RELATIVE DISTANCE AND PACKING A BODY BY HOMOTHETICAL COPIES

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We say that a set C in Euclidean n-space E^n *is packed* with sets $C_1, \dots, C_k$ if they are subsets of C and if they have pairwise disjoint interiors. There are many questions about packing convex bodies. For instance, a long-standing problem about packing a disk with k equal, as large as possible, disks. The best possible configurations are found for all $k \leq 12$. The proofs are given in papers of Fodor [3], Graham [5], Kravitz [6], Pirl [11], and also in the dissertation of Mellisen [10]. For $k > 12$ there is a number of conjectures.

We consider a more general problem about packing a planar convex body C with a number of homothetical copies. In particular cases, this question was considered also earlier; in [8], where a convex body is packed by five homothetical copies, and by Doyle, Lagarias and Randall [2], where a centrally-symmetric body is packed with a few homothetical copies. Our problem can be also considered in an equivalent form as a question about a distribution of k points in C in possibly large relative distance. In first section we recall the notion of relative distance and we discuss this equivalence. Next two sections are devoted finding configurations of points of C in possibly large C-distance. The last section presents corollaries about efficient packing of C by smaller homothetical copies.

1. Selfpacking a convex body and relative distance

Whenever we say *distance*, we mean the Euclidean distance. The distance of points x and y is denoted by $|xy|$. Let $C \subset E^n$ be a convex body. By the C-*distance* $\mathrm{dist}_C(x, y)$ of x and y we understand the ratio of $|xy|$ to the half of the maximum distance of points a and b in C such that the segments xy and ab are parallel (see [8]). When there is no doubt about the body C, we also use the term *relative distance*.

THEOREM. *Let C be a convex body in E^n and let $k \geq 2$ be an integer. If C contains k points in relative distances at least d, then we can pack C with its k homothetical copies of ratio $\frac{d}{2+d}$. Vice-versa; if we can pack C by its k homothetical copies of a positive ratio $r < 1$, then we can find k points in C in relative distances at least $\frac{2r}{1-r}$.*

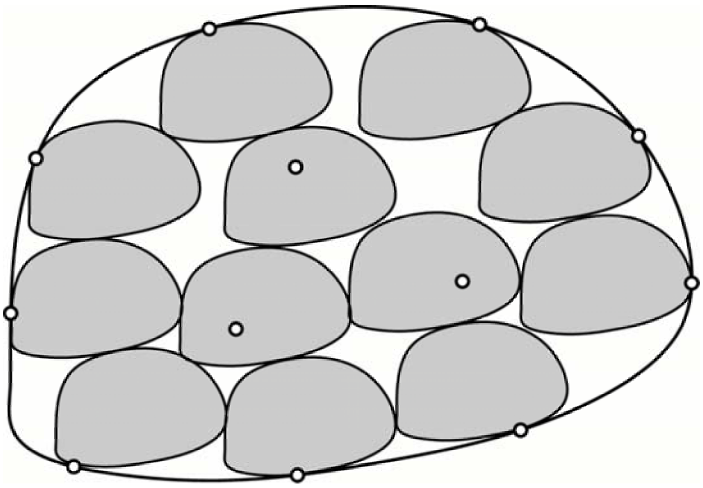


Figure 1

The above Fig. 1 illustrates our Theorem. The idea of this theorem appears yet in other papers. For instance, in [8] when Corollary is deduced from Theorem, in [2] in Theorem 3.1 in the case of centrally-symmetric bodies, and in [10] for packing a disk with disks. In order to prove Theorem we show Lemma 1, which implies Lemma 2. Our theorem is an immediate consequence of Lemma 2.

Simple compactness arguments show that for every convex body C in E^n and for every integer $k \geq 2$, the below defined numbers $d_k(C)$ and $r_k(C)$ exist. We define $d_k(C)$ as the greatest possible value d for which there are k points in C such that the relative distance of every pair of them is at least d. By $r_k(C)$ we mean the greatest possible positive ratio of k homothetical copies of C that can be packed into C.

