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Convexity Problems in Spaces of Constant Curvature and in Normed Spaces

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Abstract

We investigate various problems related to convexity in the three spaces of constant curvature (the Euclidean space, the hyperbolic space and the sphere) and in normed spaces.

Our first problem is due to Erdős and Szekeres, who, in 1935, made the conjecture that in a set S of $2^{k-2}+1$ points in the Euclidean plane, if no three points of S lie on the same line, then there are k points in convex position. In 2006, a computer-based proof of this conjecture for $k = 6$ was given by Peters and Szekeres. We give a proof of the conjecture, without the use of computers, for $k = 6$ if the convex hull of S is a pentagon.

Next, we introduce variants of convex sets and polyhedral domains in Euclidean d-space, called spindle convex sets and ball-polyhedra, respectively, and examine their properties. We prove the following theorem for the three planes of constant curvature. Among simple 'polygonal' curves of a given perimeter and with k circle arcs as 'edges', the regular one encloses the largest area. Then we disprove a conjecture of Maehara about spheres in the Euclidean n-space for $n \geq 4$, formulate variants of the conjecture for the hyperbolic and the spherical spaces, and prove similar results.

Lassak [39] proposed the problem of finding point sets in an oval at pairwise distances, measured in the norm relative to the oval, as large as possible. In Chapter 9, we show that, among seven points in an oval, there is a pair at a distance at most one, measured in the norm relative to the oval.

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Chapter 1

Introduction

We investigate problems related to convexity in the Euclidean space, in the hyperbolic space, on the sphere and in normed spaces.

In Chapters 2 and 3, we introduce the geometries we work with: the three spaces of constant curvature, and normed spaces. We present the tools we use in our investigation. In Chapter 3, we prove several theorems for the Euclidean, the hyperbolic and the spherical spaces, including a characterization of curves of constant geodesic curvature.

Our first problem involves finite point sets in the Euclidean plane $\mathbb{E}^2$, and it is discussed in Chapter 4. In 1935, Paul Erdős and George Szekeres proved the existence of a positive integer M for every integer $k \geq 3$, such that among M points in the plane in general position (that is, no three points lie on the same line), there are k points in convex position. They conjectured that the smallest such integer is equal to $2^{k-2}+1$. Their conjecture is trivial for $k=3$, was proven for $k=4$ by Esther Klein in the 1930s, and for $k=5$ by J. D. Kalbfleish, J. G. Kalbfleish and Stanton (cf. [36]) in 1970. In 2006, Peters and Szekeres gave a computer-based proof of the conjecture for $k=6$. Our main result is a proof of this conjecture, without the use of computers, for $k=6$ if the convex hull of the points is a pentagon.

In Chapter 5, we introduce a variant of the notion of convexity, called *spindle convexity.* A set S, in the Euclidean n-space, is *spindle convex* if for any two points $p, q \in S$, S contains the intersection of all the unit balls containing p and q. Spindle convexity is a special case of the concept of “Überkonvexität”, defined by Mayer in [42], who, in his definition, used translates of a given convex body instead of unit balls. Our goal is to find analogues, for spindle convex sets, of theorems from the theory of (linearly) convex sets. In Section 5.1, we prove a theorem about separating two spindle convex sets by a unit sphere. The main result of Section 5.2 is a Kirchberger-type theorem for separating two finite sets by a sphere of radius at most one. In Section 5.3, we prove counterparts of the Theorems of Carathéodory and Steinitz about a point contained in the convex hull (respectively, in the interior of the convex hull) of a point set. Finally, Section 5.4 deals with an Erdős-Szekeres type question. For $k \geq 3$, what is the smallest integer $\widehat{M}_n(k)$ satisfying the following: If S is a set of $\widehat{M}_n(k)$ points in the Euclidean n-space such that any $n+1$ points are in spindle convex position, then S contains k points in spindle convex position?

Our goal in Chapter 6 is to examine *ball-polyhedra*. A *ball-polyhedron* is the nonempty intersection of finitely many unit balls in $\mathbb{E}^n$. Our study is motivated by polyhedral domains, which were the subject of research for the ancient Greeks as well as for today's geometers. In particular, ball-polyhedra were studied, for example, by Heppes [32] and by Sallee [45]. In Section 6.1, we prove the Euler-Poincaré formula for a special class of 3-dimensional ball-polyhedra, called *standard ball-polyhedra*.

The results presented in Section 6.2 are motivated by the famous Kneser-Poulsen Conjecture. This conjecture states that the area of the intersection of finitely many Euclidean balls does not decrease under a contraction of the centres of the balls. The conjecture has been proven in the plane by K. Bezdek and Connelly [4], and is still open for $n \geq 3$. We prove that the inradius of the intersection of finitely many unit balls does not decrease under a contraction, and show that no similar statement holds for the diameter, the circumradius and the minimal width of the intersection.

In Section 6.3, we examine the edge-graphs of 3-dimensional standard ball-polyhedra. Our research is motivated by the characterization of the edge-graphs of 3-dimensional polytopes by Steinitz (cf. [50], pp. 103-126). In Section 6.4, we prove the following conjecture of K. Bezdek for the special class of ball-polyhedra: If C is a convex body in $\mathbb{E}^3$ such that any planar section of C is axially symmetric, then C is either a body of revolution or an ellipsoid. In Section 6.5, we present a variant of the Discrete Isoperimetric Inequality for 2-dimensional ball-polyhedra. The results in Chapters 5 and 6 are obtained in collaboration with K. Bezdek, M. Naszódi and P. Papez.

In Chapter 7, we generalize the isoperimetric problem discussed in Section 6.5. We consider a simple closed polygon Γ in $\mathbb{E}^2$, or $\mathbb{H}^2$ or $\mathbb{S}^2$, and replace the edges of Γ by the shortest closed curves of constant geodesic curvature $k_g \geq 0$ facing outwards (respectively, inwards). We call this object an *outer k_g-polygon* (respectively, *inner k_g-polygon*). If $\mathbf{M} = \mathbb{S}^2$, we assume also that Γ lies in an open hemisphere. We prove that, among inner k_g-polygons with a given perimeter and with a given number of vertices, the regular one has the largest area. Similarly, among outer k_g-polygons with a given perimeter ℓ and with a given number of vertices, the regular one has maximal area in the case that ℓ is not equal to the perimeter of a circle of geodesic curvature k_g. Otherwise, the area of an outer k_g-polygon is maximal if, and only if, the vertices of the k_g-polygon lie on a circle of geodesic curvature k_g. The results discussed in Chapter 7 are obtained in collaboration with B. Csikós and M. Naszódi.

Maehara [40] proved the following theorem. Let $\mathfrak{F}$ be a family of at least $n+3$ distinct $(n-1)$-spheres in $\mathbb{E}^n$. If any $n+1$ spheres in $\mathfrak{F}$ have a point in common, then all of them have a point in common. Maehara conjectured for $n \geq 3$ that the assertion is valid if $\mathfrak{F}$ is a family of $n+2$ distinct unit $(n-1)$-spheres in $\mathbb{E}^n$. In Chapter 8, we construct a family $\mathfrak{F}$ of $n+2$ distinct unit $(n-1)$-spheres, with $n \geq 4$, such that any $n+1$ members of $\mathfrak{F}$ have a point in common, but $\bigcap \mathfrak{F} = \emptyset$. We prove a variant of Maehara's theorem for the hyperbolic and the spherical space, and examine also the counterparts of his conjecture. The results in Chapter 8 are a joint work with K. Bezdek, M. Naszódi and P. Papez.

Lassak [39] proposed the problem of finding point sets in a plane convex body C at pairwise distances, measured in the norm relative to C, as large as possible. In Chapter 9, we prove that, in a set of seven points in a plane convex body C, there is a pair at a distance at most one, measured in the norm relative to C. This result verifies a conjecture stated in [11]. We give a partial characterization of the plane convex bodies and point sets for which the minimal pairwise distance of the points is equal to one. Our results yield also that if seven homothetic copies of a plane convex body C, with a positive homothety ratio λ, are packed into C, then $\lambda \leq \frac{1}{3}$. The results in this chapter are a joint work with A. Joós.

Theorems, proposition, definitions, etc. are numbered by chapter and section. In case of a theorem from the literature, the names of the authors are given in brackets following the word 'Theorem'. In case of new theorems, there are no names listed. Thus, a theorem, with names after it, is *not* officially included in the work of the author, even if his name appears in the list (e.g. Theorem 9.1.1).

Chapter 2

The Euclidean space and normed spaces

2.1 The Euclidean space and its affine properties

Consider the vector space $\mathbb{R}^n$ of the real n-tuples over the field $\mathbb{R}$ of real numbers. An element of $\mathbb{R}^n$ is a *vector* or a *point*. The zero vector $(0, 0, \ldots, 0)$ is the *origin* of $\mathbb{R}^n$, which we denote by o. For any two vectors $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $y = (\beta_1, \beta_2, \ldots, \beta_n)$, we define

$$< x, y >= \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n,$$

which we call the *standard inner product*, or simply the *inner product of x and y*. The *n-dimensional Euclidean space* $\mathbb{E}^n$ is the real vector space $\mathbb{R}^n$ equipped with the function $< ., . >: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

For $x \in \mathbb{E}^n$, we call $\|x\| = \sqrt{< x, x >}$ the *Euclidean norm of x*. The *Euclidean distance of points* $x, y \in \mathbb{E}^n$ is defined as $\mathrm{dist}(x, y) = \|y - x\|$. It is easy to see that $\mathbb{R}^n$ with the Euclidean distance function is a metric space. The topology induced by this metric function is the *usual topology* of $\mathbb{E}^n$.

The *distance* $\mathrm{dist}(A, B)$ *of the* nonempty *sets* $A, B \subset \mathbb{E}^n$ is $\inf\{\mathrm{dist}(a, b) : a \in A \text{ and } b \in B\}$. The *angle* of two nonzero vectors $x, y \in \mathbb{E}^n$ is

$$\sphericalangle(x, y) = \arccos\left(\frac{< x, y >}{\|x\| \cdot \|y\|}\right).$$

A function $f : \mathbb{E}^n \to \mathbb{E}^n$ that satisfies $\mathrm{dist}(f(x), f(y)) = \mathrm{dist}(x, y)$ for any $x, y \in \mathbb{E}^n$ is an *isometry of* $\mathbb{E}^n$.

There is a natural extension of vector operations for subsets of $\mathbb{E}^n$. Namely, for $A, B \subset \mathbb{E}^n$ distinct from the empty set and $\alpha, \beta \in \mathbb{R}$, we set

$$\alpha A + \beta B = \{\alpha a + \beta b : a \in A \text{ and } b \in B\}.$$

The set $A+B$ is called the *vector sum* or *Minkowski sum of* A *and* B (cf. Figure 2.1). For simplicity, we denote $\{a\} + B$ by $a + B$. A set $A \subset \mathbb{E}^n$ that satisfies $A = -A$ is called an o-symmetric set.

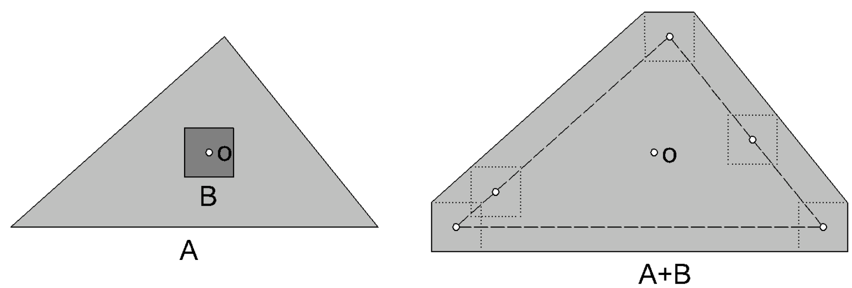

Figure 2.1: Minkowski sum

A set in the form $K = x + L$, where L is a linear subspace in $\mathbb{E}^n$ and $x \in \mathbb{E}^n$, is an *affine subspace of* $\mathbb{E}^n$ *of dimension* $\dim K = \dim L$. It is convenient to call the empty set $\emptyset$ a (-1)-dimensional subspace. Affine subspaces of dimension 0, 1, 2 and $n-1$, called, respectively, *points*, *lines*, *planes* or *hyperplanes*, often play an important role. Hyperplanes correspond to the level surfaces of nondegenerate linear functionals. *Affine subspaces* of the form $x + L$ and $y + L$, where $x, y \in \mathbb{E}^n$ and L is a linear subspace of $\mathbb{E}^n$, are *parallel.* Parallel hyperplanes are different level surfaces of the same nondegenerate linear functional.

The intersection of (finitely or infinitely many) affine subspaces is an affine subspace. For any nonempty set $A \in \mathbb{E}^n$, the intersection of all the affine subspaces that contain A is the *affine hull of* A, denoted by $\operatorname{aff} A$. For $a_1, a_2, \ldots, a_k \in \mathbb{E}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ with $\sum_{i=1}^{k} \lambda_i = 1$, the point $\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_k a_k$ is an *affine combination of* $a_1, a_2, \ldots, a_k$. It is easy to show that, for any nonempty $A \subset \mathbb{E}^n$, $\operatorname{aff} A$ consists of all the affine combinations of points of A.

The restriction of the standard inner product to an affine subspace $K \subset \mathbb{E}^n$ of dimension k, where $1 \leq k \leq n$, induces a topology. This topological space is homeomorphic to $\mathbb{E}^k$. For any set $A \subset \mathbb{E}^n$, the interior or boundary of A with respect to the topology in $\operatorname{aff} A$, is, respectively, the *relative interior of* A, denoted by $\operatorname{relint} A$, or the *relative boundary of* A, denoted by $\operatorname{relbd} A$.

A *set* $A \subset \mathbb{E}^n$ is said to be *affinely independent* if no point a of A is an affine combination of points of A distinct from a. Equivalently, A is affinely independent, if $\sum_{i=1}^{k} \lambda_i a_i = 0$ and $\sum_{i=1}^{k} \lambda_i = 0$ implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$ for any $a_1, a_2, \ldots, a_k \in A$. The *points* of a set $A \subset \mathbb{E}^n$ are *in general position*, if any $n+1$ of them are affinely independent.

Let $A \subset \mathbb{E}^n$. If $\dim \operatorname{aff} A = k$, then we say that the *dimension of* A is $\dim A = k$. In this case, an affinely independent subset $B \subset A$ contains at most $k+1$ points. Furthermore, $\operatorname{aff} B = \operatorname{aff} A$ if, and only if, B contains exactly $k+1$ points.

An *affine transformation* is a transformation $h : \mathbb{E}^n \to \mathbb{E}^n$ with the property that $h(l)$ is a line for every line l in $\mathbb{E}^n$. Every affine transformation may be written as $h(x) = y + g(x)$, where g is a linear transformation with $\det g \neq 0$ and $y \in \mathbb{E}^n$.

Let $(a_1, a_2, \ldots, a_{n+1})$ be an ordered $(n+1)$-tuple of affinely independent points in $\mathbb{E}^n$. Then the determinant $D = \det(a_2 - a_1, a_3 - a_1, \ldots, a_{n+1} - a_1)$ is not zero. The *orientation* of $(a_1, a_2, \ldots, a_{n+1})$ is the sign of the determinant D. In the plane, *three points* are *in counterclockwise*, respectively *clockwise*, *order*, if their orientation is positive, respectively negative. The *points* $a_1, a_2, \ldots, a_k = a_0 \in \mathbb{E}^2$ are *in counter-clockwise*, respectively *clockwise*, *order*, if a_{i-1}, a_i and a_{i+1} are in counterclockwise, respectively clockwise, order for $i = 1, 2, \ldots, k$.

2.2 Convex sets and their separation properties

Let $a, b \in \mathbb{E}^n$. The *closed* and *open segment with endpoints* a *and* b are, respectively, the sets $[a, b] = \{\lambda a + (1-\lambda)b : 0 \leq \lambda \leq 1\}$ and $(a, b) = \{\lambda a + (1-\lambda)b : 0 < \lambda < 1\}$. The *closed* (resp., *open*) *half line with endpoint* $p \in \mathbb{E}^n$ *and direction* $v \in \mathbb{E}^n$ is the set $\{p + \lambda v : \lambda \in [0, \infty)\}$ (resp., $\{p + \lambda v : \lambda \in (0, \infty)\}$). Two *segments* or *half lines* are *parallel*, if their affine hulls are parallel. If $[a, b], [c, d] \subset \mathbb{E}^n$ are parallel segments and $c \neq d$, then

$$\frac{\|a-b\|}{\|c-d\|} = \frac{\|h(a)-h(b)\|}{\|h(c)-h(d)\|}$$

for any affine transformation h.

A *set* $C \subset \mathbb{E}^n$ is *convex* if $a, b \in C$ implies that $[a, b] \subset C$. Clearly, affine subspaces of $\mathbb{E}^n$ are convex. Furthermore, the intersection of (finitely or infinitely many) convex sets is convex.

If A and B are two nonempty sets, then their direct sum is defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

If $A \subset \mathbb{E}^k$ and $B \subset \mathbb{E}^l$ are nonempty and convex, then $A \times B \subset \mathbb{E}^{k+l}$ is convex.

The intersection of all the convex sets that contain a given nonempty set $A \subset \mathbb{E}^n$ is the *convex hull of* A, denoted by $\operatorname{conv} A$ or $[A]$ (cf. Figure 2.2). For $A_1, A_2, \ldots, A_k \subset \mathbb{E}^n$ distinct from $\emptyset$, we define $[A_1, A_2, \ldots, A_k] = [A_1 \cup A_2 \cup \ldots \cup A_k]$ and, $(A_1, A_2, \ldots, A_k) =$

relint$[A_1 \cup A_2 \cup \ldots \cup A_k]$. Here, for simplicity, we omit the curly brackets if some of the sets are singletons. Thus, $[a, B]$ denotes $[\{a\}, B]$. If, for any $x \in A$, $x \notin \mathrm{conv}(A \backslash \{x\})$, we say that the *points* of A are *in convex position*.

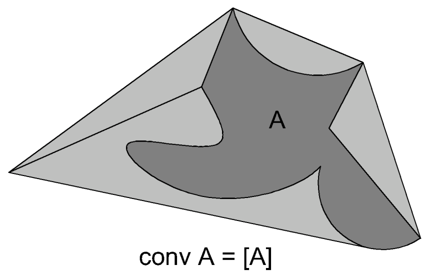

Figure 2.2: Convex hull of a set

A point in the form $\sum_{i=1}^{k} \lambda_i a_i$, where $a_i \in \mathbb{E}^n$ and $\lambda_i \geq 0$ for $i = 1, 2, \ldots, k$ such that $\sum_{i=1}^{k} \lambda_i = 1$, is called a *convex combination of* $a_1, a_2, \ldots, a_k$. For a nonempty set $A \subset \mathbb{E}^n$, $[A]$ consists of all the convex combinations of points of A. The *centroid of the* finite *set* $A = \{a_1, a_2, \ldots, a_k\}$ is the point $(\sum_{i=1}^{k} a_i)/k$.

Carathéodory's Theorem states that if $x \in [A]$ and $A \in \mathbb{E}^n$, then x is the convex combination of $k \leq n+1$ points of A. In other words, if $x \in [A]$ then $x \in [B]$ for some $B \subset A$ with card $B \leq n + 1$. If $A = \{a_1, a_2, \ldots, a_{n+1}\}$ is an affinely independent set and x is the centroid of A, we see that $n+1$ in Carathéodory's Theorem is necessary.

A similar theorem is due to Steinitz: If $x \in (A)$, then x is in the interior of the convex hull of at most $2n$ points of A. If $A = \{\pm a_1, \pm a_2, \ldots, \pm a_n\}$, where the points $a_1, a_2, \ldots, a_n$ are linearly independent, then the origin o is not in the interior of the convex hull of any proper subset of A. This shows that $2n$ in Steinitz's Theorem is necessary.

For any hyperplane H, $\mathbb{E}^n \setminus H$ consists of two convex, open, connected components, whose boundary is H. These two components are the *open half spaces bounded by* H. Their closures are the *closed half spaces bounded by* H. Two *sets* $A, B \subset \mathbb{E}^n$, contained in distinct closed (respectively, open) half spaces bounded by the hyperplane H, are said to be *separated* (respectively, *strictly separated*) by H.

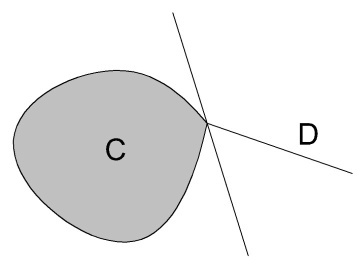

Figure 2.3: Separation of convex sets

Let C and D be nonempty convex sets. Then there is a hyperplane that separates C and D if, and only if, their relative interiors do not intersect (cf. Figure 2.3). If C and D are disjoint nonempty closed convex sets and at least one of them is compact, then there is a hyperplane that strictly separates C and D.

