$, it follows that $||a-x|| > ||e_{d-1} - x||$. Hence, there is no $r > 0$ such that $A_0 \subset \mathbf{B}^d[x,r]$ and $B \subset H_0 \setminus \operatorname{int} \mathbf{B}^d[x,r]$.

Using this argument, it is easy to show that if $\delta > 0$, then there is no homothetic copy of the ellipsoid K_0 that separates A_0 from B. Thus, by property (4) of K, there are exactly two distinct homothetic copies K_1 and K_2 of K that separate A_0 from B. These copies are in a symmetric position about the hyperplane H_0. Let $A' := \{\lambda e_d, -\lambda e_d\}$ such that $\lambda e_d \in \operatorname{int} K_1 \setminus K_2$ and $|\lambda| < 1$. Clearly, no homothetic copy of K separates $A := A' \cup A_0$ from B, but, removing any of the points from $A \cup B$, the remaining points are separable. Finally, moving the points of $A \cup B$ a little in suitable directions, we obtain two point sets $\hat{A}$ and $\hat{B}$, of cardinality $d+1$ and 2, respectively, such that $\hat{A}$ cannot be strictly separated from $\hat{B}$, but any $d+2$ points of $\hat{A} \cup \hat{B}$ are strictly separable.

5. Non–strict Separation

We may define the non–strict *separation number* $\overline{\mathrm{sep}}(\mathcal{F})$ of a family $\mathcal{F}$ of convex domains by modifying Definition 1.2 in the obvious way. Then, one might study $\overline{\mathrm{sep}}(\mathcal{H}(K))$ and $\overline{\mathrm{sep}}(\mathcal{S}(K))$ for any convex domain $K \in \mathbb{R}^d$. We list which of our results hold, or how they need to be modified, for these two quantities.

Theorem 2.1 and Corollary 2.2 are valid if K is a planar strictly convex body. The only ingredient of the proof of Theorem 2.1 that needs to be modified is Theorem 2.4

(the Topological Helly Theorem). Luckily enough, its statement remains valid if we replace the word "open" with the word "closed" everywhere (cf. [3]).

However, the example of $A := \{\pm e_1, \pm e_2\}$ and $B := \{o\}$ shows that $\overline{\mathrm{sep}}(\mathcal{H}(K)) \geq 5$ if K is an axis–parallel square.

Theorem 3.1 holds also for $\overline{\mathrm{sep}}(\mathcal{H}(K))$. With regard to Theorem 3.2, we have the following.

Theorem 5.1. *Let $k \geq 6$ be even, and let $W \subset \mathbb{R}^d$ be a wedge with an angle less than or equal to $\frac{2\pi}{k}$. Then $\overline{\mathrm{sep}}(\mathcal{S}(W)) \geq k$.*

The separation number of the family $\mathcal{B}^d$ of axis–parallel bricks is different from the strict separation number:

Theorem 5.2. *We have $\overline{\mathrm{sep}}(\mathcal{B}^d) = 2d + 1$.*

Proof. To prove that $\overline{\mathrm{sep}}(\mathcal{B}^d) \leq 2d + 1$, we consider the smallest brick $S \in \mathcal{B}$ that contains A. Then, there is a set $A_0 \subset A$ of cardinality at most $2d$ such that each facet of S contains at least one point of A_0. Clearly, A is separated from B by a brick if, and only if, $B \cap S = \emptyset$, which can be verified by checking it for sets of the form $T = A_0 \cup \{b\}$, where $b \in B$.

To show that $\overline{\mathrm{sep}}(\mathcal{B}^d) \geq 2d + 1$, we consider the sets A and B, where A is the set of vertices of the cross–polytope $\mathbf{B}_1^d$ and $B := \{o\}$. □

The results in Section 4 are valid also for $\overline{\mathrm{sep}}(\mathcal{H}(K))$.

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