We can rewrite Theorem in the following form. *For every convex body $C \subset E^n$ and for every integer $k \geq 2$ we have*

$$r_k(C) = \frac{d_k(C)}{2 + d_k(C)} \quad \text{and} \quad d_k(C) = \frac{2r_k(C)}{1 - r_k(C)}.$$

LEMMA 1. *Let xy and ab be two parallel segments in E^n. Put $d = 2(|xy|/|ab|)$. The two segments being homothetical copies of the segment ab with homothety centers x and y, and with the homothety ratio $\frac{d}{2+d}$, have exactly one common point.*

Proof. Denote by w the point of intersection of the straight lines containing segments xb and ya (see Fig. 2). We tacitly assume that the notation for a and b is taken such that the segments intersect. Through w we provide the straight line parallel to the segment xy. The intersections of this line with the segment xa is denoted by g, and with the segment yb is denoted by h. Thus $\frac{|gw|}{|ab|} = \frac{|wx|}{|bx|} = \frac{|wx|}{|bw|+|wx|} = \left(\frac{|bw|}{|wx|} + 1\right)^{-1} = \left(\frac{|ab|}{|xy|} + 1\right)^{-1}$.

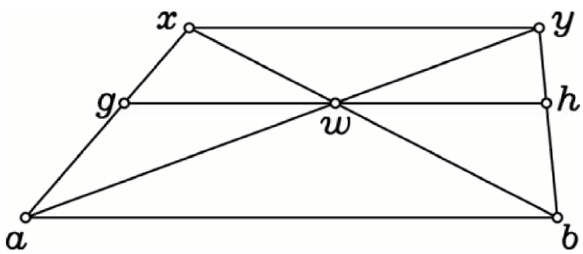


Figure 2

Analogically, $\frac{|hw|}{|ab|} = \left(\frac{|ab|}{|xy|} + 1\right)^{-1}$. Consequently, for the homothety ratio $\left(\frac{|ab|}{|xy|} + 1\right)^{-1} = \left(\frac{2}{d} + 1\right)^{-1} = \frac{d}{2+d}$, the common part of the images of the segments ab under homotheties with centers x and y is just the point w. ∎

LEMMA 2. *Let $C \subset E^n$ be a convex body and let x, y be boundary points of C. For every positive constant $d \leq 2$ the following conditions are equivalent.*

(i) $\mathrm{dist}_C(x, y) = d$,

(ii) *homothetical copies of C with homothety centers x and y, and with ratio $\frac{d}{2+d}$ have at least one common boundary point and they do not have common interior points.*

Lemma 2 follows from Lemma 1 when in the part of ab we take a longest segment contained in C which is parallel to xy (see Fig. 3). Observe that this way of proving permits to avoid using arguments of separation of the two copies of C by a straight line.

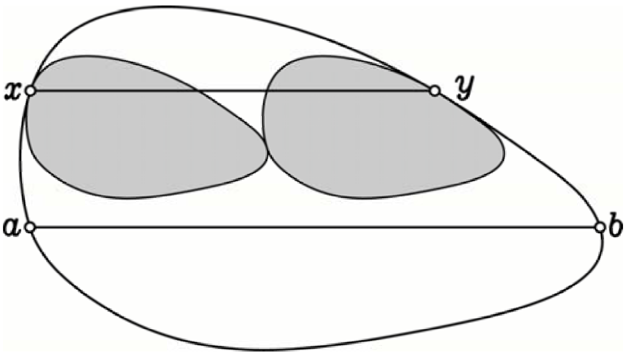


Figure 3

From the proof of Lemma 1 we see that in Lemma 2 only for the ratio $\frac{d}{2+d}$ we get exactly one point of the intersection of the two segments which are homothetical copies of ab. If the ratio is smaller, then the intersection is empty. If it is greater, then the intersection contains more than one point. So analogical equivalence like in Lemma 2 holds true if we have the inequality $\mathrm{dist}_C(x, y) < d$ in (i) and the condition about non-empty intersection of the interiors of copies in (ii).

2. Configurations of points in large relative distance

Denote by d_k the infimum of $d_k(C)$ over all convex bodies $C \subset E^2$. Compactness arguments show that this infimum is attained. Natural questions appear about the values of d_k for $k = 2, 3, \ldots$, and also for which convex bodies $C \subset E^2$ they are attained. We know only that $d_2 = 2$, and that $d_5 = 1$ see [8]. Of course, the value $d_2 = 2$ is attained for every convex body C. The value $d_5 = 1$ is attained for triangles, parallelograms and for some other polygons (e.q. for a pentagon which is obtained from a square by cutting off a triangle at a vertex).

A conjecture says that $d_3 = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ (see [8]). The example of a regular pentagon P in the part of C shows that this value cannot be replaced by a larger one.