Kirchberger's theorem states that if $A, B \in \mathbb{E}^n$ are compact and, for any $T \subset A \cup B$ with $\operatorname{card} T \leq n+2$, $T \cap A$ and $T \cap B$ are strictly separated by a hyperplane, then A and B are strictly separated by a hyperplane. The proof of Kirchberger's theorem is based on Helly's theorem, one of the most fundamental theorems in the theory of convex sets. According to Helly's theorem, if $C_1, C_2, \ldots, C_k$, $k \geq n+1$, are convex sets such that any $n+1$ of them have a nonempty intersection, then $\bigcap_{i=1}^k C + i \neq \emptyset$. There is an infinite version of Helly's theorem: if $\{C_i : i \in I\}$ is an infinite family of closed convex sets at least one of which is compact, and any $n+1$ of them have a nonempty intersection, then $\bigcap_{i \in I} C_i \neq \emptyset$.

2.3 Convex bodies

An important class of convex sets is the family of *convex bodies*. A convex body is a compact, convex set with nonempty interior. We call a plane convex body an *oval*. We denote the family of ovals by $\mathfrak{C}$ and the family of o-symmetric ovals by $\mathfrak{M}$. Let $C, C_1, C_2, \ldots, C_k \in \mathfrak{C}$. If $\bigcup_{i=1}^{k} C_i \subset C$, and $C_i \cap C_j = \emptyset$ for every $i \neq j$, we say that C is *packed* by $C_1, C_2, \ldots, C_k$. If $C \subset \bigcup_{i=1}^{k} C_i$, we say that $C_1, C_2, \ldots, C_k$ *cover* C.

An important convex body in $\mathbb{E}^n$ is the *closed n-dimensional unit ball with centre o*: $\mathbf{B}^n[o] = \{x \in \mathbb{E}^n : \|x\| \leq 1\}$. For $c \in \mathbb{E}^n$ and $r > 0$, $\mathbf{B}^n[c, r] = c + r\mathbf{B}^n[o]$ is the *closed n-dimensional ball of radius r and centre c*, $\mathbf{B}^n(c, r) = \operatorname{int} \mathbf{B}^n[c, r]$ is the *open n-dimensional ball of radius r and centre c* and $\mathbb{S}^{n-1}(c, r) = \operatorname{bd} \mathbf{B}^n[c, r]$ is the *$(n-1)$-dimensional sphere of radius r and centre c*. A *k-dimensional ball of radius r and centre c* is the intersection of $\mathbf{B}^n[c, r]$ with an affine k-space that contains c. We define lower dimensional spheres similarly. Note that a 0-dimensional sphere is a pair of points, and a 0-dimensional ball is a singleton.

For simplicity, we set $\mathbf{B}^n[c] = \mathbf{B}^n[c, 1]$, $\mathbf{B}^n(c) = \mathbf{B}^n(c, 1)$ and $\mathbb{S}^{n-1}(c) = \mathbb{S}^{n-1}(c, 1)$. Furthermore, for a nonempty set $X \subset \mathbb{E}^n$, $\mathbf{B}[X] = \bigcap_{x \in X} \mathbf{B}^n[x]$ and $\mathbf{B}(X) = \bigcap_{x \in X} \mathbf{B}^n(x)$. We define $\mathbf{B}[\emptyset)$ and $\mathbf{B}(\emptyset)$ as the ambient space $\mathbb{E}^n$. If $x \in \mathbb{S}^{n-1}(c_1, r_1) \cap \mathbb{S}^{n-1}(c_2, r_2)$ then the *angle* of $\mathbb{S}^{n-1}(c_1, r_1)$ and $\mathbb{S}^{n-1}(c_2, r_2)$ is $\pi - \sphericalangle(x - c_1, x - c_2)$.

Consider a convex set $C \subset \mathbb{E}^n$ and a point $x \in \operatorname{bd} C$. Due to the separation properties of convex sets, there is a hyperplane H that separates x and C. Clearly, $x \in H$ and $H \cap \operatorname{int} C = \emptyset$. We say that H *supports C at x*. Given a hyperplane H and a convex body C in $\mathbb{E}^n$, there are exactly two hyperplanes parallel to H that support C.

Let $C, D \subset \mathbb{E}^n$ be convex bodies. The *Hausdorff distance of C and D* is

$$\min\{\delta \in [0, \infty) : C \subset D + \delta \mathbf{B}^n(o) \text{ and } D \subset C + \delta \mathbf{B}^n(o)\}.$$

It is easy to see that the Hausdorff distance of convex bodies defines a metric on the family of n-dimensional convex bodies. We call this metric the *Hausdorff metric*.

Blaschke's Selection Theorem states that, in every infinite family $\mathfrak{F} = \{C_i : i \in I\}$ of n-dimensional convex bodies contained in a given ball $\mathbf{B}^n(c, r)$, there is a subfamily $\{C_k : k = 1, 2, 3 \ldots\} \subset \mathfrak{F}$ such that $\lim_{k\to\infty} C_k$ exists with respect to Hausdorff distance.

Let $C \subset \mathbb{E}^n$ be a convex body. If $x, y \in C$ implies that $(x, y) \subset \operatorname{int} C$, then we say that C is *strictly convex*. In other words, C is strictly convex if it does not contain a segment in its boundary. If, for every $x \in \operatorname{bd} C$, there is a unique hyperplane that supports C at x, we say that C is *smooth*.

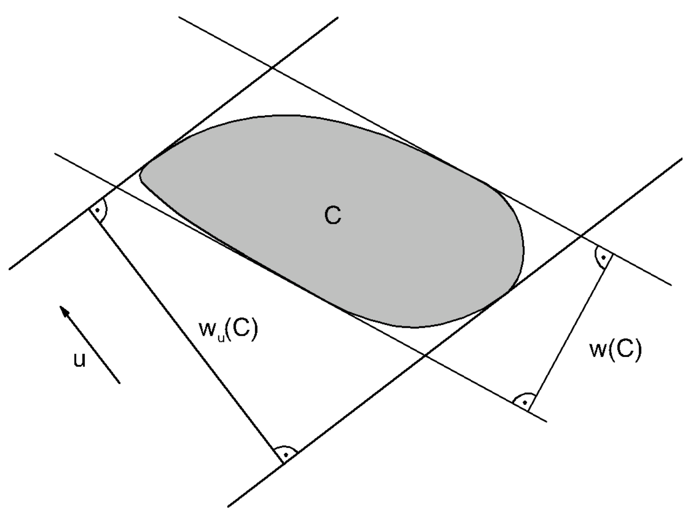

Figure 2.4: Width in the direction u and minimal width

The *diameter of a* bounded *set* P is

$$\operatorname{diam} P = \sup\{\operatorname{dist}(x, y) : x, y \in P\}.$$

If C is a convex body and $u \in \mathbb{S}^{n-1}(o)$, then

$$\mathrm{w}_u(C) = \max\{< x, u >: x \in C\} - \min\{< x, u >: x \in C\}$$

is the *width of* C *in the direction* u. This number is equal to the distance between the two supporting hyperplanes of C, perpendicular to u (cf. Figure 2.4). The *minimal width of* C is

$$\mathrm{w}(C) = \min\{\mathrm{w}_u(C) : u \in \mathbb{S}^{n-1}(o)\}.$$

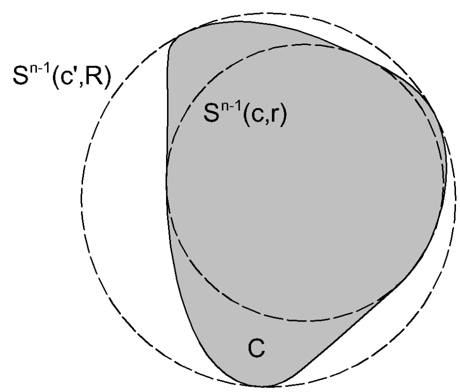

Figure 2.5: Circumsphere and insphere

For every bounded set $C \subset \mathbb{E}^n$, there is a unique smallest ball $\mathbf{B}^n[c', R]$ that contains C. The *circumball, circumsphere, circumradius* and *circumcentre of* C are $\mathbf{B}^n[c', R]$, $\mathbb{S}^{n-1}(c', R)$, R and c', respectively. We denote the circumradius of C by $\mathrm{cr}(C)$.

The radius of a largest open ball contained in the bounded, convex set $C \subset \mathbb{E}^d$ is the *inradius* of C, denoted by $\mathrm{ir}(C)$. If there is a unique largest ball $\mathbf{B}^n(c,r)$ contained in C, then $\mathbf{B}^n[c,r]$, $\mathbb{S}^{n-1}(c,r)$ and c are called the *inball, insphere* and *incenter of* C, respectively (cf. Figure 2.5).

If H is a hyperplane that supports a convex body C then $H \cap C$ is a *face of* C. We regard C as a face of itself. A face F of C with $\dim \operatorname{aff} F = k$ is a *k-face of* C. The empty set is a (-1)-dimensional face of C. If $\{x\}$ is a face of C, then x is an *exposed point of* C. A point $x \in C$ that is not contained in any open segment $(y,z) \subset C$ is an *extreme point of* C. We denote the set of extreme points of C by $\operatorname{ext} C$ and the set of exposed points of C by $\exp C$. Clearly, $\exp C \subset \operatorname{ext} C$, but these sets are generally not equal (cf. Figure 2.6). The Krein-Milman Theorem states that $C = [\operatorname{ext} C] = [\operatorname{cl} \exp C]$.

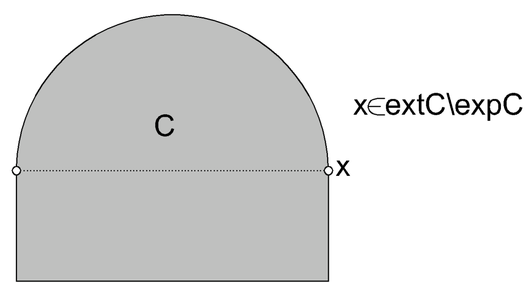

Figure 2.6: An extreme point which is not an exposed point

2.4 Convex polytopes

The convex hull of a finite point set S is a *convex polytope* or simply a *polytope*. The intersection of finitely many closed half spaces is a *polyhedral domain*. A set is a polytope if, and only if, it is a bounded polyhedral domain. Two-dimensional polytopes are called *polygons*.

A 0-face, or a 1-face, or an $(n-1)$-face of a polytope $P \in \mathbb{E}^n$, is a *vertex*, or *edge*, or *facet* of P, respectively. The set of the vertices of P is called the vertex set of P, and is denoted by $V(P)$. A face of a polytope P is $[S]$ for some $S \subset V(P)$. Furthermore, if $F_1 = [S_1]$ and $F_2 = [S_2]$ are faces of a polytope P, $S_1, S_2 \subset V(P)$, then $F_1 \cap F_2 = [S_1 \cap S_2]$ is a face of P. Hence, the family of faces of P forms a (bounded) lattice.

Let f_k denote the number of k-faces of P. The Euler-Poincaré Formula is:

$$\sum_{i=0}^{n-1} f_k = 1 + (-1)^{n-1}.$$

A sequence of faces of P, $F_0, F_1, \ldots, F_{n-1}, F_n = P$ such that $\dim F_k = k$ and $F_k \subset F_{k+1}$, is called a *flag of* P. If, for every two flags $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of P, there is an isometry $h : \mathbb{E}^n \to \mathbb{E}^n$ that maps $\mathfrak{F}_1$ into $\mathfrak{F}_2$, then P is a *regular polytope*. In particular, a polygon is regular if, and only if, the lengths of its edges and its angles are equal.

2.5 Volume and surface area

Let $[\alpha_i, \beta_i]$ be an interval in the ith coordinate axis. Then $[\alpha_1, \beta_1] + [\alpha_2, \beta_2] + \ldots + [\alpha_n, \beta_n] \subset \mathbb{E}^n$ is an *elementary brick*. We set $\lambda_n(B) = \prod_{i=1}^{n}(\beta_i - \alpha_i)$ and call this quantity the *volume of* B.

Let $\{B_i : i = 1, 2, \ldots, k\}$ be a family of elementary bricks such that any two elements have disjoint interiors. We say that $\{B_i : i = 1, 2, \ldots, k\}$ is a *packing of elementary bricks*. We define $\lambda_n(\bigcup_{i=1}^{k} B_i) = \sum_{i=1}^{k} \lambda_n(B_i)$. It is easy to check that this quantity is independent of the way $\bigcup_{i=1}^{k} B_i$ is decomposed into a packing of elementary bricks. We note also that the union of finitely many elementary bricks may be decomposed into a packing of elementary bricks, which enables us to define the volume of the union of elementary bricks.

If $A \subset \mathbb{E}^n$ is a bounded set, then the infimum of $\lambda_n(\bigcup_{i=1}^{k} B_i)$, over all finite families $\{B_i : i = 1, 2, \ldots, k\}$ of elementary bricks whose union contains A, is the *outer measure* $\lambda_n^{out}(A)$ *of* A. Similarly, the supremum of $\lambda_n(\bigcup_{i=1}^{k} B_i)$ over all finite families of elementary bricks whose union is contained in A, is the *inner measure* $\lambda_n^{in}(A)$ *of* A. If $\operatorname{int} A = \emptyset$, we set $\lambda_n^{in}(A) = 0$. If $\lambda_n^{out}(A) = \lambda_n^{in}(A)$, then we say that the *$n$-dimensional volume of* A is $\operatorname{vol}_n(A) = \lambda_n^{out}(A)$. Note that $\operatorname{vol}_n(U) = \lambda_n(U)$, if U is a union of elementary bricks. It is known that every n-dimensional bounded convex set has n-dimensional volume. The 2-dimensional volume of a set $A \subset \mathbb{E}^2$ is called the *area of* A, denoted by $\operatorname{area}(A)$.

Some properties of the volumes of convex sets are the following.

- If $A, B \subset \mathbb{E}^n$ are bounded, convex sets with disjoint interiors and $A \cup B$ is convex, then $\operatorname{vol}_n(A \cup B) = \operatorname{vol}_n(A) + \operatorname{vol}_n(B)$.

- If $A \subset \mathbb{E}^n$ is bounded and convex, and $\alpha \in \mathbb{R}$, then $\operatorname{vol}_n(\alpha A) = |\alpha|^n \operatorname{vol}_n(A)$.

- For every isometry $h : \mathbb{E}^n \to \mathbb{E}^n$ and every bounded, convex set $A \subset \mathbb{E}^n$, $\operatorname{vol}_n(h(A)) = \operatorname{vol}_n(A)$.

- If $h : \mathbb{E}^n \to \mathbb{E}^n$ is an affine transformation, $A, B \subset \mathbb{E}^n$ are bounded and convex, and $\operatorname{vol}_n(B) > 0$, then
$$\frac{\operatorname{vol}_n(h(A))}{\operatorname{vol}_n(h(B))} = \frac{\operatorname{vol}_n(A)}{\operatorname{vol}_n(B)}.$$

Let $C \subset \mathbb{E}^n$ be a convex body. The limit

$$\mathrm{surf}(C) = \lim_{t \to 0} \frac{\mathrm{vol}_n(C + t\mathbf{B}^n[o]) - \mathrm{vol}_n(C)}{t}$$

is the *surface area of* C. This limit exists for every convex body. For an oval C, we may use the term *perimeter* $\mathrm{perim}(C)$ *of* C for $\mathrm{surf}(C)$.

There is another, equivalent way to define $\mathrm{perim}(C)$. Let $\gamma : [0,1] \to \mathbb{E}^n$ be a continuous curve. If the set $S = \{\sum_{i=1}^{k} \|\gamma(t_i) - \gamma(t_{i-1})\| : k = 1, 2, 3, \ldots$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_k = 1\}$ is bounded, we say that γ is *rectifiable* and the *arc length* of γ is $\mathrm{arclength}(\gamma) = \sup S$. If γ is continuously differentiable, then

$$\mathrm{arclength}(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt.$$

The boundary of an oval C is a simple, closed, continuous curve with arc length $\mathrm{perim}(C)$.

2.6 Normed spaces

Observe that the Euclidean norm $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ is a map $\|.\| : \mathbb{E}^n \to \mathbb{R}$ with the following properties:

- $\|x\| \geq 0$ for $x \in \mathbb{E}^n$ with equality if, and only if, $x = 0$;
- $\|\lambda x\| = |\lambda| \cdot \|x\|$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{E}^n$;
- $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in \mathbb{E}^n$.

This suggests the following definition.

Definition 2.6.1. Consider the n-dimensional real vector space $\mathbb{R}^n$. If a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

- $f(x) \geq 0$ for $x \in \mathbb{R}^n$ with equality if, and only if, $x = 0$;
- $f(\lambda x) = |\lambda| f(x)$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$;
- $f(x + y) \leq f(x) + f(y)$ for $x, y \in \mathbb{R}^n$;

then f is a *norm* in $\mathbb{R}^n$. The vector space $\mathbb{R}^n$ equipped with a norm is a *normed space* or, if $n = 2$, a *normed plane*.

Obviously, the Euclidean norm is a norm. Furthermore, if C is an o-symmetric convex body in $\mathbb{E}^n$, then the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \min\{\lambda : x \in \lambda C, \lambda \geq 0\}$ is a norm, which we denote by $\|.\|_C : \mathbb{R}^n \to \mathbb{R}$. On the other hand, consider a norm $f : \mathbb{R}^n \to \mathbb{R}$. The set $C = \{x \in \mathbb{R}^n : f(x) \leq 1\}$ is the *unit ball of the norm*. The unit ball of any norm is an o-symmetric convex body in $\mathbb{E}^n$. Hence any norm f is $\|.\|_C$, where C is the unit ball of f.

A *norm* $\|.\|_C$ is *strictly convex* if its unit ball C is strictly convex in $\mathbb{E}^n$. A *norm* is *smooth* if its unit ball is smooth. Observe that $\mathrm{dist}_C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\mathrm{dist}_C(x, y) = \|y - x\|_C$ is a metric, which we call the *metric defined by the norm* $\|.\|_C$. The quantity $\mathrm{dist}_C(x, y)$ is the *normed distance of points x and y*. The *normed distance of the nonempty sets $X, Y \subset \mathbb{R}^n$* is $\mathrm{dist}_C(X, Y) = \min\{\mathrm{dist}_C(x, y) : x \in X, y \in Y\}$. We may denote the Euclidean norm $\|.\|$ and distance $\mathrm{dist}(.,.)$ by $\|.\|_{\mathbb{E}}$ and $\mathrm{dist}_{\mathbb{E}}(.,.)$, respectively.

If $X \subset \mathbb{R}^n$ is bounded and $\|.\|_C : \mathbb{R}^n \to \mathbb{R}$ is a norm, then the *diameter of X with respect to the norm* $\|.\|_C$ is $\mathrm{diam}_C(X) = \sup\{\mathrm{dist}_C(x, y) : x, y \in X\}$. Consider a convex body $D \in \mathbb{E}^n$. If, for any $u \in \mathbb{S}^{n-1}(o)$, the normed distance of the two supporting hyperplanes of D, orthogonal to u, is equal to a constant d, then D is *of constant width d with respect to the norm* $\|.\|_C$. We note that D is of constant width two with respect to $\|.\|_C$ if, and only if, $1/2(D-D)=C$. The convex body $1/2(D-D)$ is the *central symmetral of* D.

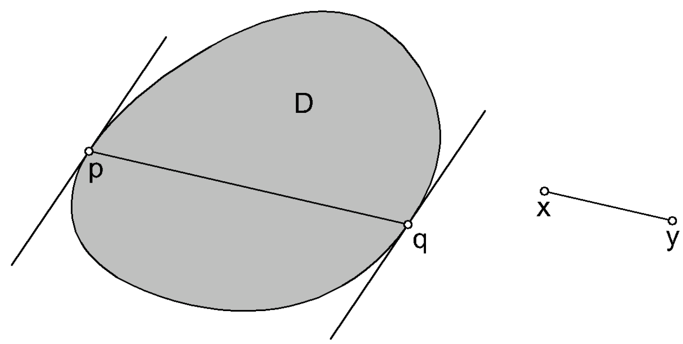

Figure 2.7: Relative distance of points x and y

The norm with unit ball $C=1/2(D-D)$ is the *relative norm of* D. The *D-distance of points* $x,y\in\mathbb{R}^n$ is $\mathrm{dist}_D(x,y)=\mathrm{dist}_C(x,y)$, where $C=1/2(D-D)$. In other words,

$$\mathrm{dist}_D(x,y)=\frac{2\|y-x\|_{\mathbb{E}}}{\|p-q\|_{\mathbb{E}}},$$

where $p,q\in D$ such that $q-p$ is parallel to $y-x$, and there are no points $p',q'\in D$ such that $\|q'-p'\|_{\mathbb{E}}>\|q-p\|_{\mathbb{E}}$ and $q'-p'$ is parallel to $y-x$ (cf. Figure 2.7). If it is clear which convex body D we write about, we may also use the term *relative distance*.

Observe that if $C\subset D\subset\mathbb{E}^n$ are convex bodies, then $d_C(x,y)\geq d_D(x,y)$ for every $x,y\in\mathbb{E}^n$. Furthermore, a convex body C is a body of constant width two in its relative norm. More generally, for any $\lambda\in[0,1]$, $\lambda C+(1-\lambda)(-C)$ is a body of constant width two in the relative norm of C.

Chapter 3

Hyperbolic and spherical spaces

3.1 Riemannian manifolds

Definition 3.1.1. Let S be a nonempty set and $\mathfrak{F}$ be a subfamily of the family $P(S)$ of the subsets of S. If

1. $\emptyset, S \in \mathfrak{F}$,

2. any intersection of (finitely or infinitely many) elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and

3. the union of any finitely many elements of $\mathfrak{F}$ is in $\mathfrak{F}$,

then we say that S is a *topological space with the topology* $\mathfrak{F}$. Elements of $\mathfrak{F}$ are called *open subsets of* S. A *neighborhood of* $p \in S$ is a subset of S which has an open subset containing p. The complement of an open set is a *closed* set.

Let S_1 and S_2 be topological spaces. A function $f : S_1 \to S_2$ is called *continuous*, if the pre-image of any open subset of S_2 is open in S_1. The topological spaces S_1 and S_2 are *homeomorphic*, if there is a continuous bijection $f : S_1 \to S_2$ such that its inverse is also continuous. Then f is called a *homeomorphism.*

Let S be a topological space and $A \subset S$. The topology $\mathfrak{F}' = \{F \cap A : F \in \mathfrak{F}\}$ is called the *topology induced by the topology of* S. If any pair of points $p, q \in S$ have disjoint neighborhoods, we say that S is a *Hausdorff topological space.* A topological space is *connected*, if it is not the union of two disjoint, open subsets. If every point of a connected Hausdorff topological space S is contained in an open subset U which is homeomorphic to an open subset of $\mathbb{E}^n$, then S is *locally n-dimensional Euclidean.*

A pair (ϕ, U), where $U \subset S$ is open and ϕ is a homeomorphism to an open subset of $\mathbb{E}^n$, is an *n-coordinate pair.* Let $u_i : \mathbb{E}^n \to \mathbb{R}$ be defined as $u_i((x_1, x_2, \ldots, x_n)) = x_i$. Then, with a little abuse of notation, $x_i = u_i \circ \phi$ is called a *coordinate function* and $x_1, x_2, \ldots, x_n$ is the *coordinate system defined by ϕ*. We say that U is the *domain* of this coordinate system.

Let M be a locally n-dimensional Euclidean topological space. Two n-coordinate pairs (ϕ, U) and (θ, V) are *C^r-related*, if $\phi \circ \theta^{-1}$ and $\theta \circ \phi^{-1}$ possess continuous rth partial derivatives on their domains. In this case, we say that $\phi \circ \theta^{-1}$ and $\theta \circ \phi^{-1}$ are *C^r functions.* A *C^r n-subatlas on M* is a family $\mathfrak{F} = \{(\phi_i, U_i) : i \in I\}$ of n-coordinate pairs, any two of which are C^r-related, and $\bigcup_{i \in I} U_i = M$. A C^r n-subatlas maximal with respect to containment is a *C^r n-atlas.* The space M equipped with a C^r n-atlas is a *C^r-manifold.* A C^∞-manifold is a *smooth manifold.* An example of a smooth manifold is the n-dimensional Euclidean space.

Let $f : M \to N$, where M and N are smooth manifolds. If f has the property that, for any n-coordinate pairs (ϕ, U) of M and (θ, V) of N, the function $\theta \circ f \circ \phi^{-1}$ is C^∞, we say that f is *smooth.* Smooth functions $f : M \to \mathbb{E}^n$, where $\dim M = n$, form an algebra over the field of real numbers. We denote this algebra by $C^\infty(M)$. If $f : M \to N$ is smooth, bijective and its inverse is smooth, then f is a *diffeomorphism*, and M and N are *diffeomorph.*

In the following part of this section, we deal only with smooth manifolds. Let $p \in M$. We introduce an equivalence relation on the family of smooth functions defined on a neighborhood of p. We say that

$$f : U \to \mathbb{R} \sim g : V \to \mathbb{R} \quad \text{if, and only if, } p \text{ has a neighborhood } W \subset U \cap V$$

$$\text{such that } f|_W = g|_W.$$

The equivalence classes of this relation are the *germs at p of smooth functions*. Germs form an algebra $C_p^\infty(M)$ over the field of real numbers.

A derivation on $C_p^\infty(M)$ is a linear map $\sigma : C_p^\infty(M) \to \mathbb{R}$ such that

$$\sigma(fg) = g\sigma(f) + f\sigma(g) \tag{3.1}$$

for any $f, g \in C_p^\infty(M)$. A derivation on $C_p^\infty(M)$ is also called a *tangent vector to M at p*. Every derivation σ is of the form

$$\sigma(f) = \sum_{i=1}^{n} \sigma(x_i) \left. \frac{\partial f}{\partial x_i} \right|_0 ,$$

where n is the dimension of M. We denote the set of tangent vectors to M at p by T_pM. Note that T_pM has an n-dimensional vector space structure inherited from the vector space structure of $\mathbb{E}^n$.

A derivation on $C^\infty(M)$ is a linear map $\sigma : C^\infty(M) \to \mathbb{R}$ which has the property in (3.1) for any $f, g \in C^\infty(M)$. The disjoint union of the tangent spaces T_pM over all the points p of M is the *vector bundle TM of M*. The vector bundle of an n-dimensional smooth manifold M may be equipped with a $(2n)$-dimensional smooth manifold structure. A *vector field on M* is a smooth function $V : M \to TM$. The set of vector fields of M is denoted by $\Gamma(TM)$. This set coincides with the set of derivations on $C^\infty(M)$. Note that the composition of derivations might not be a derivation. On the other hand, it is easy to show that $[X, Y] = XY - YX$ is a derivation for any $X, Y \in \Gamma(TM)$. We call the vector field $[X, Y]$ the *Lie bracket* of X, Y. Note that $[.,.] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is an antisymmetric, bilinear function.

Let $f : M \to N$ be smooth, where $\dim M = m$ and $\dim N = n$. Consider a point $p \in M$, an m-coordinate pair (ϕ, U) of M and an n-coordinate pair (θ, V) of N, such that $p \in U$ and $f(p) \in V$. The differential map $T_p f$ of f at $p \in M$ is the map

$$T_p f : T_p M \to T_{f(p)} N$$

defined as

$$D_{\phi(p)}(\theta \circ f \circ \phi^{-1}),$$

where $D_{\phi(p)}(\theta \circ f \circ \phi^{-1})$ denotes the differential map of the real function $\theta \circ f \circ \phi^{-1}$ at the point $\phi(p) \in \mathbb{E}^m$. This definition does not depend on the choice of the two coordinate pairs. If f is injective, and $T_p f$ is injective for every $p \in M$, then we say that f is an *injective immersion.* If f is a homeomorphism between M and $f(M)$, then f is an *embedding.* If $M \subset N$ and the identity map $\mathrm{id} : M \to N$ is an injective immersion, then M is a *submanifold of* N.

Definition 3.1.2. Let M be a smooth manifold of dimension n, and $g_p : T_p(M) \times T_p(M) \to \mathbb{R}$ be a positive definite, symmetric bilinear form such that the function $p \mapsto g_p(X(p), Y(p))$ is smooth for any $X, Y \in \Gamma(TM)$. The function $g(p) = g_p$ is called the *metric tensor of* M, and M is a *Riemannian manifold.*

For $u, v \in T_p M$, we may write $u = \sum_{i=1}^n u_i \left.\frac{\partial}{\partial x_i}\right|_p$ $v = \sum_{i=1}^n v_i \left.\frac{\partial}{\partial x_i}\right|_p$. Then $g_p(u, v) = \sum_{i,j=1}^n g_{ij}(p) u_i v_j$, where

$$g_{ij}(p) = g_p\left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right).$$

Note that the matrix $[g_{ij}(p)]$ is symmetric.

If M is a submanifold of N, and for $p \in M$, g_p is the restriction of h_p to T_pM, we say that (M, g) is a *Riemannian submanifold* of (N, h). If $u \in T_pM$, then the *length of* u is $\sqrt{g_p(u,u)}$. If $u, v \in T_pM$ are not zero, the *angle between* u *and* v is

$$\sphericalangle(u,v) = \arccos \frac{g_p(u,v)}{\sqrt{g_p(u,u)}\sqrt{g_p(v,v)}}.$$

If $\gamma : [0,a] \to M$ is a curve of class C^1, the arc length of γ is defined as

$$\text{arclength}(\gamma) = \int_0^a \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt.$$

We define the arc length of piecewise C^1 class curves similarly. We say that c is an *arc-length parametrized curve*, if $g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) = 1$ for $t \in [0,a]$. For points $p, q \in M$, the *distance* $\text{dist}_M(p,q)$ is defined as the infimum of the arc lengths of piecewise C^1 curves connecting p and q.

Let (M, g) and (N, h) be Riemannian manifolds. A map $f : M \to N$ is an *isometry*, if f is a diffeomeorphism, and for every $p \in M$, $u, v \in T_pM$, we have

$$h_{f(p)}(T_pf(u), T_pf(v)) = g_p(u,v).$$

If a family of diffeomorphisms f_t, where $t \in I$ for an interval I that contains 0, satisfies $f_t \circ f_{t'} = f_{t+t'}$ for every $t, t', t+t' \in I$, then $\{f_t : t \in I\}$ is a *one parameter group.*

A *connection* on a smooth manifold M is a bilinear map $D : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that, for any $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, we have

1. $D_{fX}Y = fD_XY$;

2. $D_X(fY) = (Xf)Y + fD_XY$;

3. $D_X Y - D_Y X = [X, Y]$.

Note that $D_{fX}Y = fD_X Y$ implies that, given Y, $D_X Y|_p$ depends only on $X|_p$. Hence, we may regard a connection as a bilinear map $D : T_pM \times \Gamma(TM) \to \Gamma(TM)$, and talk about $D_v Y$, where $v \in T_pM$, $p \in M$ and $Y \in \Gamma(TM)$.

For every Riemannian manifold (M, g), there is a unique connection D, called the *canonical connection* or *Levi-Civita connection*, which satisfies the following:

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z).$$

In the following, D denotes only the canonical connection.

For an n-coordinate pair (ϕ, U), where $p \in U \subset M$, we may write a vector field as a linear combination of the derivations $\frac{\partial}{\partial x_i}$, where $i = 1, 2, \ldots, n$. In particular,

$$D_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \sum_{i=1}^{n} \Gamma^i_{jk} \frac{\partial}{\partial x_i}.$$

The quantities Γ^i_{jk} are called the *Christoffel symbols* of M. Note that $\Gamma^i_{jk} = \Gamma^i_{kj}$. It is known (see [25]) that

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{s=1}^{n} g_{il} \left(\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l} \right).$$

Let $\gamma : I \to M$ be a smooth curve on the interval $I \subset \mathbb{R}$. A vector field A along the curve γ is a smooth function $A : I \to TM$ such that $A(t) \in T_{\gamma(t)}M$ for every $t \in I$. If $X, Y \in \Gamma(TM)$ and $X \circ \gamma = X' \circ \gamma$, then $D_{\dot\gamma(t)}X = D_{\dot\gamma(t)}Y$ for any $\tau \in I$. Hence, we may talk about $D_{\dot\gamma}X$ for a vector field X along the curve γ. If $D_{\dot\gamma}\dot\gamma = 0$, then γ is called a *geodesic*. A geodesic is *complete*, if it is not a proper subset of a geodesic. A *geodesically complete* Riemannian manifold is a manifold in which all the geodesics may be extended to geodesics defined on $\mathbb{R}$. A *totally geodesic submanifold* of M is a submanifold in which all the geodesics are geodesics of M.

The trilinear form $R_p : T_p(M) \times T_p(M) \times T_p(M) \to T_p(M)$ defined as

$$R(x, y)z = D_Y(D_X Z) - D_X(D_Y Z) + D_{[X,Y]}Z$$

where $X, Y, Z \in \Gamma(TM)$, $x = X_p$, $y = Y_p$ and $z = Z_p$, is called the *curvature tensor* of M at $p \in M$. Let $P \subset T_pM$ be a given plane. Then

$$K(x, y) = \frac{g_p(R(x, y)x, y)}{g_p(x, x)g_p(y, y) - g_p(x, y)^2},$$

where $x, y \in P$ are linearly independent, is independent of x and y. The quantity $K(x, y)$ is called the *sectional curvature* of M at p in the plane spanned by $x, y \in T_pM$.

Now we define the geodesic curvature of a curve in a 2-dimensional Riemannian manifold M in two steps. Let M be a 2-dimensional submanifold of $\mathbb{E}^3$, and let $\gamma : I \to M$ be an arc-length parametrized curve, where I is an interval. Then the *geodesic curvature of* γ *at point* $\gamma(t)$ is the length of the orthogonal projection of $\ddot{\gamma}(t)$ onto the tangent plane of M at $\gamma(t)$. In particular, $k_g = \|\ddot{\gamma}(t)\|$ if $M = \mathbb{E}^2$.

If $\gamma(t) = (x(t), y(t))$ in a 2-coordinate pair (ϕ, U), where $\gamma(t) \in U$, then the orthogonal projection of $\ddot{\gamma}(t)$ is of the form

$$\ddot{\gamma}(t)_{proj} = \lambda \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial y}, \tag{3.2}$$

where

$$\lambda = \ddot{x}(t) + \Gamma^1_{11}\dot{x}^2(t) + 2\Gamma^1_{12}\dot{x}(t)\dot{y}(t) + \Gamma^1_{22}\dot{y}^2(t) \tag{3.3}$$

and

$$\nu = \ddot{y}(t) + \Gamma^2_{11}\dot{x}^2(t) + 2\Gamma^2_{12}\dot{x}(t)\dot{y}(t) + \Gamma^2_{22}\dot{y}^2(t) \tag{3.4}$$

(see, for example, [49]).

Note that the expressions in (3.3) and (3.4) use only intrinsic quantities of M. This allows us to define the geodesic curvature of an arc-length parametrized curve $\gamma(x(t), y(t))$ in a 2-coordinate pair of any 2-dimensional Riemannian manifold M. More specifically, we define k_g as the length of the vector in (3.2):

$$k_g = \sqrt{\lambda^2 g_{11} + 2\lambda\nu g_{12} + \nu^2 g_{22}}, \tag{3.5}$$

where λ and ν are defined by (3.3) and (3.4). In the literature, geodesic curvature is often defined as a signed quantity in a so-called oriented Riemannian manifold. For this definition, the reader is referred to [28].

A *variation of a smooth curve* $\gamma : [a, b] \to M$ is a smooth function $H : [a, b] \times [-\varepsilon, \varepsilon] \to M$ such that $H(s, 0) = \gamma(s)$ for any $s \in [a, b]$. The partial derivative $\frac{\partial H}{\partial t}\big|_{(s,0)}$ is a vector field along γ, called the *inital speed vector field of H*.

3.2 Hyperbolic space

For two thousand years, many mathematicians tried to derive Euclid's parallel postulate from his other four postulates. The idea that the parallel postulate is an independent one appeared first in a treatise of Lobachevsky in 1830 and the famous Appendix of J. Bolyai in 1832. Bolyai described a non-Euclidean geometry based on the negation of the parallel postulate. The first models of this geometry were introduced by Beltrami in 1868. His models are known as the projective disk model, the conformal disk model and the conformal half-plane model. He used his models to show that this new geometry, called *hyperbolic geometry*, is equiconsistent with the Euclidean geometry. In this section, we introduce and describe hyperbolic geometry by means of a higher dimensional analogue of the conformal disk model.

Consider the open unit ball $\mathbf{B}^n = \mathbf{B}^n(o)$ as a submanifold of the smooth manifold $\mathbb{E}^n$. We define the inner product

$$g_p^H(x, y) = \frac{4}{(1 - \|p\|^2)^2} \left(\sum_{i=1}^{n} \alpha_i \beta_i \right), \tag{3.6}$$

where $p \in \mathbf{B}^n$ and $x = (\alpha_1, \alpha_2, \ldots, \alpha_n), y = (\beta_1, \beta_2, \ldots, \beta_n) \in T_p\mathbf{B}^n$. This yields a Riemannian manifold, which we call the *n-dimensional hyperbolic space* $\mathbb{H}^n$. The sphere $\mathbb{S}^{n-1} = \operatorname{bd} \mathbf{B}^n$ is called the *sphere at infinity*. This space has a constant sectional curvature -1.

Note that since $g_p^H(x, y)$ is pointwise proportional to the standard inner product of $\mathbb{E}^n$, the hyperbolic angle of any two vectors is the same as their Euclidean angle when we regard them as vectors of the Riemannian manifold $\mathbb{E}^n$. In other words, we say that this model of hyperbolic geometry is *conformal*.

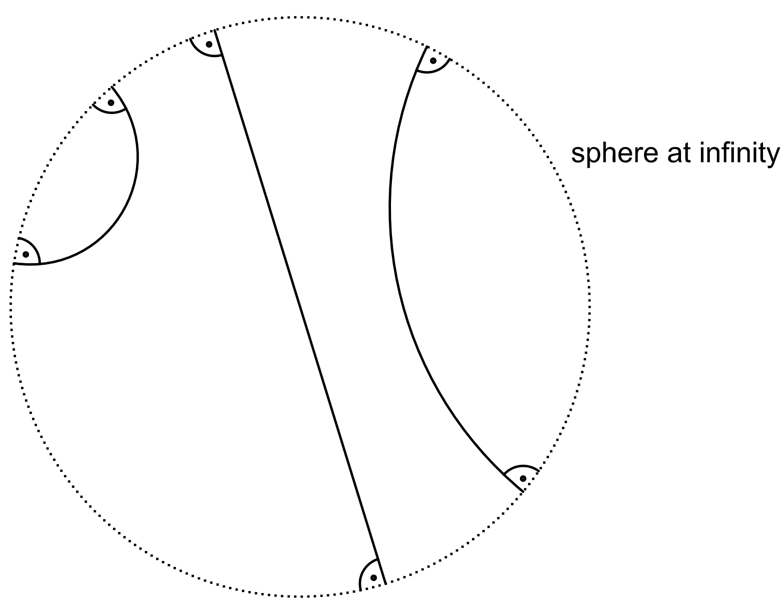

Figure 3.1: Hyperbolic lines

A complete geodesic of a hyperbolic space is a *hyperbolic line* (cf. Figure 3.1). It is the intersection of $\mathbf{B}^n$ with either a line of $\mathbb{E}^n$ passing through o or a Euclidean circle that is orthogonal to the sphere at infinity. Similarly, the intersection of $\mathbf{B}^n$ with either a k-dimensional affine subspace passing through o or a k-dimensional sphere $\mathbb{S}^k(c, r)$ which is orthogonal to $\mathbb{S}^{n-1}$ is a *hyperbolic k-space.* The totally geodesic, geodesically complete submanifolds of $\mathbb{H}^n$ are exactly the hyperbolic k-spaces of $\mathbb{H}^n$. A hyperbolic $(n-1)$-space is also called a *hyperbolic hyperplane.* A hyperbolic hyperplane dissects $\mathbb{H}^n$ into two open, connected components, which are called *open hyperbolic half spaces.* Their closures are *closed hyperbolic half spaces.*

The *hyperbolic arc length* of a continuously differentiable curve $\gamma : [0, 1] \to \mathbb{H}^n$ is

$$\text{arclength}_H(\gamma) = \int_0^1 \frac{2\dot{\gamma}(t)}{1 - \|\gamma(t)\|^2} dt. \tag{3.7}$$

From this, we have that the *hyperbolic distance* $\text{dist}_H(p, q)$ *of points* $p, q \in \mathbb{H}^n$ is

$$\cosh(\text{dist}_H(p, q)) = 1 + \frac{2\|p - q\|}{(1 - \|p\|^2)(1 - \|q\|^2)}.$$

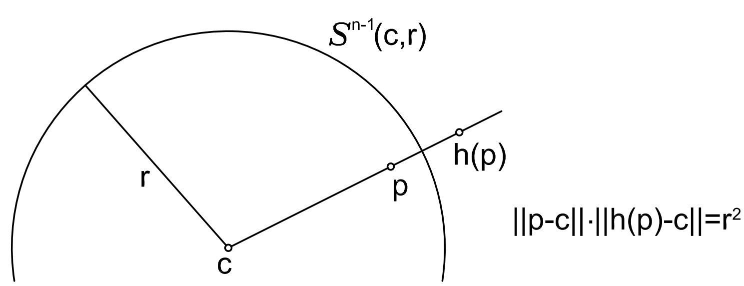

Figure 3.2: Inversion with respect to the sphere $\mathbb{S}^{n-1}(c, r)$

To describe the isometries of $\mathbb{H}^n$, we need a little preparation. Consider a sphere

$\mathbb{S}^{n-1}(c,r) \subset \mathbb{E}^n$. The transformation $h : \mathbb{E}^n \setminus \{c\} \to \mathbb{E}^n \setminus \{c\}$ defined as

$$h(p) = c + \frac{r^2}{\|p-c\|^2}(p-c)$$

is the *inversion with respect to the sphere* $\mathbb{S}^{n-1}(c,r)$ or the *reflection about the sphere* $\mathbb{S}^{n-1}(c,r)$ (cf. Figure 3.2). The reflection about a sphere $\mathbb{S}^{n-1}(c,r)$ is a conformal mapping. In other words, the angle of two objects is the same as the angle of their images. The reflection about $\mathbb{S}^{n-1}(c,r)$ has also the property that the image of a sphere or a hyperplane is a sphere or a hyperplane. Furthermore, the image of a hyperplane or a sphere is itself if, and only if, it is orthogonal to $\mathbb{S}^{n-1}(c,r)$. A function $f : \mathbb{H}^n \to \mathbb{H}^n$ is an isometry of $\mathbb{H}^n$ if, and only if, f is the restriction to $\mathbf{B}^n$ of the composition of finitely many reflections about a sphere or hyperplane, each of which is orthogonal to the sphere at infinity. It is easy to see that the group of isometries acts transitively on $\mathbb{H}^n$.