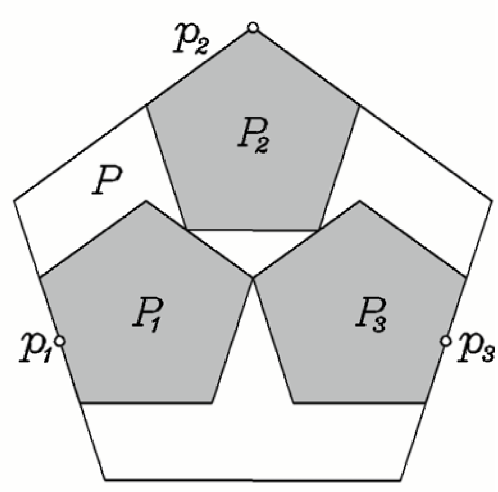


Figure 4

In Fig. 4 we see three homothetical copies P_1, P_2, P_3 of P and corresponding centers p_1, p_2, p_3 of homotheties. Observe that they can be moved step by step around P so that the relative distance between pairs of them is always $\frac{1}{2}(1 + \sqrt{5})$. Having in mind Theorem, instead of this we can say that P_1,

P_2, P_3 may be moved around such that they touch themselves and the boundary of P all the time. In Fig. 4 we first move p_1 up to the lower end of the corresponding side of P. This means that P_1 moves and makes some space which permits to move P_2. Simultaneously, p_2 moves on the boundary of P. Then we can move P_3 and so on. We conclude that for the regular pentagon the ratio $\frac{1+\sqrt{5}}{2}$ cannot be lessened.

In general case we know only some estimates. Namely, from [1] we see that $d_3 \geq \frac{4}{3}$.

We conjecture that $d_4 = \sqrt{5} - 1$. The estimate $d_4 \geq \frac{1}{3}(\sqrt{5} + 1)$ has been proved in [7]. Moreover, we conjecture that $d_6 = 1$. This value is attained for all centrally symmetric convex bodies (see [2]) and also for triangles. We also conjecture that $d_7 = \frac{4}{5}$, which is attained for triangles (see Fig. 5).

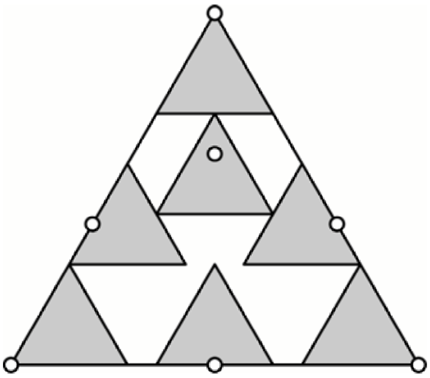


Figure 5

The following Proposition and its proof present a method of distribution of points in a planar convex body C in possibly large C-distances.

PROPOSITION. *Let C be a planar convex body and let $t \geq 2$ be an integer. In C we can find at least $\frac{1}{8}(t^2 + 4t + q)$ points in pairwise relative distances at least $\frac{4}{t}$, where $q = 3$ for t odd, where $q = 4$ for every even t which is not a multiple of 4, and where $q = 8$ if t is a multiple of 4.*

Proof. By Lemma 1 from [8] there is a parallelogram P circumscribed about C such that the midpoints of two its parallel sides belong to C (see Fig. 6). Denote them by a and c. Let b and d be points of C in two remaining sides of P. Denote by D the quadrangle $abcd$.

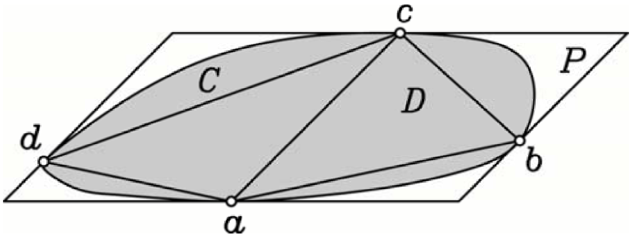


Figure 6

Put $w = t/2$ for t even, and $w = (t-1)/2$ for t odd. We provide segments $S_0, \ldots, S_w$ with endpoints in the boundary of the quadrangle D which are parallel to the segment ac; the line containing S_i should be in the C-distance $4i/t$ from d, where $i = 0, \ldots, w$. So the C-distances of those lines are at least $4/t$.