A set $C \subset \mathbb{H}^n$ is *hyperbolic convex*, if the hyperbolic segment with endpoints p and q is contained in S for any $p, q \in C$. The *hyperbolic convex hull* of a set $S \subset \mathbb{H}^n$ is the intersection of the hyperbolic convex sets that contain S. Hyperbolic polytopes, polygons, edges, faces, etc. are defined similarly to their Euclidean counterparts.

Let $S \subset \mathbf{B}^n$ be a closed set which has Euclidean volume. Then the following integral exists and is finite:

$$\mathrm{vol}_H(S) = \int_S \frac{2^n}{(1-\|x\|^2)^n} dA,$$

which we call the *hyperbolic volume of* S. Two-dimensional volume is called *area*. For the definition of volume in a Riemannian manifold, the reader is referred to [25].

If T is a hyperbolic triangle with angles α, β, γ, then the area of T is:

$$\mathrm{area}_H(T) = \pi - \alpha - \beta - \gamma. \tag{3.8}$$

3.3 Spheres, horospheres and hyperspheres in $\mathbb{H}^n$

Let $c_H \in \mathbb{H}^n, r_H \in (0, \infty)$. The set $A = \{x \in \mathbb{H}^n : \mathrm{dist}_H(x, c_H) = r_H\}$ is called the *hyperbolic sphere of radius* r_H *and centre* c_H. Hyperbolic spheres correspond to the spheres of $\mathbb{E}^n$ that are contained in $\mathbf{B}^n$, but generally the hyperbolic centre and radius of a sphere $\mathbb{S}^{n-1}(c, r) \subset \mathbf{B}^n$ are not c and r. The area of a hyperbolic disk D of radius r_H is

$$\mathrm{area}_H(D) = 4\pi \sinh^2 \frac{r_H}{2}. \tag{3.9}$$

Consider an $(n-1)$-sphere or a hyperplane $S \subset \mathbb{E}^n$ and assume that $A = S \cap \mathbf{B}^n \neq \emptyset$. If S is tangent to $\mathbb{S}^{n-1}$, A is called a *horosphere*. If S intersects $\mathbb{S}^{n-1}$ at an angle α, where $0 < \alpha < \pi/2$, then A is a *hypersphere* (cf. Figure 3.3).

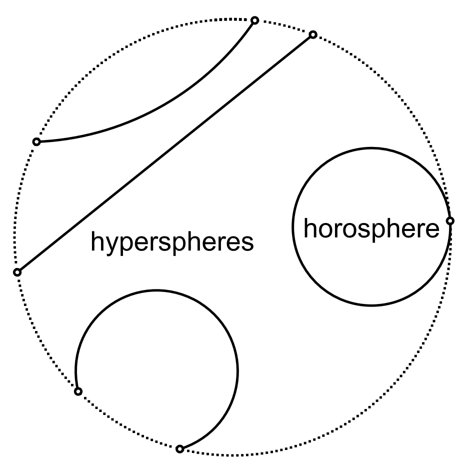

Figure 3.3: Hyperspheres and horospheres

Since reflections about spheres are conformal transformations, it is easy to see that any two horospheres are congruent, and two hyperspheres are congruent if, and only if, the Euclidean $(n-1)$-spheres that they are contained in intersect the sphere at infinity at the same angle.

Proposition 3.3.1. *A set $S \in \mathbb{H}^n$ is a hypersphere if, and only if, there is a hyperbolic hyperplane H and a real number $d > 0$ such that $S = \{x \in H_1 : \mathrm{dist}_H(x, H) = d\}$, where H_1 is one of the two open hyperbolic half spaces bounded by H.*

Proof. First, we prove the "only if" direction.

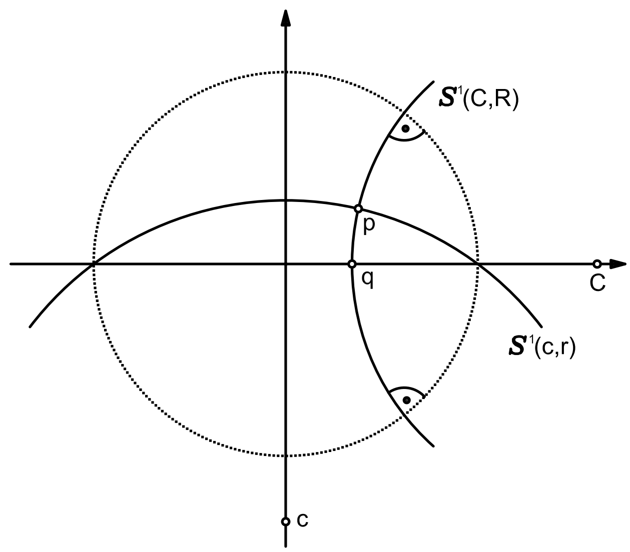

Figure 3.4: Hyperspheres are distance-hypersurfaces

Let $S = \mathbb{S}^{n-1}(c, r) \cap \mathbf{B}^n$ be a hypersphere and H be the hyperbolic hyperplane that intersects $\mathbb{S}^{n-1}$ in $\mathbb{S}^{n-1}(c, r) \cap \mathbb{S}^{n-1}$. Using the transitivity of the isometry group of $\mathbb{H}^n$, we may assume that H passes through the origin o, and hence it can be extended to a Euclidean hyperplane. Consider the intersection q of S and the hyperbolic line orthogonal to H and passing through a point p of H. We show that $\mathrm{dist}_H(p, q)$ is independent of p. Using the spherical symmetry of the conformal ball model, we may assume that $n = 2$ and H is the open segment with endpoints $a = (-1, 0)$ and

$b = (1, 0)$. We may also assume that the Euclidean centre c of the circle $\mathbb{S}^1(c, r)$ is on the negative half of the y-axis. This yields that $c = (0, -\sqrt{r^2 - 1})$. We may also assume that p has a nonnegative x-coordinate.

If p is the origin then $q = (0, r - \sqrt{r^2 - 1})$ whence $\cosh \operatorname{dist}_H(p, q) = r/\sqrt{r^2 - 1}$. If p is not the origin, then the hyperbolic segment with endpoints p and q may be extended to a Euclidean circle $\mathbb{S}^1(C, R)$ with C on the x-axis (cf. Figure 3.4). An elementary calculation yields that

$$p = (\sqrt{R^2 + 1} - R, 0), \quad \text{and} \tag{3.10}$$

$$q = \left(\frac{r}{r\sqrt{R^2 + 1} + R\sqrt{r^2 - 1}}, \frac{R}{r\sqrt{R^2 + 1} + R\sqrt{r^2 - 1}} \right). \tag{3.11}$$

From this, we obtain that $\cosh \operatorname{dist}_H(p, q) = r/\sqrt{r^2 - 1}$ is independent of R. We note that $f(r) = \operatorname{arccosh}(r/\sqrt{r^2 - 1})$ is a bijective mapping from $(1, \infty)$ to $(0, \infty)$. From this follows the "'if'' direction of Proposition 3.3.1. □

Even though the parallel postulate does not hold for $\mathbb{H}^n$, it is often useful to talk about parallel lines in the following sense. Let $L_1, L_2 \subset \mathbb{H}^n$ be directed lines, which are represented by the open circular arcs C_1 and C_2 in $\mathbb{E}^n$. Let r_i and s_i denote the starting point and the endpoint of C_i. We say that L_1 and L_2 are *parallel*, if $s_1 = s_2$.

It is easy to see that if L_1 is a directed line passing through a point p, then, on every directed line L_2 parallel to L_1, there is a unique point q such that the hyperbolic segment with endpoints p and q meets L_1 and L_2 at equal angles. This point q is *conjugate to p with respect to L_1*.

Proposition 3.3.2. *Let $S \subset \mathbb{H}^n$ containing a point p. Then S is a horosphere if, and only if, there is a directed hyperbolic line L passing through p such that S consists of the points conjugate to p with respect to L.*

Proof. First, we prove the "only if " direction. We may assume that $n = 2$, $p = o$ and $S = \mathbb{S}^1((0, 1/2), 1/2) \cap \mathbf{B}^2$. We show that the hyperbolic line L is the intersection of $\mathbf{B}^2$ with the Euclidean line $x = 0$.

Consider a hyperbolic line $L' = \mathbb{S}^1(c, r) \cap \mathbf{B}^2$ such that $\mathbb{S}^1(c, r)$ contains the point $(0, 1)$. We may assume that c has a positive x-coordinate. Denote the point of $\mathbb{S}^1(c, r) \cap \mathbb{S}^1((0, 1/2), 1/2)$, distinct from $(1, 0)$, by q. We show that q is conjugate to p with respect to L.

The equation of $\mathbb{S}^1(c, r)$ is $(x - r)^2 + (y - 1)^2 = r^2$ for some $r > 0$. Thus,

$$q = \left(\frac{2r}{4r^2 + 1}, \frac{1}{4r^2 + 1} \right).$$

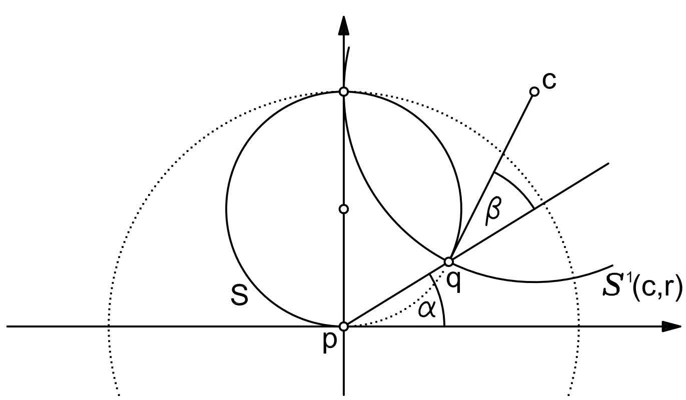

Figure 3.5: Horospheres consist of conjugate points

Observe that the Euclidean segment connecting p and q is also a hyperbolic segment. Denote the angle between the segment $[p, q]$ and the x-axis by α, and the angle between the Euclidean line passing through p, q, and the segment $[q, c]$ by β (cf. Figure 3.5). We note that q is conjugate to p if, and only if, $\alpha = \beta$. An easy calculation yields

$$\cos\beta - \cos\alpha = \frac{< c-q, q >}{\|c-q\|\|q\|} - \frac{2r}{\sqrt{4r^2+1}} = 0,$$

which implies the "only if" direction of our assertion. The "if" part immediately follows from the uniqueness of the conjugate point on any directed line parallel to L. □

Propositions 3.3.1 and 3.3.2 allow us to define horospheres and hyperspheres in an alternative way, using only their model independent, geometric properties. This approach is followed, for example, in the Appendix of J. Bolyai.

3.4 The spherical space

Consider the unit sphere $\mathbb{S}^n = \mathbb{S}^n(o)$ as an n-dimensional Riemannian submanifold of the $(n+1)$-dimensional Riemannian manifold $\mathbb{E}^{n+1}$. Then, for $p \in \mathbb{S}^n$ and $x, y \in T_p\mathbb{S}^n$, we have

$$g_p^S(x, y) =< x, y > .$$

The sphere $\mathbb{S}^n$ is a Riemannian manifold of constant sectional curvature 1, which we call *n-dimensional spherical space.*

The *spherical distance of points* $p, q \in \mathbb{S}^n$, which is derived from the inner product g_p^S of $\mathbb{S}^n$, is the angle of p and q in the Euclidean space $\mathbb{E}^{n+1}$; that is,

$$\mathrm{dist}_S(p, q) = \sphericalangle(p, q).$$

Two points at spherical distance π are *antipodal.* Note that $\mathbb{S}^n$ is conformal: the angle between the vectors $x, y \in T_p\mathbb{S}^n$ is the same as their angle when regarded as Euclidean vectors.

The set $\mathbb{S}^n \cap L$, where L is a $(k+1)$-dimensional linear subspace of $\mathbb{E}^{n+1}$, is called a *k-dimensional great-sphere of* $\mathbb{S}^n$. In particular, $\mathbb{S}^n$ is a great-sphere of itself. A 1-dimensional great-sphere is a *great-circle*. The complete geodesics of $\mathbb{S}^n$ are the great-circles, and the totally geodesic, geodesically complete submanifolds of $\mathbb{S}^n$ are the great-spheres of $\mathbb{S}^n$. If p and q are not antipodal, then the shorter geodesic that connects them is the *closed spherical segment with endpoints p and q*. Removing p and q from this geodesic, we obtain the *open spherical segment with endpoints p and q*.

An $(n-1)$-dimensional great-sphere $\mathbb{S}^{n-1}(o)$ of $\mathbb{S}^n$ dissects $\mathbb{S}^n$ into two connected components, called *open hemispheres*, the boundaries of which are $\mathbb{S}^{n-1}(o)$. The closure of an open hemisphere is a *closed hemisphere*. An open hemisphere is the intersection of $\mathbb{S}^n$ with an open half space of $\mathbb{E}^{n+1}$ that contains o in its boundary. The *spherical centre* of the open hemisphere $\mathbb{S}^n \cap \{x \in \mathbb{E}^{n+1} :< x, u >> 0, u \in \mathbb{S}^n\}$ is u.

Assume that the dimension of the affine subspace of smallest dimension, containing a set $A \subset \mathbb{S}^n$, is k. Then we say that the *dimension of A with respect to* $\mathbb{S}^n$ is $k-1$.

A set $C \subset \mathbb{S}^n$ is *spherically convex*, if C is contained in an open hemisphere of $\mathbb{S}^n$, and $p, q \in C$ implies that the spherical segment connecting p and q is contained in C. If $S \subset \mathbb{S}^n$ is contained in an open hemisphere, then the *spherical convex hull* of S is the intersection of all the spherically convex sets that contain S.

Remark 3.4.1. There is a natural extension of the notions of spherically convex sets and spherical convex hulls for subsets of any given k-sphere $\mathbb{S}^k(c, r)$, where

$0 \leq k \leq n-1$. The spherical convex hull of $S \subset \mathbb{S}^k(c,r)$ with respect to $\mathbb{S}^k(c,r)$ is denoted by $\mathrm{Sconv}\big(S, \mathbb{S}^k(c,r)\big)$.

Let $= c \in \mathbb{S}^n$ and $r > 0$. An *open spherical cap of radius r and centre c* is the intersection $\mathbf{B}^{n+1}(c, 2\sin(r/2)) \cap \mathbb{S}^n$. A closed spherical cap is the closure of the open spherical cap in the relative topology of $\mathbb{S}^n$.

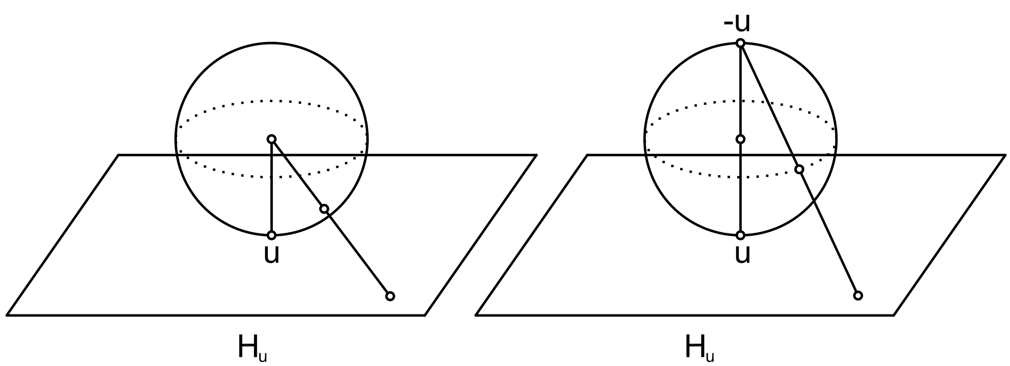

Figure 3.6: Central and stereographic projections

Consider the open hemisphere $S_u = \mathbb{S}^n \cap \{x \in \mathbb{E}^{n+1} :< x, u >> 0, u \in \mathbb{S}^n\}$ and the hyperplane $H_u = \{x \in \mathbb{E}^{n+1} :< x, u >= 1\}$. Note that H_u is the hyperplane tangent to $\mathbb{S}^n$ at u. The *central projection* of S_u onto H_u is the mapping

$$f : S_u \to H_u, \qquad f(x) = \frac{x}{< x, u >}.$$

Note that a set $C \subset S_u$ is spherically convex if, and only if, $f(C)$ is convex in H_u. The *stereographic projection* of $\mathbb{S}^n$ from $-u$ onto H_u is

$$g : \mathbb{S}^n \setminus \{-u\} \to H_u, \qquad g(x) = -u + \frac{4}{\|x + u\|}$$

(cf. Figure 3.6). Observe that g is the restriction to $\mathbb{S}^n$ of the reflection about the sphere $\mathbb{S}^n(-u, 2)$, and thus, it is conformal.

Note that if $C \subset \mathbb{S}^n$ is spherically convex, then the set $[o, C]$ is convex in $\mathbb{E}^{n+1}$. Hence, it makes sense to define the *n-dimensional spherical volume* of C as

$$\mathrm{vol}_n^S(C) = \frac{\omega_n}{\kappa_{n+1}} \mathrm{vol}_{n+1}([o, C]), \tag{3.12}$$

where $\omega_n = \mathrm{surf}(\mathbb{S}^n)$ and $\kappa_{n+1} = \mathrm{vol}_{n+1}(\mathbf{B}^{n+1})$. Two-dimensional spherical volume is called *spherical area.* The area of a spherical triangle T with angles α, β, γ is

$$\mathrm{area}_S(T) = \alpha + \beta + \gamma - \pi.$$

The area of a spherical cap of spherical radius r is

$$4\pi \sin^2 \frac{r}{2}.$$

3.5 Curves of constant geodesic curvature

The aim of this section is to characterize curves of constant geodesic curvature in the planes of constant sectional curvature K : $\mathbb{E}^2$ $(K = 0)$, $\mathbb{H}^2$ $(K = -1)$ and $\mathbb{S}^2$ $(K = 1)$.

Theorem 3.5.1. *A curve of constant geodesic curvature k_g is a segment or a circle arc in $\mathbb{E}^2$, a hyperbolic segment, or a hypercycle arc, or a horocycle arc or a hyperbolic circle arc in $\mathbb{H}^2$, a spherical segment or a spherical circle arc in $\mathbb{S}^2$.*

The following chart shows a connection between k_g and geometric property of the curve γ.

<table>
<tr><td></td><td>$\mathbb{E}^2$</td><td>$\mathbb{H}^2$</td><td>$\mathbb{S}^2$</td></tr>
<tr><td>$k_g = 0$</td><td>Euclidean segment</td><td>hyperbolic segment</td><td>spherical segment</td></tr>
<tr><td>$0 < k_g < 1$</td><td rowspan="3">circle arc of radius r $k_g = 1/r$</td><td>hypercycle arc with distance d from a line, $k_g = \tanh d$</td><td rowspan="3">circle arc of spherical radius r $k_g = \cot r$</td></tr>
<tr><td>$k_g = 1$</td><td>horocycle arc</td></tr>
<tr><td>$1 < k_g$</td><td>circle arc of radius r, $k_g = \coth r$</td></tr>
</table>

Proof. First, we prove the assertion for $\mathbb{E}^2$. Consider an arc-length parametrized curve $\gamma : [0, \alpha] \to \mathbb{E}^2$ of geodesic curvature k_g. We may assume that γ is twice continuously differentiable. Since γ is parametrized with arc length, we have $\|\dot{\gamma}(s)\| = 1$. Hence $\dot{\gamma}(s) = (\cos f(s), \sin f(s))$ for some continuously differentiable function f. The geodesic curvature of this curve is $k_g = \|\ddot{\gamma}(s)\| = |\dot{f}(s)|$. This geodesic curvature is constant if, and only if, f is a linear function.

Next, we prove the statement for $\mathbb{H}^2$. Since k_g, $\gamma(0)$ and $\dot{\gamma}(0)$ determine the arc-length parametrized curve $\gamma : [0, \alpha] \to \mathbb{H}^2$ of constant geodesic curvature k_g, it is sufficient to show that hypercycles, horocycles and circles have constant geodesic curvature as in the chart above.

We show this for hypercycles and circles. For horocycles it is the immediate consequence of a limit argument.

From (3.6), the Christoffel symbols of $\mathbb{H}^2$ are

$$\begin{aligned}\frac{2x}{1-x^2-y^2} &= \Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22}, \quad \text{and} \\ \frac{2y}{1-x^2-y^2} &= \Gamma^2_{22} = \Gamma^1_{12} = -\Gamma^2_{11}\end{aligned}$$

Assume that γ is a hypercycle such that the angle, between γ and the circle at infinity, is α. Under a suitable isometry of $\mathbb{H}^2$, the image of γ is the intersection of $\mathbf{B}^2$ with the horizontal line $y = \cos\alpha$. Hence, we may assume that $\gamma(s) = (x(s), \cos\alpha)$ where s is hyperbolic arc length. Then $dx/ds = (1 - x^2 - \cos^2\alpha)/2 = 1/\sqrt{E}$, $dy/ds = 0$, and (3.3) and (3.4) simplify to

$$\lambda = 0, \qquad \nu = -\cos\alpha \frac{1 - x^2 - \cos^2\alpha}{2}.$$

Hence, from (3.5) we obtain $k_g = \cos\alpha$, and it is independent of x.

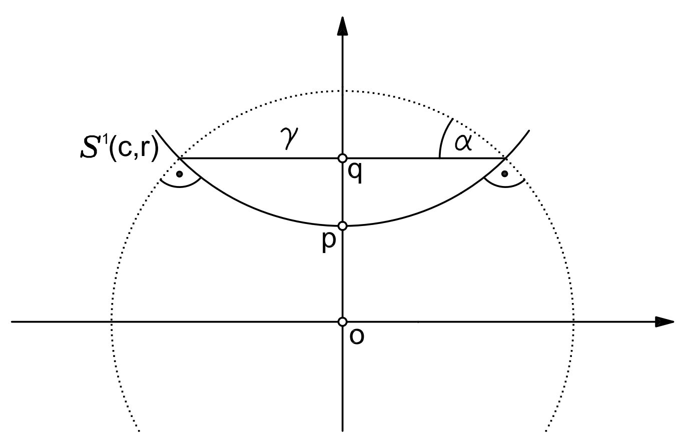

Figure 3.7: Geodesic curvature of a hypercycle

Consider the hyperbolic line $L = \mathbb{S}^1(c, r) \cap \mathbf{B}^1$ that intersects the sphere at infinity at the points $(-\sin\alpha, \cos\alpha)$ and $(\sin\alpha, \cos\alpha)$. Clearly, this is the line such that every point g of γ is at a distance d from L independently of g. Let p, respectively q, be the point of L, respectively γ, with 0 as the x-coordinate (cf. Figure 3.7). Then $d = \mathrm{dist}_H(p, q)$. We note that the Euclidean radius and centre of the circle $\mathbb{S}^1(c, r)$ are, respectively, $r = \tan\alpha$ and $c = (0, \sqrt{\tan^2\alpha + 1})$. Hence, $p = (0, (1 - \sin\alpha)/\cos\alpha)$ and

$$\cosh d = 1 + \frac{2\|p - q\|^2}{(1 - \|p\|^2)(1 - \|q\|^2)} = \frac{1}{\sin\alpha} = \frac{1}{\sqrt{1 - k_g^2}}.$$

From this, an identity about hyperbolic functions yields $k_g = \tanh d$.

Now we assume that γ is a circle of hyperbolic radius r_H. We may also assume that the hyperbolic centre of γ is o. Then γ coincides with a Euclidean circle $\mathbb{S}^1(o, r)$. By symmetry, it is clear that γ is of constant geodesic curvature k_g in $\mathbb{H}^2$. It remains to determine the value of k_g as a function of r_H. We note that the connection between r_H and the Euclidean radius r is $\cosh r_H = (1 + r^2)/(1 - r^2)$, which implies that $\tanh(r_H/2) = r$.

Clearly, a hyperbolic arc-length parametrized form of the circle γ is $\gamma(s) = (r\cos(ks), r\sin(ks))$ for some constant $k > 0$. The value of k is computed from $g_H(\dot\gamma(s), \dot\gamma(s)) = 1$, and it is $k = (1 - r^2)/(2r)$. We substitute $\gamma(s)$ into (3.3), (3.4) and (3.5) and simplify to obtain that

$$k_g = \frac{1 + r^2}{2r} = \coth r_H.$$

Finally, we show that, in $\mathbb{S}^2 \subset \mathbb{E}^3$, a circle C of spherical radius θ has constant geodesic curvature $\cot\theta$. It is easy to see that the arc length parametrization of C may be chosen as $\gamma : [0, 2\pi] \to \mathbb{S}^2 \subset \mathbb{E}^3$, $\gamma(t) = (\sin\theta\cos(kt), \sin\theta\sin(kt), \cos\theta)$, where $k = 1/\sin\theta$. The geodesic curvature of γ in $\mathbb{S}^2$ at $\gamma(t)$ is the Euclidean norm of the orthogonal projection of $\ddot\gamma(t)$ onto the tangent plane $T_{\gamma(t)}\mathbb{S}^2$. We note that

$\gamma(t)$ is a unit normal vector of $\mathbb{S}^2$ at $\gamma(t)$. Thus,

$$k_g = \|\ddot{\gamma}_{tangent}(t)\| = \|\ddot{\gamma}(t) - <\ddot{\gamma}(t), \gamma(t)> \gamma(t)\| = \cot\theta.$$

□

3.6 Some hyperbolic and spherical formulae

The theorems of sines and cosines are well known from high school. Here we present their hyperbolic and spherical counterparts.

Theorem 3.6.1. *Let* $\mathbf{M} \in \{\mathbb{H}^2, \mathbb{S}^2\}$. *Consider a triangle* $T \subset \mathbf{M}$ *with side lengths* a, b *and* c. *Denote by* α, β *and* γ *the angle of* T *at the vertex opposite of the side of length* a, b *and* c, *respectively.*

If $\mathbf{M} = \mathbb{H}^2$, *then*

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}, \tag{3.13}$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma, \quad \textit{and} \tag{3.14}$$

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c. \tag{3.15}$$

If $\mathbf{M} = \mathbb{S}^2$, *then*

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma}, \tag{3.16}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos\gamma, \quad \textit{and} \tag{3.17}$$

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c. \tag{3.18}$$

The following hyperbolic variant of Heron's formula is a consequence of the formulae (3.8), (3.13), (3.14) and (3.15).

Proposition 3.6.2. *Let $T \subset \mathbb{H}^2$ be a hyperbolic triangle with side lengths a, b and c. Then*

$$\tan \frac{\mathrm{area}_H(T)}{2} = \frac{\Delta}{1+x+y+z}, \tag{3.19}$$

where $x = \cosh a, y = \cosh b, z = \cosh c$ *and* $\Delta = \sqrt{1 - x^2 - y^2 - z^2 + 2xyz}$.

Remark 3.6.3. Formulae 3.13 to 3.18 are simplified in the following equations:

$$\frac{\sin(\sqrt{K}a)}{\sin\alpha} = \frac{\sin(\sqrt{K}b)}{\sin\beta} = \frac{\sin(\sqrt{K}c)}{\sin\gamma}, \tag{3.20}$$

$$\cos(\sqrt{K}c) = \cos(\sqrt{K}a)\cos(\sqrt{K}b) + \sin(\sqrt{K}a)\sin(\sqrt{K}b)\cos\gamma, \tag{3.21}$$

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos(\sqrt{K}c), \tag{3.22}$$

where K is the sectional curvature of the space $\mathbf{M}$ of constant sectional curvature. The hyperbolic formulae are also derived using the identities $\cos ix = \cosh x$ and $\sin ix = i\sinh x$. To obtain the Euclidean formulae, we may use algebraic transformations and the limits $\lim_{x\to 0} \sin x/x = 1$ and $\lim_{x\to 0}(1-\cos x)/x^2 = 1/2$.

We state our next proposition using the unified notation appearing in (3.20), (3.21) and (3.22).

Proposition 3.6.4. *Let γ be a curve of constant geodesic curvature k_g and of arc length s, with endpoints p and q in the space $\mathbf{M}$ of constant sectional curvature $K \in \{1, 0, -1\}$. Assume that if γ is a circle arc then it is shorter than a semicircle. In other words, assume that $s < \frac{\pi}{\sqrt{k_g^2+K}}$ if $k_g^2 + K > 0$. Let $\sigma(s)$ denote the angle between γ and the segment with endpoints p and q, and let $d(s)$ denote the distance of p and q (cf. Figure 3.8).*

Then

$$\tan\sigma(s) = \frac{k_g}{\sqrt{k_g^2+K}}\tan\frac{s\sqrt{k_g^2+K}}{2}, \tag{3.23}$$

$$\sin\frac{\sqrt{K}d(s)}{2} = \sqrt{\frac{K}{k_g^2+K}}\sin\frac{s\sqrt{k_g^2+K}}{2}, \quad \textit{and} \tag{3.24}$$

$$\frac{\tan\frac{\sqrt{K}d(s)}{2}}{\sqrt{K}\cos\sigma(s)} = \frac{\tan\frac{s\sqrt{k_g^2+K}}{2}}{\sqrt{k_g^2+K}}. \tag{3.25}$$

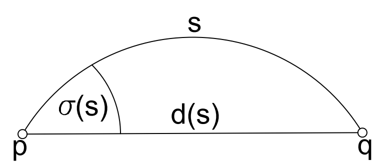

Figure 3.8: $d(s)$ and $\sigma(s)$

Proof. Note that (3.25) is a straightforward consequence of (3.23) and (3.24). First, we prove (3.23) and (3.24) for $\mathbf{M} = \mathbb{H}^2$.

Assume that γ is a hypercycle arc. We may assume that this arc is in the form $\gamma : [-a, a] \to \mathbb{H}^2$ $\gamma(t) = (t, \cos\alpha)$, where α is the angle between the hypercycle containing γ and the circle at infinity. Recall from the proof of Theorem 3.5.1 that $k_g = \cos\alpha$.

The arc length of γ is

$$s = 2\int_0^a \frac{2}{1-x^2-\cos^2\alpha}dx = \frac{2}{\sin\alpha}\operatorname{arctanh}\frac{a}{\sin\alpha},$$

which yields

$$a = \sin\alpha\tanh\frac{s\sin\alpha}{4} \tag{3.26}$$

From (3.26) and

$$\cosh d(s) = \cosh \operatorname{dist}_H(p,q) = 1 + \frac{8a^2}{(1-a^2-\cos^2\alpha)^2}, \quad \cosh 2x = 1 + 2\sinh^2 x,$$

we obtain

$$\sinh\frac{d(s)}{2} = \frac{2a}{\sin^2\alpha - a^2} = \frac{1}{\sin\alpha}\sinh\frac{s\sin\alpha}{2} = \frac{1}{\sqrt{1-k_g^2}}\sinh\frac{s\sqrt{1-k_g^2}}{2}. \qquad (3.27)$$

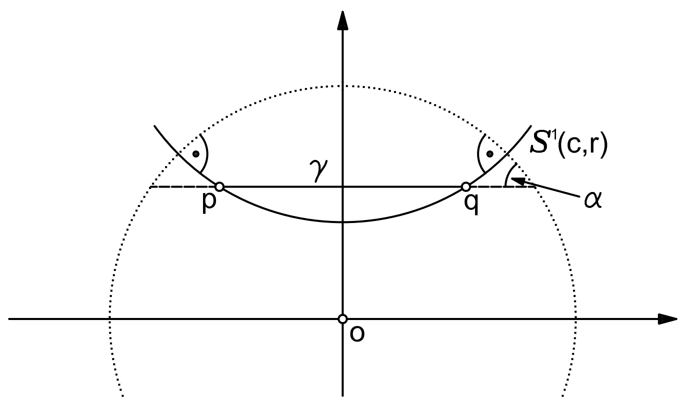

Figure 3.9: $\sigma(s)$ for a hypercycle

Let $\mathbb{S}^1(c,r)$ denote the Euclidean circle which is orthogonal to $\mathbb{S}^1$ and contains p and q (cf. Figure 3.9). The equation of $\mathbb{S}^1(c,r)$ is $x^2 + (y - \sqrt{r^2+1})^2 = r^2$ for some $r > 0$. To obtain r, we substitute the coordinates of q. Hence,

$$\sqrt{r^2+1} = \frac{2\cosh^2\frac{s\sin\alpha}{4} - \sin^2\alpha}{2\cos\alpha\cosh^2\frac{s\sin\alpha}{4}}$$

Note that $\sigma(s)$ is the angle between $\mathbb{S}^1(c,r)$ and the Euclidean segment $[p,q]$, which is equal to $\sphericalangle(q-c,-c)$. Thus,

$$\tan\sigma(s) = \frac{a}{\sqrt{r^2+1}-\cos\alpha} = \cot\alpha\tanh\frac{s\sin\alpha}{2} = \frac{k_g}{\sqrt{1-k_g^2}}\tanh\frac{s\sqrt{1-k_g^2}}{2}.$$

Assume that γ is a hyperbolic circle of radius r_H. We may assume that the hyperbolic centre of γ is o. Then γ coincides with a Euclidean circle $\mathbb{S}^1(o,r)$. The connection between r_H and r is $\cosh r_H = 1 + 2r^2/(1-r^2)$, which implies that $\tanh(r_h/2) = r$. We have seen that the hyperbolic arc length parametrization of γ is

$$\gamma : [0, 2\pi/k] \to \mathbb{H}^2, \gamma(s) = (r\cos(ks), r\sin(ks)),$$

where $k = (1-r^2)/(2r) = 1/\sinh r_H$. This yields also that $\mathrm{arclength}_H(\gamma) = 2\pi \sinh r_H$. Consider the hyperbolic triangle T with vertices o , q and the midpoint r of the hyperbolic segment with endpoints p and q (cf. Figure 3.10). Note that the angle of T at r is $\pi/2$, the angle of T at q is $\pi/2 - \sigma(s)$, the angle at o is $s/(2\sinh r_H)$, $\mathrm{dist}_H(q,r) = d(s)/2$ and $\mathrm{dist}_H(o,q) = r_H$. Let $m = \mathrm{dist}_H(o,r)$. Applying (3.13) and $k_g = \coth r_H$, we obtain

$$\sinh \frac{d(s)}{2} = \sinh r_H \sin \frac{s}{2\sinh r_H} = \frac{1}{\sqrt{k_g^2-1}} \sin \frac{s\sqrt{k_g^2-1}}{2}.$$

Formula (3.24) now follows from (3.13) and (3.14).

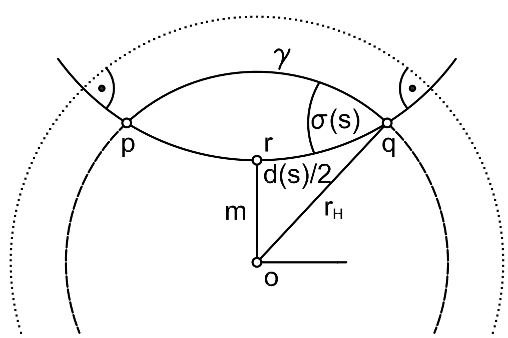

Figure 3.10: $d(s)$ for a hyperbolic circle

If $\mathbf{M} = \mathbb{E}^2$, the assertion is easy to prove. The proof for $\mathbf{M} = \mathbb{S}^2$ is similar to the proof for hyperbolic circles. □

Chapter 4

The Erdős-Szekeres Hexagon Problem

4.1 Introduction and preliminaries

In the 1930s Esther Klein asked the following question.

For every $k \geq 3$, is there an integer M such that any planar set S of at least M points in general position contains k points in convex position?

Erdős and Szekeres [22] showed that the answer is yes. They proved not only the existence of such an integer, but also that there is a solution satisfying the inequality $M \leq \binom{2k-4}{k-2} + 1$. Since their joint work led to the marriage of Esther Klein and George Szekeres, Erdős referred to it later as the "happy ending problem".

The problem that arose naturally was to find the smallest value of $\operatorname{card} S$ with the mentioned property for each k.

Definition 4.1.1. Let $k \geq 3$ and $M(k)$ denote the smallest integer such that if $S \subset \mathbb{E}^2$ is a set of points in general position and $\operatorname{card} S \geq M(k)$, then S contains k points in convex position.

From [22], we know that $M(k) \leq \binom{2k-4}{k-2} + 1$. Considering the known values $M(3) = 3$ and $M(4) = 5$, Erdős and Szekeres conjectured that $M(k) = 2^{k-2} + 1$.

Conjecture 4.1.2. *(Erdős-Szekeres) In any planar set S of $2^{k-2}+1$ points in general position, there are k points in convex position.*

In [23], Erdős and Szekeres constructed a planar set of 2^{k-2} points in general position that does not contain k points in convex position. Presently, the best known bounds for card S are

$$2^{k-2} + 1 \leq M(k) \leq \binom{2k-5}{k-2} + 1. \tag{4.1}$$

The upper bound was proven by G. Tóth and Valtr in [47] in 2005.

Let us see what is known about $M(k)$ for small values of k. Since three points are in general position if, and only if, they are in convex position, we have $M(3) = 3$. We show that $M(4) = 5$. Indeed, if, among five points in general position, there are no four points in convex position then the convex hull of the points is a triangle. This triangle contains two points, say a and b, in its interior. Then one of the two half-planes bounded by the line passing through a and b contains two additional points, say c and d. So, a, b, c and d are in convex position (cf. Figure 4.1). This consideration is due to Klein and Szekeres.

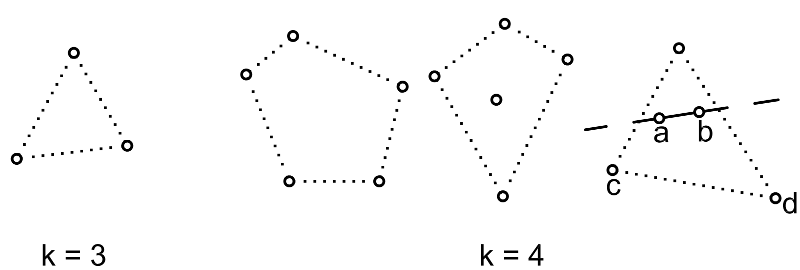

Figure 4.1: Configurations in Klein's proof

According to [46], Makai was the first to prove the equality $M(5) = 9$ but he has never published his result. The first published proof appeared in [36] and is due to J. D. Kalbfleisch, J. G. Kalbfleisch and Stanton. In 1974, Bonnice [10] gave a simple and elegant proof of the same result. Bisztriczky and G. Fejes Tóth [9] also mention an unpublished proof by Böröczky and Stahl.

To prove that $M(6) = 17$ seems considerably more complicated. Bonnice [10] makes the following comparison. In a set of nine points, we have $\binom{9}{5} = 126$ possibilities for five points to be in convex position, whereas in a set of seventeen points, we have $\binom{17}{6} = 12376$ possibilities for six points to be in convex position. For this case, a computer-based proof has been given by Szekeres and Peters [46] recently, which allows us to make another comparison. They state that their program proved the case of convex pentagons in less than one second on a 1.5GHz computer, but to check the case of convex hexagons required approximately 1500 hours. For other results related to the Erdős-Szekeres Conjecture, the reader is referred to the survey [44] of Morris and V. Soltan, or the book [12] of Brass, Moser and Pach.

Our aim is to examine the $k = 6$ case of the Erdős-Szekeres Conjecture. For simplicity, by an m-gon, $m \geq 3$, we mean a convex m-gon. We collect our results in Theorems 4.1.3 and 4.1.4.

Theorem 4.1.3. *Let $S \subset \mathbb{E}^2$ be a set of seventeen points in general position and $P = [S]$ be a pentagon. Then S contains six points in convex position.*

We note that a different proof of the same statement appeared in the diploma thesis [18] of Knut Dehnhardt.

Theorem 4.1.4. *Let $S \subset \mathbb{E}^2$ be a set of twenty five points in general position. Then S contains six points in convex position.*

There are two well-known sets of sixteen points in general position that do not contain (the vertices of) a hexagon: cf. [23] and pp. 331-332 of [12] (Figure 4.2). We note that in both examples the convex hull of the points is a pentagon.

With reference to Figure 4.3, there is a set $S \subset \mathbb{R}^2$ of points with $\operatorname{card} S = 32$ such that the convex hull of any seventeen or more points of S is a 3-gon or a 4-gon.