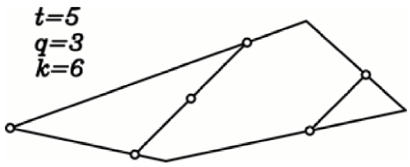
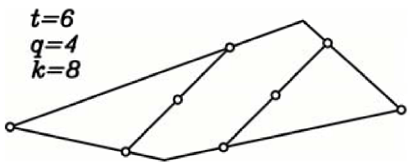


Figure 7



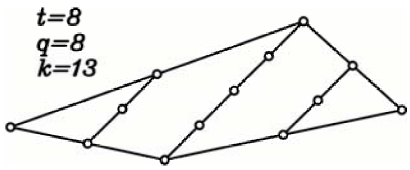


Figure 8

Figure 9

In Figures 7-9 we see the cases when $t = 5$, $t = 6$ and $t = 8$. They illustrate the three cases in Proposition 1. If $4i/t \leq 1$, then S_i contains $k = 2i+1$ points in pairwise relative distances at least $4/t$. If $4i/t > 1$, then S_i contains $k = 2(w-i)+1$ points in pairwise relative distances at least $4/t$ when t is even (see Fig. 8 and 9), and S_i contains $k = 2(w-i)+2$ points in pairwise relative distances at least $4/t$ when t is odd (see

Fig. 7). An easy calculation shows that the total number of those points in all the segments $S_0, \dots, S_w$ is exactly like in the formulation of Proposition 1. ∎

From Proposition we obtain a number of reasonable estimates for d_k when k is not very large: $d_3 \geq \frac{4}{3}$, $d_4 \geq d_5 \geq 1$, $d_6 \geq \frac{4}{5}$, $d_7 \geq d_8 \geq \frac{2}{3}$, $d_9 \geq d_{10} \geq \frac{4}{7}$, $d_{11} \geq d_{12} \geq d_{13} \geq \frac{1}{2}$. Pay attention that for $k = 3$ we get nothing else but the estimate from [1] and that for $k = 5$ we get again the estimate from [8]. Observe that the above estimate $d_4 \geq 1$ is weaker than the estimate $d_4 \geq \frac{1}{3}(\sqrt{5}+1) = 1.079\dots$ from [7], which is still far from the conjectured value $\sqrt{5} - 1 \approx 1.236$. The estimate $d_8 \geq \frac{2}{3}$ together with inequality $d_8 \leq \frac{2}{3}$ resulting from the example of a square in part of C leads to the equality $d_8 = \frac{2}{3}$

3. The case of centrally-symmetric bodies

If our body is centrally-symmetric, the problem of finding systems of far points can be regarded as looking for configurations of points in the unit disk M of a Minkowski space in possibly large Minkowski-distances. Such a problem is considered in [2]. The authors pay special attention to the case when all the points are required to be on the boundary of the unit disk.

A conjecture says that $d_3(M) \leq 1 + \frac{1}{2}\sqrt{2} \approx 1.707$ for every centrally-symmetric convex body M and that this value cannot be lessened (see [8] and [2]). The example of a regular octagon Q in the part of C shows that this value cannot be replaced by a larger one. In Fig. 10 we see three homothetical copies Q_1, Q_2, Q_3 of Q and corresponding centers q_1, q_2, q_3 of homotheties.

Similarly like for the pentagon in Fig. 4, the three points can be moved step by step around Q so that the relative distance between pairs of them is always $\frac{1+\sqrt{5}}{2}$. Thus the copies Q_1, Q_2, Q_3 move around such that they touch themselves and the boundary of Q all the time. We see that for Q the ratio $\frac{1+\sqrt{5}}{2}$ cannot be lessened. The paper [1] shows that $d_3(M) \geq$

1.546 for every centrally-symmetric convex body M. This recently has been improved up to $d_3(M) \geq 1 + \frac{1}{3}\sqrt{3} \approx 1.577$ (see [9]).

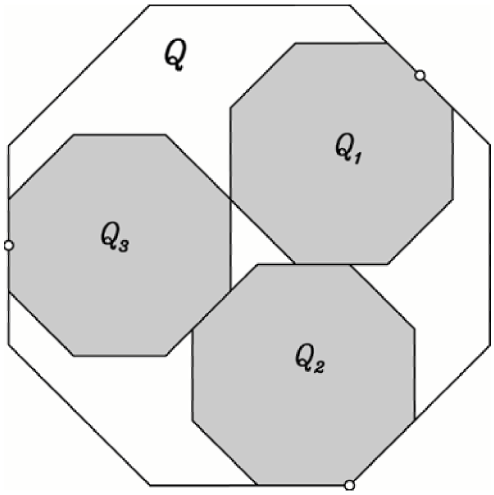


Figure 10

In [8] (see p. 247) and in [2] it is shown that $d_4(M) \geq \sqrt{2}$ for every centrally-symmetric convex body M. This value cannot be improved because of the example of the usual disk.