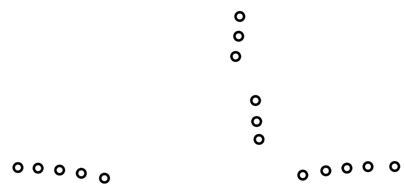

Figure 4.2: Sixteen points with no hexagon

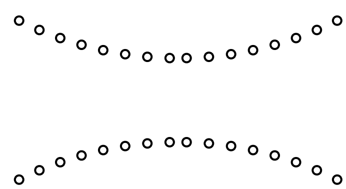

Figure 4.3: Thirty two points with no large subset whose convex hull is a pentagon

In the proof of Theorem 4.1.4, our method of argument is to assume that S does not contain (the vertices of) a hexagon. We show that there exist seventeen points in S whose convex hull is a pentagon, and then apply Theorem 4.1.3.

Definition 4.1.5. Let $\{a, b\} \subset \mathbb{E}^2$. Then $[a, b]$, $L(a, b)$, $L^+(a, b)$ and $L^-(a, b)$ denote, respectively, the closed segment with endpoints a and b, the line containing a and b, the closed ray emanating from a and containing b, and the closed ray emanating from a in $L(a, b)$ that does not contain b.

Definition 4.1.6. Let $s \geq 3$ and P be an s-gon. We say that b is *beyond* (respectively, *beneath*) the edge $[p, q]$ of P, if b is in the open half plane, bounded by aff$[p, q]$, that contains (respectively, does not contain) P. We say that $b \in \mathbb{E}^2$ *is beyond exactly one edge of* P if b is beyond one edge and beneath $s-1$ edges of P.

4.2 Proof of Theorem 4.1.3

We begin the proof with a series of lemmas.

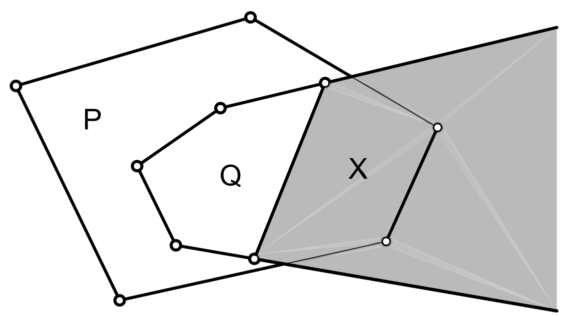

Figure 4.4: $V(Q) \cup X$ is a convex hexagon

Lemma 4.2.1. *Let P and Q be polygons with $Q \subset \operatorname{int} P \subset \mathbb{E}^2$. Let $X \subset V(P)$ be a set of points beyond exactly the same edge of Q. Then $V(Q) \cup X$ is a set of points in convex position (cf. Figure 4.4).*

Lemma 4.2.2. *Let $\{P_i : i = 1, 2, \ldots, m\}$ be a family of t triangles, q quadrangles and p pentagons such that $p + q + t = m$ and $M = [P_1, P_2, P_3, \ldots, P_m]$ is an m-gon $[x_1, x_2, x_3, \ldots, x_m]$. Suppose that $[x_i, x_{i+1}]$ is an edge of P_i, and P_i and P_{i+1} do not overlap for $i = 1, 2, 3, \ldots, m$. Let P_0 be a u-gon that contains M in its interior and assume that the points of $W = \bigcup_{i=0}^{m} V(P_i)$ are in general position. If $q + 2t < u$, then W contains a hexagon.*

Proof. Let us denote by X_i the set of points that are beyond exactly the edge $[x_i, x_{i+1}]$ of P_i and observe that every vertex of P_0 is contained in X_i for some value of i. If $\mathrm{card}(X_i \cap V(P_0)) + \mathrm{card}(V(P_i)) \geq 6$ for some P_i, then the assertion follows (cf. Figure 4.5). Since $\mathrm{card}(X_i \cap V(P_0)) + \mathrm{card}(V(P_i)) \leq 5$ for each P_i yields that $u = \mathrm{card}(V(P_0)) \leq 0 \cdot p + 1 \cdot q + 2 \cdot t$, we are done. □

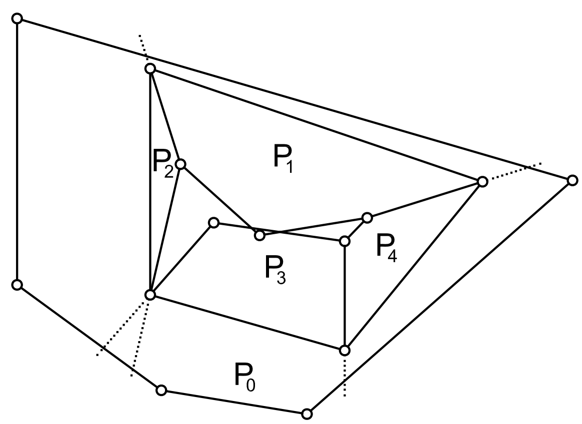

Figure 4.5: An illustration for Lemma 4.2.2

We use Lemma 4.2.2 often during the proof with $u = 5$. For simplicity in such cases, we use the notation $P_1 * P_2 * \ldots * P_m$.

Lemma 4.2.3. *Let $S \subset \mathbb{E}^2$ be a set of eleven points in general position such that $P = [S]$ is a pentagon, $Q = [S \setminus V(P)]$ is a triangle and $[S \setminus (V(P) \cup \{q\})]$ is a quadrilateral for every $q \in V(Q)$. Then S contains a hexagon.*

Proof. Let $Q = [q_1, q_2, q_3]$ and $R = \left[S \setminus \left(V(P) \cup V(Q)\right)\right] = [r_1, r_2, r_3]$. Observe that, for any $i \neq j$, the straight line $L(r_i, r_j)$ strictly separates the third vertex of R from a unique vertex of Q. We may label our points in a way that q_1, q_2 and q_3 are in counterclockwise cyclic order, and $L(r_i, r_j)$ separates r_k and q_k for any $i \neq j \neq k \neq i$. Let us denote by Q_k the open convex domain bounded by $L^-(q_k, r_i)$ and $L^-(q_k, r_j)$ for every $i \neq j \neq k \neq i$. For every $i \neq j$, let us denote by Q_{ij} the open convex domain that is bounded by the rays $L^-(q_i, r_j)$, $L^-(q_j, r_i)$ and the segment $[q_i, q_j]$ (cf. Figure 4.6).

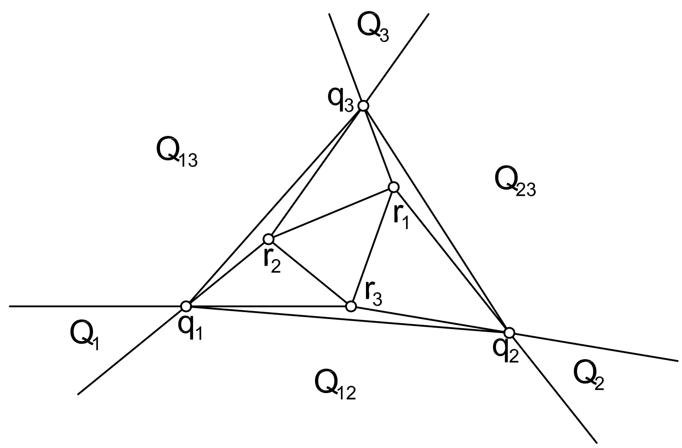

Figure 4.6: An illustration for the domains in the proof of Lemma 4.2.3

Observe that if Q_{12} contains at least two vertices of P, then these vertices together with q_1, q_2, r_1, r_2 are vertices of a hexagon. Similarly, if $Q_1 \cup Q_{13} \cup Q_3$ contains at least three vertices of P, or $Q_2 \cup Q_{23} \cup Q_3$ contains at least three vertices of P, then S contains a hexagon. Since P is a pentagon, we may assume that Q_{12} contains one, $Q_1 \cup Q_{13}$ and $Q_2 \cup Q_{23}$ both contain two, and Q_3 contains no vertex of P. By symmetry, we obtain that S contains a hexagon unless $\operatorname{card}(Q_i \cap V(P)) = 0$ and $\operatorname{card}(Q_{ij} \cap V(P)) = 1$ for every $i \neq j$. Since the latter case contradicts the condition that P is a pentagon, S contains a hexagon. $\square$

Lemma 4.2.4. *Let $S \subset \mathbb{E}^2$ be a set of thirteen points in general position such that*

$P = [S]$ is a pentagon and $Q = [S \setminus V(P)]$ is a triangle. Then S contains a hexagon.

Proof. Let q_1, q_2 and $q_3 = q_0$ be the vertices of Q in counterclockwise cyclic order and let $R = S \setminus (V(P) \cup V(Q))$. Observe that $\operatorname{card} R = 5$. Using an idea similar to that used by Klein and Szekeres, we obtain that R contains an empty quadrilateral. In other words, there is a quadrilateral U that satisfies $V(U) \subset R$ and $U \cap R = V(U)$.

We show that if U has no sideline that separates U from an edge of Q, then S contains a hexagon. Indeed, if every sideline of U separates U from exactly one vertex of Q then, by the Pigeon-Hole Principle, Q has a vertex, say q_3, such that at least two sidelines of U separate U from it. This yields that there are two sidelines passing through consecutive edges of U that separate U from only q_3. Let these edges be $[r_{i-1}, r_i]$ and $[r_i, r_{i+1}]$. Then we have $[q_1, r_{i+1}, r_i, r_{i-1}, q_2] * [q_2, r_{i-1}, q_3] * [q_3, r_{i+1}, q_1]$. Hence, we may assume that U has a sideline that separates U from an edge of Q. Without loss of generality, let this sideline pass through the edge $[r_1, r_2]$ and let it separate U from $[q_1, q_2]$.

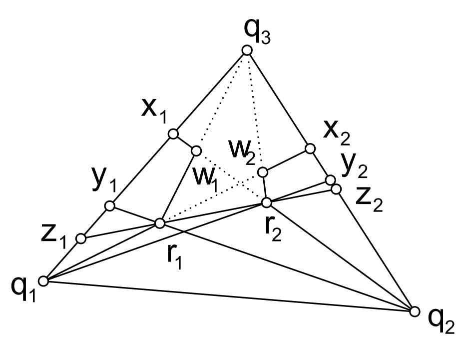

Figure 4.7: An illustration for the proof of Lemma 4.2.4

For every $3 \neq i \neq j \neq 3$, let x_i, y_i and z_i denote the intersection point of the segment $[q_i, q_3]$ with the line $L(q_j, r_j)$, $L(q_j, r_i)$ and $L(r_1, r_2)$, respectively, and let w_i

denote the intersection point of $[r_i, q_3]$ and $L(q_j, r_j)$ (cf. Figure 4.7). If some point $u \in R$ is beyond exactly the edge $[r_1, r_2]$ of $[q_1, q_2, r_2, r_1]$, then we have $[q_1, r_1, u, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r_1, q_1]$. If $u \in R$ is beyond exactly the edge $[r_1, q_3]$ of $[q_1, r_1, q_3]$, then $[q_1, r_1, u, q_3] * [q_3, s, q_2] * [q_2, r_2, r_1, q_1]$ for $s = u$ or $s = r_2$. Hence, by symmetry, we may assume that r_3, r_4 and r are in one of the quadrangles $[r_i, w_i, x_i, z_i]$ for $i = 1$ or 2, or in $[q_1, q_2, z_2, z_1]$.

Assume that $r_3 \in [r_1, w_1, x_1, z_1]$. If $L^+(r_4, r_3) \cap [q_1, q_3] \neq \emptyset$ then $[q_3, r_3, r_4, r_1, r_2] * [r_2, r_1, q_1] * [q_1, r_3, q_3]$. If $L^-(r_4, r_3) \cap [q_1, q_3] \neq \emptyset$ then $[q_1, r_4, r_3, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r_4, q_1]$. If $L(r_4, r_3) \cap [q_1, q_3] = \emptyset$ then $[q_1, r_1, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r_3, r_4, q_1]$. Thus we may assume that $r_3 \in [r_2, z_2, x_2, w_2]$. Since $r_3 \in [r_2, y_2, z_2]$ yields $[q_3, r_4, r_1, r_2, r_3] * [r_3, r_2, q_1] * [q_1, r_4, q_3]$, we may assume also that $r_3 \in [r_2, w_2, x_2, y_2]$, and (by symmetry) that $r_4 \in [r_1, w_1, x_1, y_1]$.

Assume that $r \in [r_1, w_1, x_1, y_1]$. If $[r_1, r_2, r_4, r]$ is a quadrilateral, then we may apply an argument similar to that in the previous paragraph. Thus we may assume that $r_4 \in [r_1, r_2, r]$. This yields $[r, r_4, r_1, q_1] * [q_1, r_1, r_2, q_2] * [q_2, r_3, q_3] * [q_3, r_3, r_2, r_4, r]$. Hence, $r \in [q_1, q_2, z_2, z_1]$.

If $r \in [q_1, r_1, z_1]$ then $[q_1, r, r_1, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r, q_1]$. Let $r \in [q_1, q_2, r_2, r_1]$. If $L(q_3, r_4)$ does not separate q_1 and r then $[q_1, r, q_2] * [q_2, r_1, r_4, q_3] * [q_3, r_4, r, q_1]$. We suppose otherwise, which yields that $L^+(r, r_4) \cap [q_1, q_3] \neq \emptyset$. By symmetry, we obtain also that $L^+(r, r_3) \cap [q_2, q_3] \neq \emptyset$.

Assume that $r \in [q_1, r_1, r_2]$. Then we observe that $U' = [r, r_2, r_3, r_1]$ is an empty quadrilateral, and $L(r, r_2)$ separates U' from $[q_1, q_2]$. Since $R \cap [q_1, r, r_2, q_2] = \emptyset$, an argument applied for U', similar to that applied for U, yields a hexagon. Hence $r \in [q_1, r_1, q_2] \cap [q_1, r_2, q_2]$. Then $L^+(r_3, r) \cap [q_1, q_2] \neq \emptyset \neq L^+(r_4, r) \cap [q_1, q_2]$. Now, we apply Lemma 4.2.3 with $V(P) \cup V(Q) \cup \{r_3, r_4, r\}$ as S. $\square$

We summarize our consideration in the following way. We show that if $\tilde{S}$ is a set of at least eight points in general position such that $[\tilde{S}]$ is a triangle, then $\tilde{S}$ contains a subset X that satisfies one of the following (cf. Figure 4.8):

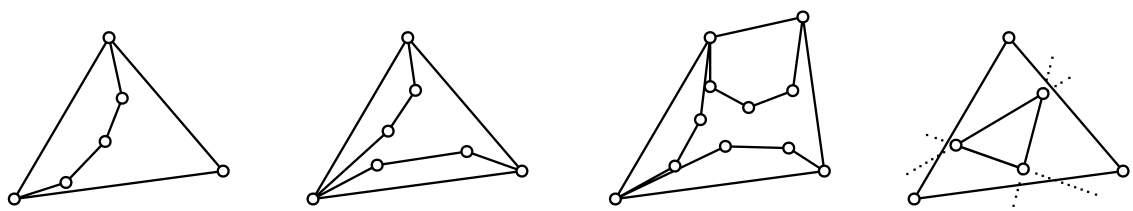

Figure 4.8: Forbidden configurations

(1) $T = [X]$ is a triangle and X contains a pentagon which has an edge in common with T.

(2) $T = [X]$ is a triangle and X contains two nonoverlapping quadrilaterals, each of which has an edge in common with T.

(3) $Q = [X]$ is a quadrilateral and X contains two quadrilaterals and a pentagon that are pairwise nonoverlapping and each has an edge in common with Q.

(4) $T = [X]$ is a triangle and, for any $t \in V(T)$, $[X \setminus \{t\}]$ is a quadrilateral.

Lemma 4.2.4 follows from Lemma 4.2.2 in the first three cases and from Lemma 4.2.3 in the last case.

Definition 4.2.5. Let $A, B \subset \mathbb{E}^2$ be sets of points in general position. Suppose that there is a bijective function $f : A \to B$ such that, for any $a_1, a_2, a_3 \in A$, the ordered triples (a_1, a_2, a_3) and $(f(a_1), f(a_2), f(a_3))$ have the same orientation. Then we say that A and B are *identical*.

We note that if A and B are identical, then $A' \subset A$ is a k-gon if, and only if, $f(A')$ is a k-gon.

Using Lemma 4.2.4 and a little more effort, we characterize all the sets $\tilde{S}$ that have no subset identical to one of the four sets in Figure 4.8. Lemma 4.2.6 summarizes our work. We omit its straightforward proof.

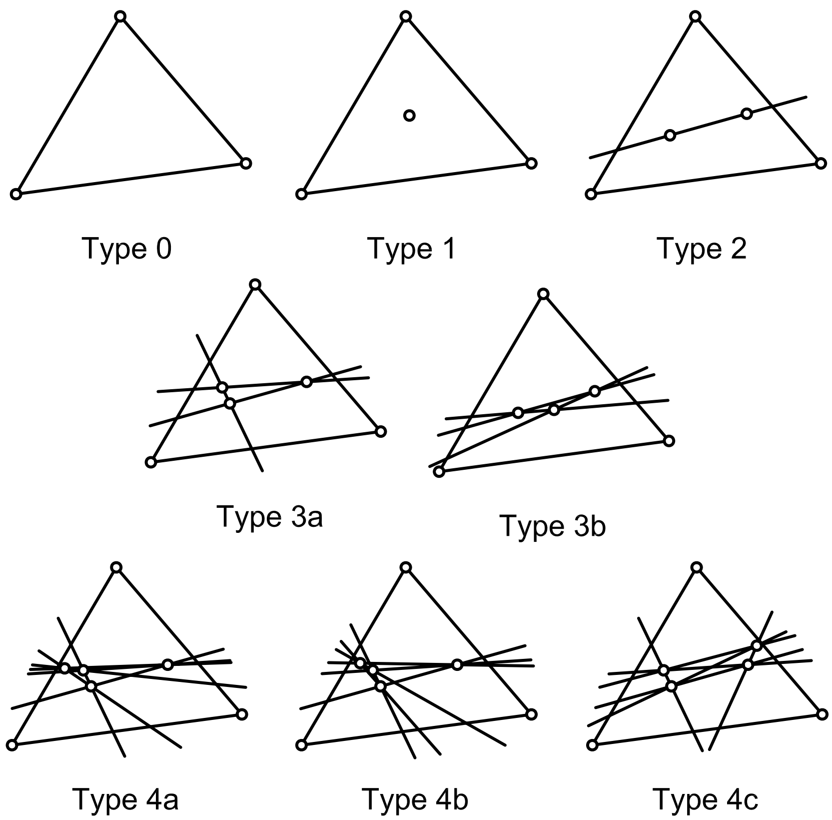

Figure 4.9: Configurations that do not imply the existence of a hexagon

Lemma 4.2.6. *Let $\tilde{S} \subset \mathbb{E}^2$ be a set of points in general position such that $[\tilde{S}]$ is a triangle and $\tilde{S}$ has no subset identical to one of the four sets in Figure 4.8. Then $\tilde{S}$ is identical to one of the sets in Figure 4.9.*

This list will help us to exclude some other cases from our investigation. If a set is identical to one of the sets in Figure 4.9, we say that its type is the type of the corresponding set in the figure.

Lemma 4.2.7. *Let $S \subset \mathbb{E}^2$ be a set of seventeen points in general position such that $P = [S]$ is a pentagon and $Q = [S \setminus V(P)]$ is a quadrilateral. Then S contains a hexagon.*

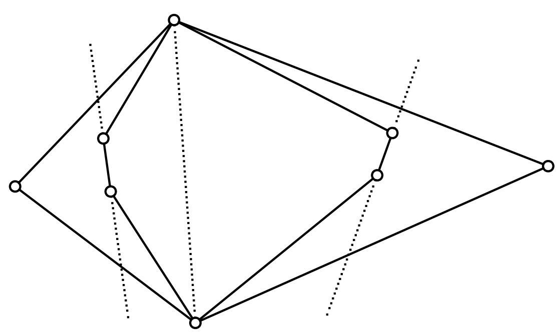

Figure 4.10: Two lines that do not intersect a diagonal

Proof. By Lemma 4.2.4, we may assume that any diagonal of Q divides Q into two triangles that contain exactly four points of S in their interiors. Furthermore, both these triangles have to be either type 4a, or 4b, or 4c. Let us observe that if both triangles contain a pair of points such that the line passing through them does not intersect the diagonal, then these two pairs of points and the two endpoints of the diagonal are in convex position (cf. Figure 4.10). Hence we may assume that, in at least one of the triangles, each line passing through two points intersects the diagonal.

Since there is, in a type 4c set, no edge of the convex hull that meets all the lines that pass through two of its points, we may assume that the set of the points in one of the triangles is type 4a or 4b, and that the diagonal is the left edge of one of the triangles in Figure 4.9. We observe also that configurations of type 4a or 4b are almost identical, the only difference is that the line passing through the two points closest to the left edge of the triangle intersect the bottom or the right edge of the triangle. Thus, we may handle these two cases together if we leave it open whether this line intersects the bottom or the right edge of the triangle.

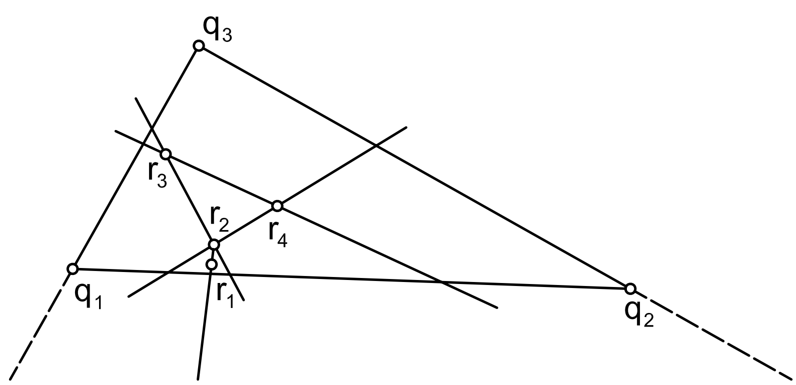

Figure 4.11: An illustration for the main case in the proof of Lemma 4.2.7

We denote our points as in Figure 4.11, and let $L = L(r_1, r_2)$. Observe that L divides the set of points, beyond exactly the edge $[q_1, q_2]$ of $[q_1, q_2, q_3]$, into two connected components. If a point p is in the component that contains q_1, respectively q_2, in its boundary, then we say that p is on the *left-hand side*, respectively *right-hand* *side*, of L. Let $B = (Q \setminus [q_1, q_2, q_3]) \cap S$. Observe that $\operatorname{card} B = 5$ and that every point of B is either on the left-hand side, or on the right-hand side of L. By the Pigeon-Hole Principle, there are three points of B that are on the same side of L. Let us denote these points by s_1, s_2 and s_3.

Assume that s_1, s_2 and s_3 are on the left-hand side of L. Observe that if $L(s_i, s_j)$ and $[q_1, r_1]$ are disjoint for some $i \neq j$, then $[q_1, s_i, s_j, r_1, r_2, r_3]$ is a hexagon. Thus we may relabel s_1, s_2 and s_3 such that $s_3 \in [q_1, r_1, s_2] \subset [q_1, r_1, s_1]$. This yields that either $[s_1, s_2, s_3, q_1]$ or $[s_1, s_2, s_3, r_1]$ is a quadrilateral. If $[s_1, s_2, s_3, q_1]$ is a quadrilateral then $[s_1, s_2, s_3, q_1] * [q_1, s_3, r_1, r_2, r_3] * [r_3, r_4, q_2] * [q_2, r_1, s_2, s_1]$. If $[s_1, s_2, s_3, r_1]$ is a quadrilateral then $[s_1, s_2, s_3, r_1, q_2] * [q_2, r_4, r_3] * [r_3, r_2, r_1, s_3, q_1] * [q_1, s_2, s_1]$.

Let s_1, s_2 and s_3 be on the right-hand side of L. Observe that if $L(s_i, s_j)$ and $[q_2, r_1]$ are disjoint for some $i \neq j$, then $[q_2, s_i, s_j, r_1, r_2, r_4]$ is a hexagon. Hence we may assume that $s_3 \in [q_2, r_1, s_2] \subset [q_2, r_1, s_1]$. Then $[s_1, s_2, s_3, q_2]$ or $[s_1, s_2, s_3, r_1]$ is a quadrangle. If $[s_1, s_2, s_3, q_2]$ is a quadrilateral then $[s_1, s_2, s_3, q_2] * [q_2, s_3, r_1, r_2, r_4] * [r_4, r_2, q_1] * [q_1, r_1, s_2, s_1]$. If $[s_1, s_2, s_3, r_1]$ is a quadrilateral then $[s_1, s_2, s_3, r_1, q_1] * [q_1, r_2, r_4] * [r_4, r_2, r_1, s_3, q_2] * [q_2, s_2, s_1]$. $\square$

Lemma 4.2.8. *Let $S \subset \mathbb{E}^2$ be a set of points in general position such that $P = [S]$ and $Q = [S \setminus V(P)]$ are pentagons, and $S \setminus (V(P) \cup V(Q))$ has a subset of type 3a, or a subset identical to the point set in Figure 4.13, or 4.14 or 4.15. Then S contains a hexagon.*

Proof. Let R denote the subset of $S \setminus (V(P) \cup V(Q))$ that is either of type 3a, or is identical to the point set in Figure 4.13, or 4.14 or 4.15. Let q_1, q_2, q_3, q_4 and q_5 denote the vertices of Q in counterclockwise cyclic order.

Assume that R is of type 3a. Let us denote the points of R as in Figure 4.12. Let R_{12}, R_{23} and R_{13} denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_1, t_3, t_2, r_2]$, the edge $[r_2, r_3]$ of $[r_2, t_1, t_3, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_3, r_3]$. If $\text{card}(R_{12} \cap V(Q)) \geq 2$, or $\text{card}(R_{23} \cap V(Q)) \geq 2$ or $\text{card}(R_{13} \cap V(Q)) \geq 3$, then S contains a convex hexagon. Otherwise, there is a vertex q_i of Q in the convex domain bounded by the half-lines $L^-(r_2, t_1)$ and $L^-(r_2, t_2)$, from which we obtain $[r_1, t_1, r_2, q_i] * [q_i, r_2, t_2, r_3] * [r_3, t_3, r_1]$.

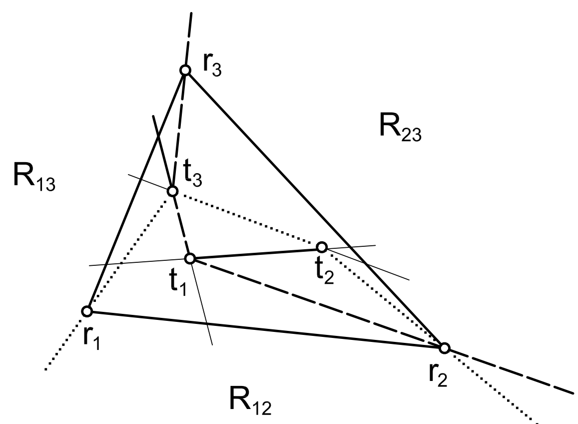

Figure 4.12: A type 3a point set with the notation of the proof of Lemma 4.2.8

Let us assume that R is the set in Figure 4.13 and denote the points of R as indicated. Let R_{12}, R_{23} and R_{13} denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_2, t_2, t_1, r_1]$, the edge $[r_2, r_3]$ of $[r_2, t_2, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_1, r_3]$. If $\text{card}(R_{12} \cap V(Q)) \geq 2$, or $\text{card}(R_{23} \cap V(Q)) \geq 3$ or $\text{card}(R_{13} \cap V(Q)) \geq 3$ then S contains a hexagon. Hence, we may assume that $q_1 \in R_{12}$, $\{q_2, q_3\} \subset R_{23}$, $\{q_4, q_5\} \subset R_{13}$ and there is no vertex of Q in $R_{23} \cap R_{13}$. If $L(q_1, r_4)$ does not intersect the interior of $[R]$, then $[t_1, t_2, r_2, r_4, q_1, r_1]$ is a hexagon.

Let $r_4 \in [q_1, r_1, r_2]$. If $L(r_4, r_1)$ does not separate q_5 and q_1, and $L(r_4, r_2)$ does not separate q_2 and q_1, then $[q_1, r_4, r_2, q_2] * [q_2, r_2, t_2, r_3] * [r_3, t_1, r_1, q_5] * [q_5, r_1, r_4, q_1]$. Thus we may assume that, say, $L(r_4, r_1)$ separates q_5 and q_1. If $L(r_2, r_3)$ separates q_4 and R then $[q_4, r_3, r_2] * [r_2, t_2, t_1, r_1] * [r_1, t_1, r_3, q_4]$. If $L(r_2, r_3)$ does not separate q_4 and R then $[r_4, r_2, r_3, q_4, q_5, r_1]$ is a hexagon.

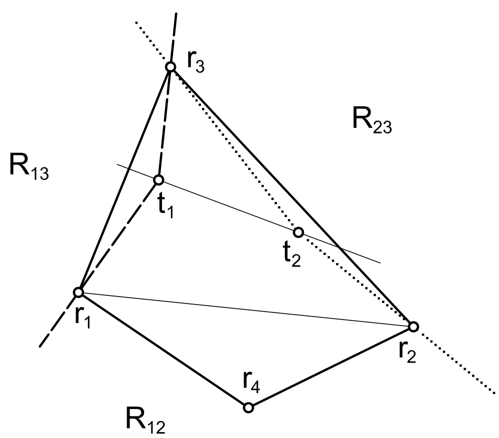

Figure 4.13: Another case in Lemma 4.2.8

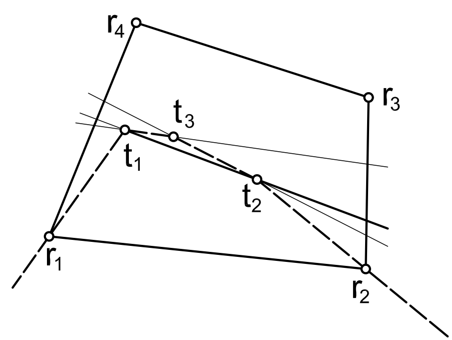

Figure 4.14: One more case in Lemma 4.2.8

Assume that R is the set in Figure 4.14 and denote the points of R as indicated. We may clearly assume that there is no vertex of Q beyond exactly the edge $[r_1, r_2]$ of $[r_1, r_2, t_2, t_3, t_1]$. Hence there is an edge, say $[q_1, q_2]$, that intersects both rays $L^-(r_1, t_1)$ and $L^-(r_2, t_2)$. If $L(r_1, r_2)$ separates R from both q_1 and q_2 then $[q_1, r_1, r_2, q_2] * [q_2, r_2, t_2, r_3] * [r_3, t_2, t_1, r_4] * [r_4, t_1, r_1, q_1]$. Hence we may assume that $L(r_1, r_2)$ does not separate R, say, from q_2. If $L(t_2, t_3)$ does not separate r_2 and q_2, then $[r_1, r_2, q_2, t_2, t_3, t_1]$ is a hexagon. If $L(t_2, t_3)$ separates r_2 and q_2, then $[q_1, r_2, q_2] * [q_2, t_2, t_3, r_4] * [r_4, t_1, r_1, q_1]$.

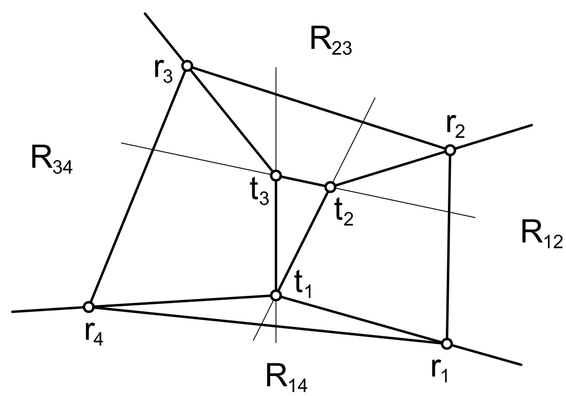

Figure 4.15: The last case in Lemma 4.2.8

We are left with the case when R is the set in Figure 4.15 with points as indicated. Let R_{12}, R_{23}, R_{34} and R_{14} denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_1, r_2, t_2, t_1]$, the edge $[r_2, r_3]$ of $[r_2, t_2, t_3, r_3]$, the edge $[r_3, r_4]$ of $[r_3, t_3, t_1, r_4]$, and the edge $[r_1, r_4]$ of $[r_4, t_1, r_1]$. If $\mathrm{card}(R_{i(i+1)} \cap V(Q)) \geq 2$ for some $i \in \{1, 2, 3\}$, then S contains a hexagon. Otherwise, R_{14} contains at least two vertices of Q, which we denote by q_1 and q_2. If both q_1 and q_2 are beyond exactly the edge $[r_1, r_4]$ of $[r_1, t_2, t_3, r_4]$ then $[t_2, t_3, r_4, q_1, q_2, r_1]$ is a hexagon. Thus we may assume that, say, q_1 is beyond exactly the edge $[r_3, r_4]$ of $[r_3, t_3, r_4]$. From this, it follows that $[r_3, t_3, r_4, q_1] * [q_1, r_4, t_1, r_1] * [r_1, t_1, t_2, r_2] * [r_2, t_2, t_3, r_3]$. $\square$

Now we are ready to prove Theorem 4.1.3.

Let $Q = [S \setminus V(P)]$, $R = [S \setminus (V(P) \cup V(Q))]$ and $T = S \setminus (V(P) \cup V(Q) \cup V(R))$. If Q is a triangle, then we apply Lemma 4.2.4. If Q is a quadrilateral, we apply Lemma 4.2.7. Let Q be a pentagon. If R is a triangle or a quadrilateral then it contains a subset identical to the subset R of $S \setminus (V(P) \cup V(Q))$ in Lemma 4.2.8. Let R be a pentagon. We note that T contains two points, say, t_1 and t_2.

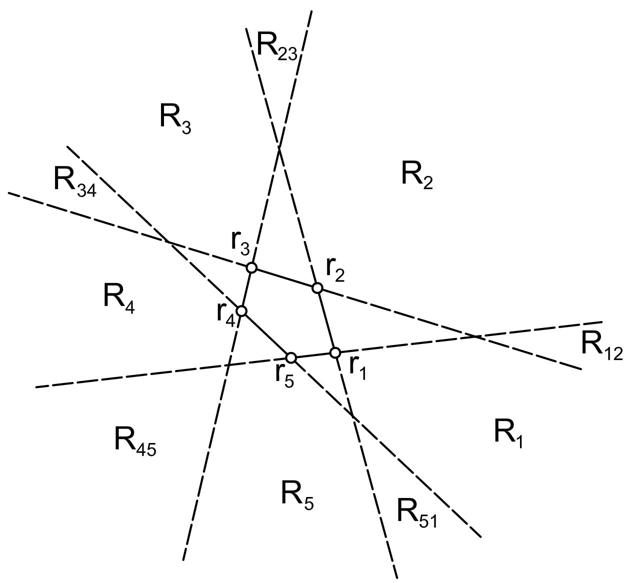

Figure 4.16: The regions around R in the proof of Theorem 4.1.3

Let q_1, q_2, q_3, q_4, q_5, and r_1, r_2, r_3, r_4, r_5 denote, respectively, the vertices of Q and R, in counterclockwise cyclic order. If some q_i is beyond exactly one edge of R, then $[R, q_i]$ is a hexagon. Thus we may assume that every vertex of Q is beyond at least two edges of R. Observe that there is no point on the plane that is beyond all five edges of R. If some q_i is beyond all edges of R but one, say $[r_1, r_5]$, then we obtain $[r_1, r_2, r_3, r_4, r_5] * [r_5, r_4, q_i] * [q_i, r_2, r_1]$. Hence we may assume that every vertex of Q

is beneath at least two edges of R.

For $1 \leq i \leq 5$, let R_i denote the set of points that are beyond the two edges of R that contain r_i and beneath the other three edges of R, and let $R_{i(i+1)}$ denote the set of points that are beyond the edges of R that contain r_i or r_{i+1}, and beneath the other two edges of R (cf. Figure 4.16). We call $R_{(i-1)i}$ and $R_{i(i+1)}$ *consecutive regions*.

Assume that two distinct and nonconsecutive regions contain vertices of Q, say, $q_k \in R_{51}$ and $q_l \in R_{23}$. Since every vertex of Q is beneath at least two edges of R, q_k and q_l are distinct points. If there is a vertex q_h of Q in $R_{34} \cup R_4 \cup R_{45}$, then $[q_l, r_3, r_4, q_h] * [q_h, r_4, r_5, q_k] * [q_k, r_1, r_2, q_l]$. Let $V(Q) \cap (R_{34} \cup R_4 \cup R_{45}) = \emptyset$. Then exactly one edge of Q intersects $R_{34} \cup R_4 \cup R_{45}$. Let us denote this edge by $[q_m, q_{m+1}]$. If $q_m \in R_{23}$ then $[q_{m+1}, r_4, q_m] * [q_m, r_2, r_1] * [r_1, r_2, r_3, r_4, q_{m+1}]$. Let $q_m \in R_3$ and, by symmetry, $q_{m+1} \in R_5$. If there are at least three vertices of Q in $R_2 \cup R_{23} \cup R_3$ or in $R_1 \cup R_{15} \cup R_5$ then $V(Q) \cup V(R)$ contains a hexagon. Hence we may assume that a vertex q_g of Q is in R_{12}. Since every vertex of Q is beneath at least two edges of Q, the sum of the angles of R at r_1 and r_2 is greater than π, which implies that $L(r_1, r_2)$ separates R and q_g. Thus we have $[q_g, r_2, r_3, q_m] * [q_m, r_4, q_{m+1}] * [q_{m+1}, r_5, r_1, q_g]$.

Assume that two consecutive regions contain vertices of Q, say $q_k \in R_{51}$ and $q_l \in R_{12}$. If $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) \neq \emptyset$, then we may apply the argument in the previous paragraph. Let $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) = \emptyset$. If at least four vertices of Q are beneath the edge $[r_3, r_4]$ of R then these vertices, together with r_3 and r_4, are six points in convex position. Hence we may assume that $R_3 \cup R_4$ contains at least two vertices of Q. Let us denote these vertices by q_e and q_f. If $q_e, q_f \in R_3$ then $[r_1, r_2, q_e, q_f, r_4, r_5]$ is a hexagon. Thus, we may clearly assume that, say, $q_e \in R_3$ and

$q_f \in R_4$. Then we have $[q_l, r_2, r_3, q_e] * [q_e, r_3, r_4, q_f] * [q_f, r_4, r_5, q_k] * [q_k, r_5, r_1, q_l]$.

Assume that $R_{i(i+1)}$ contains a vertex of Q for some i, say $q_1 \in R_{51}$. By the preceeding, no vertex of Q is in $R_{12} \cup R_{23} \cup R_{34} \cup R_{45}$. An argument similar to that used in the previous paragraph yields the existence of a hexagon if R_2 or R_3 or R_4 contains no vertex of Q. Let $q_k \in R_2$, $q_l \in R_3$ and $q_m \in R_4$. Then $[q_1, r_1, r_2, q_k] * [q_k, r_2, r_3, q_l] * [q_l, r_3, r_4, q_m] * [q_m, r_4, r_5, q_1]$.

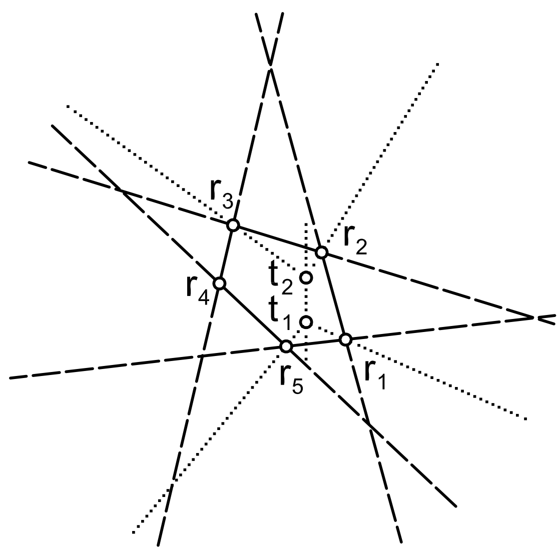

Figure 4.17: The last case in the proof of Theorem 4.1.3

We have now arrived at the case that each vertex of Q is beyond exactly two edges of R. Clearly, we may assume that $q_i \in R_i$ for each i. If $L(t_1, t_2)$ intersects two consecutive edges of R, then S contains a hexagon. Hence we may assume that, say, $L^+(t_1, t_2) \cap [r_2, r_3] \neq \emptyset \neq L^-(t_1, t_2) \cap [r_5, r_1]$ (cf. Figure 4.17). If both q_1 and q_2 are beyond exactly the edge $[r_1, r_2]$ of $[r_1, t_1, t_2, r_2]$, then we have a hexagon. If neither point is beyond exactly that edge, then $[q_1, r_1, r_2, q_2] * [q_2, r_2, t_2, r_3] * [r_3, t_2, t_1, r_5] *$

$[r_5, t_1, r_1, q_1]$. Thus, we may assume that q_1 is beyond exactly the edge $[r_1, r_2]$ and q_2 is not. If q_5 is beyond exactly the edge $[r_4, r_5]$ of $[r_4, r_5, t_1, t_2, r_3]$ then $[q_5, r_5, t_1, t_2, r_3, r_4]$ is a hexagon. Hence, we may assume that q_5 is beyond exactly the edge $[r_1, r_5]$ of $[r_1, t_1, r_5]$ and, similarly, that q_3 is beyond exactly the edge $[r_2, r_3]$ of $[r_2, t_2, r_3]$. From this, we obtain that $[q_3, r_3, r_4, q_4] * [q_4, r_4, r_5, q_5] * [q_5, r_5, r_1, q_1] * [q_1, r_1, r_2, q_2] * [q_2, r_2, t_2, r_3, q_3]$.