We have $d_5(M) \geq 1$ for every centrally symmetric convex body M, and this estimate cannot be improved in general. This immediately follows from $d_5 = 1$ and since $d_5(P) = 1$ for each parallelogram P.

From [8] (see p. 246) and from [2] we know that $d_6(M) \geq 1$, that $d_7(M) \geq 1$, and that both the estimates are the best possible.

CLAIM. *Let M be a planar centrally-symmetric convex body and let s be a positive integer. In M we can find at least $3s^2 + 3s + 1$ points in pairwise relative distances at least $\frac{1}{s}$.*

Proof. It is well known that we can inscribe in M an affine-regular hexagon H (under the assumption of the centrall-symmetry this was proved in many papers; the earliest of them seems to be [4]).

The central symmetry and convexity of M implies that for every diagonal of H there is no longer parallel segment in M. Take a hexagonal configuration of points in H like in Fig. 11.

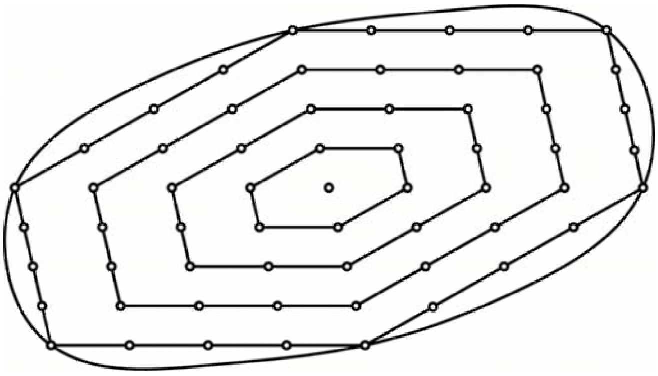


Figure 11

Considering s hexagons containing them on the boundaries we easily evaluate the number of those points: $1+6+\ldots+6s = 1+6\cdot\frac{s(s+1)}{2} = 3s^2+3s+1$. ∎

Observe that the thesis of Claim does not depend on the area of C like the estimates in the last section of [2]. From Claim, in particular, we obtain that for every centrally-symmetric body M we have $d_7(M) \geq 1$ (which has been observed in [8] and [2]), $d_i(M) \geq \frac{2}{3}$ for $i \leq 13$, and $d_i(M) \geq \frac{1}{2}$ for $i \leq 19$.

4. Corollaries about efficient selfpacking

Thanks to Theorem, we can reformulate the estimates about the values of d_k in terms of homothety ratios under which a number of homothetical copies of C may be packed in C. In particular, we can reformulate our Proposition and Claim in those terms. Of course, in every situation, the distribution of relatively far points shows the positions of the packed homothetical copies in C.

In particular, we have $r_2 = \frac{1}{2}$, $r_3 \geq \frac{2}{5}$, $r_4 \geq r_5 = \frac{1}{3}$, $r_7 \geq r_8 = \frac{1}{4}$, $r_9 \geq r_{10} \geq \frac{2}{9}$, $r_{11} \geq r_{12} \geq r_{13} \geq \frac{1}{5}$.

For every centrally-symmetric convex body M we have $r_2(M) = \frac{1}{2}$, $r_3(M) \geq \frac{4+\sqrt{3}}{13} \approx 0.441$, $r_4(M) \geq \sqrt{2} - 1 \approx 0.414$ (it cannot be improved for a disk), $r_5(M) \geq \frac{1}{3}$ (it cannot be improved for a parallelogram), $r_6(M) = r_7(M) = \frac{1}{3}$, $r_8(M) \geq \ldots \geq r_{13}(M) \geq \frac{1}{4}$, $r_{14}(M) \geq \ldots \geq r_{19}(M) \geq \frac{1}{5}$.

Also a few conjectures about the values of d_k (mentioned in preceding sections) can be reformulated in terms of the homothety ratios under which a number of homothetical copies of C can be packed in C. Below they are collected in one Conjecture.

CONJECTURE. *Every planar convex body can be packed with three homothetical copies of ratio $\frac{1}{5}\sqrt{5}$, with four copies of ratio $\frac{3}{2} - \frac{1}{2}\sqrt{5}$, with six copies of ratio $\frac{1}{3}$, and with seven copies of ratio $\frac{2}{7}$. Every centrally-symmetric convex body can be packed with three copies of ratio $\frac{5}{17} + \frac{5}{17}\sqrt{2}$.*

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