Chapter 5

Spindle convexity

5.1 Spindle convex sets and their separation properties

In 1935, Mayer [42] introduced a new notion of convexity, called "Überkonvexität". Unfortunately, his definition was too general to raise the interest of other mathematicians, so the concept of Überkonvexität has been forgotten.

The primary aim of this chapter is to introduce a special case of Mayer's convexity, which we call *spindle convexity.* We investigate properties of spindle convex sets, and establish a theory similar to that of (linearly) convex sets. We begin with a definition.

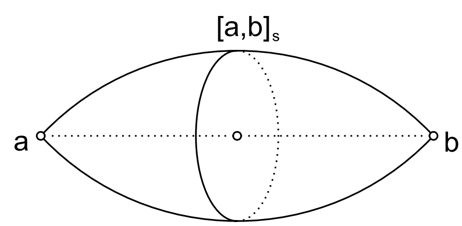

Figure 5.1: The closed spindle of a and b

Definition 5.1.1. Let $a, b \in \mathbb{E}^n$. If $||a-b|| < 2$, then the *closed spindle* of a and b, denoted by $[a,b]_s$, is the union of $[a,b]$ and the arcs of circles of radii at least one that have endpoints a and b and that are shorter than a semicircle. If $||a-b|| = 2$, then $[a,b]_s = \mathbf{B}^n[\frac{a+b}{2}]$. If $||a-b|| > 2$, then we define $[a,b]_s$ to be $\mathbb{E}^n$. The *open spindle* of a and b, denoted by $(a,b)_s$, is the interior of $[a,b]_s$.

In the next remark, we rephrase the definition of spindle. For the definition of $\mathbf{B}[X]$, see Section 2.3.

Remark 5.1.2. For any $a, b \in \mathbb{E}^n$ with $\|a-b\| \leq 2$, we have

$$[a,b]_s = \mathbf{B}\left[\mathbf{B}[\{a,b\}]\right] \quad \text{and} \quad (a,b)_s = \mathbf{B}\left(\mathbf{B}[\{a,b\}]\right).$$

The main definition of this chapter is the following.

Definition 5.1.3. A set $C \subset \mathbb{E}^n$ is *spindle convex* if $[a,b]_s \subset C$ for any distinct points a and b in C.

We collect elementary properties of spindle convex sets.

Remark 5.1.4. We note that

(1) a spindle convex set is convex;

(2) a spindle convex set is (-1)-dimensional if it is the empty set, 0-dimensional if it is one point, and full-dimensional otherwise;

(3) the intersection of spindle convex sets is spindle convex.

Motivated by Remark 5.1.4, we make the following definition.

Definition 5.1.5. Let X be a set in $\mathbb{E}^n$. Then the *spindle convex hull* of X in $\mathbb{E}^n$ is $\operatorname{conv}_{\mathrm{s}} X = \bigcap\{C \subset \mathbb{E}^n : X \subset C \text{ and } C \text{ is spindle convex in } \mathbb{E}^n\}$ (cf. Figure 5.2). If $x \notin \operatorname{conv}_{\mathrm{s}}(X \setminus \{x\})$ for any $x \in X$, we say that the points of X are in *spindle convex position*.

Note that the (linear) convex hull of a point set is independent of the dimension of the ambient space. Although it is not true for spindle convex hull, the following proposition holds. Recall that $\mathrm{cr}(X)$ denotes the circumradius of the bounded set $X \subset \mathbb{E}^n$ (cf. Section 2.3).

Proposition 5.1.6. *Let H be an affine subspace of $\mathbb{E}^n$ and X be a bounded set in H with $\mathrm{cr}(X) \leq 1$. Then the spindle convex hull of X in H is the intersection of H and the spindle convex hull of X in $\mathbb{E}^n$.*

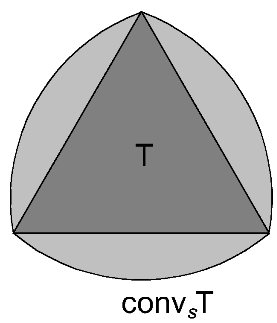

Figure 5.2: The spindle convex hull of a triangle

Definition 5.1.7. Let $a, b \in \mathbb{E}^n$ be two points with $||a - b|| \leq 2$. The *arc distance* $\rho(a, b)$ of a and b is the arc length of a shorter unit circular arc connecting a and b; that is,

$$\rho(a, b) = 2 \arcsin \frac{||a - b||}{2}.$$

Remark 5.1.8. $\rho(a, b)$ is a strictly increasing function of $||a - b||$.

In general, arc distance is not a metric. Lemma 5.1.9 describes when the triangle inequality holds or fails for arc distance.

Lemma 5.1.9 (K. Bezdek, Connelly, Csikós). *Let a, b and c be points in $\mathbb{E}^2$ such that each of $||a - b||, ||a - c||$ and $||b - c||$ is at most 2. Then*

$$\begin{array}{rlcll}
\text{(i)} & \rho(a,b)+\rho(b,c) & > & \rho(a,c) & \iff b \notin [a,c]_s; \\
\text{(ii)} & \rho(a,b)+\rho(b,c) & = & \rho(a,c) & \iff b \in \mathrm{bd}[a,c]_s; \\
\text{(iii)} & \rho(a,b)+\rho(b,c) & < & \rho(a,c) & \iff b \in (a,c)_s.
\end{array}$$

As a sample of this new geometry, we present Lemma 5.1.10. Unlike in the Euclidean case, the proof of this statement is not trivial. Lemmas 5.1.9 and 5.1.10 are both proven in [6].

Lemma 5.1.10 (K. Bezdek, Connelly, Csikós). *Let a, b, c and d be the vertices, in cyclic order, of a quadrilateral in $\mathbb{E}^2$. If a, b, c and d are in spindle convex position, then*

$$\rho(a,c)+\rho(b,d) > \rho(a,b)+\rho(c,d);$$

that is, the sum of the arc lengths of the diagonals is greater than the sum of the arc lengths of an opposite pair of sides.

Our next aim is to investigate the separability of spindle convex sets by unit spheres, motivated by the separation properties of convex sets.

Lemma 5.1.11. *Let a spindle convex set $C \subset \mathbb{E}^n$ be supported by the hyperplane H at $x \in \mathrm{bd}\, C$. Then the closed unit ball, supported by H at x and lying in the same side as C, contains C.*

Proof. Let $\mathbf{B}^n[q]$ denote the unit ball, supported by H at x, such that C and $\mathbf{B}^n(q)$ are contained in the same closed half space determined by H. Let H^+ be this closed half space, and let $y \notin \mathbf{B}^n[q]$ be a point with $\|x-y\| \le 2$.

Note that, in the plane determined by the points x, y and q, there is a shorter unit circle arc that connects x and y and that does not intersect $\mathbf{B}^n(q)$ (cf. Figure 5.3).

This arc is not contained in H^+, which implies that $[x,y]_s \not\subset H^+$. Since C is spindle convex and H supports C, we have that $y \notin C$. As y is arbitrary, $C \subset \mathbf{B}^n[q]$ follows. $\square$

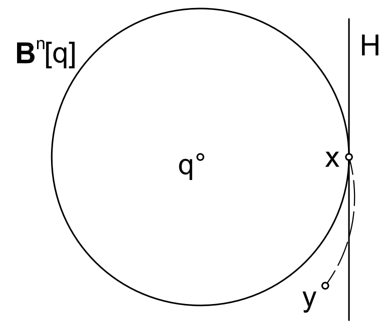

Figure 5.3: An illustration for the proof Lemma 5.1.11

Definition 5.1.12. Let $C \subset \mathbf{B}^n[q] \subset \mathbb{E}^n$ and $x \in \operatorname{bd} C$. If $x \in \mathbb{S}^{n-1}(q)$, we say that $\mathbb{S}^{n-1}(q)$ or $\mathbf{B}^n[q]$ *supports* C at x.

Corollary 5.1.13. *Let C be a closed convex set in $\mathbb{E}^n$. Then the following are equivalent.*

(i) *C is spindle convex.*

(ii) *C is the intersection of unit balls containing it; that is, $C = \mathbf{B}[\mathbf{B}[C]]$.*

(iii) *For every $x \in \operatorname{bd} C$, there is a unit ball that supports C at x.*

Corollary 5.1.13 is a straightforward consequence of Lemma 5.1.11.

Corollary 5.1.14. *Let C be a closed spindle convex set in $\mathbb{E}^n$. If $\operatorname{cr}(C) = 1$, then $C = \mathbf{B}^n[q]$ for some $q \in \mathbb{E}^n$. If $\operatorname{cr}(C) > 1$ then $C = \mathbb{E}^n$.*

Proof. The second assertion is simple. To show the first assertion, we note that if C has two distinct supporting unit balls, then $\mathrm{cr}(C) < 1$, and refer to (ii) in Corollary 5.1.13. $\square$

The main theorem in this section is the following.

Theorem 5.1.15. *Let C and D be spindle convex sets in $\mathbb{E}^n$ with disjoint relative interiors. Then there is a unit ball $\mathbf{B}^n[q]$ such that $C \subset \mathbf{B}^n[q]$ and $D \subset \mathbb{E}^n \setminus \mathbf{B}^n(q)$ (cf. Figure 5.4).*

Furthermore; if C and D have disjoint closures and C is contained in an open unit ball, then there is a unit ball $\mathbf{B}^n[q]$ such that $C \subset \mathbf{B}^n(q)$ and $D \subset \mathbb{E}^n \setminus \mathbf{B}^n[q]$.

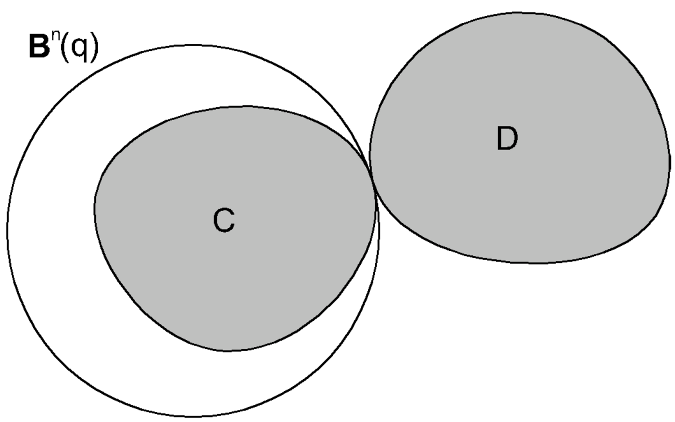

Figure 5.4: Separating by a unit sphere

Proof. Since C and D are spindle convex, they are convex, bounded sets. From this and the hypothesis, we obtain that their closures are convex, compact sets with disjoint relative interiors. Thus, there is a hyperplane H that separates them and that supports C at a point, say $x \in \operatorname{bd} C$. The closed unit ball $\mathbf{B}^n[q]$ that supports C at x satisfies the conditions of the first statement.

Next, we assume that C and D have disjoint closures. Thus, $\mathbf{B}^n[q]$ from above is disjoint from the closure of D and remains so even after a sufficiently small translation. If $C \subset \mathbf{B}^n(q)$, we are done. Let $C \not\subset \mathbf{B}^n(q)$. We note that C is contained in an open unit ball and $\operatorname{cr}(C) < 1$. Hence, there is a sufficiently small translation of $\mathbf{B}^n[q]$, in the direction of $x - q$, that satisfies the second statement. $\square$

Definition 5.1.16. Let $C, D \subset \mathbb{E}^n, q \in \mathbb{E}^n$ and $r > 0$. We say that $\mathbb{S}^{n-1}(q, r)$ *separates* C and D, if one of the sets is contained in $\mathbf{B}^n[q, r]$ and the other is contained in $\mathbb{E}^n \setminus \mathbf{B}^n(q, r)$. We say that C and D are *strictly separated* by $\mathbb{S}^{n-1}(q, r)$, if one of the sets is contained in $\mathbf{B}^n(q, r)$ and the other is contained in $\mathbb{E}^n \setminus \mathbf{B}^n[q, r]$.

5.2 A Kirchberger-type theorem for separation by spheres

Kirchberger's Theorem states the following. Let A and B be compact sets in $\mathbb{E}^n$. Then there is a hyperplane strictly separating A and B if, and only if, for any set $T \subset A \cup B$ of cardinality at most $n + 2$, there is a hyperplane strictly separating $A \cap T$ and $B \cap T$ (cf. Section 2.2). We show that this statement does not remain true if we replace hyperplanes by unit spheres, even if we also replace $n + 2$ by an arbitrarily large positive integer. To show this, we construct two sets A and B as follows.

Let $A = \{a\} \subset \mathbb{E}^n$ be a singleton, and let $b_0 \in \mathbb{E}^n$ be a point with $0 < ||a - b_0|| = \delta < 1$. Then $\mathbf{B}^n[a] \setminus \mathbf{B}^n(b_0)$ is a non-convex, closed set bounded by two closed spherical caps: an inner one K that is contained in $\mathbb{S}^{n-1}(b_0)$ and an outer one that is contained in $\mathbb{S}^{n-1}(a)$ (cf. Figure 5.5). We choose points $b_1, b_2, \ldots, b_{k-1}$ such that $\mathbf{B}^n[b_i] \cap K$ is a spherical cap of radius ε for every value of i, and

$$K \subset \bigcup_{j=1}^{k-1} \mathbf{B}^n[b_j] \quad \text{and} \quad K \not\subset \bigcup_{j=1, j\neq i}^{k-1} \mathbf{B}^n[b_j] \quad \text{for} \quad i = 1, 2, \ldots, k-1. \tag{5.1}$$

Let $B = \{b_0, b_1, \ldots, b_{k-1}\}$. From (5.1), it easily follows that

$$\mathbf{B}^n(a) \subset \bigcup_{j=0}^{k-1} \mathbf{B}^n[b_j] \quad \text{and} \quad \mathbf{B}^n(a) \not\subset \bigcup_{j=0, j\neq i}^{k-1} \mathbf{B}^n[b_j] \quad \text{for} \quad i = 0, 1, \ldots, k-1. \tag{5.2}$$

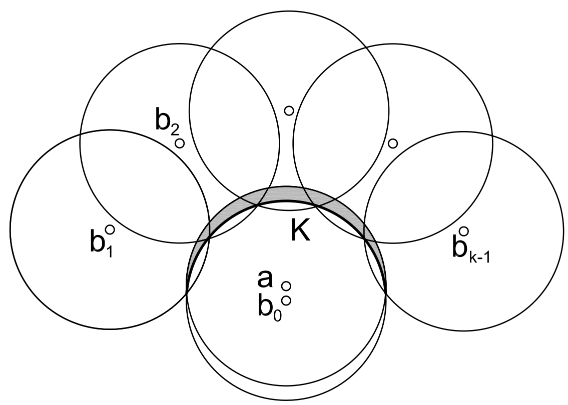

Figure 5.5: A counterexample to a Kirchberger-type theorem with separation by unit spheres

By the first part of (5.2), it is clear that there is no point $q \in \mathbb{E}^n$ such that $a \in \mathbf{B}^n(q)$ and $B \subset \mathbb{E}^n \setminus \mathbf{B}^n[q]$. On the other hand, if ε is sufficiently small, then $a \in [B]$. Hence, there is no $q \in \mathbb{E}^n$ such that $B \subset \mathbf{B}^n(q)$ and $a \notin \mathbf{B}^n[q]$. In summary, there is no unit sphere that strictly separates A and B.

However, by the second part of (5.2), for any $T \subset A \cup B$ of cardinality at most k, there is a point $q_T \in \mathbb{E}^n$ such that $T \cap A \subset \mathbf{B}^n(q_T)$ and $T \cap B \subset \mathbb{E}^n \setminus \mathbf{B}^n[q_T]$.

In Theorem 5.2.4, we provide a weaker analogue of Kirchberger's theorem using a special case of Theorem 3.4 of Houle [33], and Lemma 5.2.2.

Theorem 5.2.1 (Houle). *Let A and B be finite sets in $\mathbb{E}^n$. There is a sphere $\mathbb{S}^{n-1}(q,r)$ strictly separating A and B such that $A \subset \mathbf{B}^n(q,r)$ if, and only if, for every $T \subset A \cup B$ with $\operatorname{card} T \leq n+2$, there is a sphere $\mathbb{S}^{n-1}(q_T, r_T)$ strictly separating $T \cap A$ and $T \cap B$ such that $T \cap A \subset \mathbf{B}^n(q_T, r_T)$.*

Lemma 5.2.2. *Let A and B be finite sets in $\mathbb{E}^n$, and suppose that $\mathbb{S}^{n-1}(o)$ is the smallest sphere that separates A and B such that $A \subset \mathbf{B}^n[o]$. Then there is a set $T \subset A \cup B$ with $\operatorname{card} T \leq n+1$ such that $\mathbb{S}^{n-1}(o)$ is the smallest sphere $\mathbb{S}^{n-1}(q,r)$ that separates $T \cap A$ and $T \cap B$ and satisfies $T \cap A \subset \mathbf{B}^n[q,r]$.*

Proof. First, observe that $A \neq \emptyset$. Assume that $\mathbb{S}^{n-1}(o)$ separates A and B such that $A \subset \mathbf{B}^n[o]$. Note that $\mathbb{S}^{n-1}(o)$ is the smallest sphere separating A and B such that $A \subset \mathbf{B}^n[o]$ if, and only if, there is no closed spherical cap of radius less than $\pi/2$ that contains $A \cap \mathbb{S}^{n-1}(o)$ and whose interior with respect to $\mathbb{S}^{n-1}(o)$ is disjoint from $B \cap \mathbb{S}^{n-1}(o)$. Indeed, if there is a sphere $\mathbb{S}^{n-1}(q,r)$, of radius $r<1$, that separates A and B such that $A \subset \mathbf{B}^n[q,r]$, then we may choose $\mathbb{S}^{n-1}(o) \cap \mathbf{B}^n[q,r]$ as such a spherical cap; a contradiction. On the other hand, if K is such a closed spherical cap then, by the finiteness of A and B, we may transform $\mathbb{S}^{n-1}(o)$ into a sphere $\mathbb{S}^{n-1}(q,r)$, of radius $r<1$, that separates A and B such that $\mathbf{B}^n[q,r] \cap \mathbb{S}^{n-1}(o) = K$; a contradiction. In particular, we may assume that $A \cup B \subset \mathbb{S}^{n-1}(o)$.

Consider a point $y \in \mathbf{B}^n[o] \setminus \{o\}$. Observe that the closed half space $\{w \in \mathbb{E}^n : \langle w, y \rangle \geq \|y\|^2\}$ intersects $\mathbb{S}^{n-1}(o)$ in a closed spherical cap of radius less than $\pi/2$. Let us denote this spherical cap and its interior with respect to $\mathbb{S}^{n-1}(o)$ by K_y and L_y, respectively. Note that we have defined a one-to-one correspondence between $\mathbf{B}^n[o] \setminus \{o\}$ and the family of closed spherical caps of $\mathbb{S}^{n-1}(o)$ with radius less than

$\pi/2$.

Let us consider a point $x \in \mathbb{S}^{n-1}(o)$. Observe that $x \in K_y$ if, and only if, the straight line passing through x and y intersects $\mathbf{B}^n[o]$ in a segment of length at least $2\|x-y\|$ (cf. Figure 5.6).

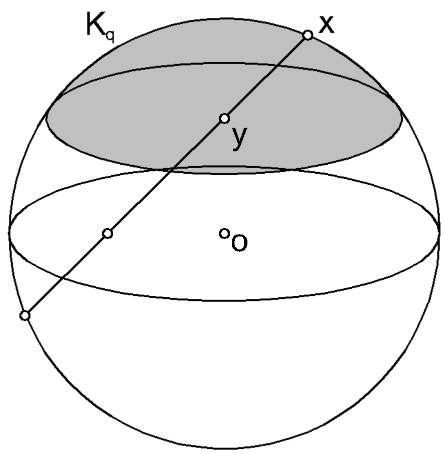

Figure 5.6: An illustration for the proof of Lemma 5.2.2

Let

$$\begin{aligned} F_x &= \{y \in \mathbf{B}^n[o] \setminus \{o\} : x \in K_y\} \text{ and} \\ G_x &= \{y \in \mathbf{B}^n[o] \setminus \{o\} : x \notin L_y\}. \end{aligned} \tag{5.3}$$

It is easy to see that

$$F_x = \mathbf{B}^n[x/2, 1/2] \setminus \{o\} \text{ and } G_x = \mathbf{B}^n[o] \setminus \Big(\mathbf{B}^n(x/2, 1/2) \cup \{o\}\Big). \tag{5.4}$$

By the first paragraph of this proof, $\mathbb{S}^{n-1}(o)$ is the smallest sphere separating A and B such that $A \subset \mathbf{B}^n[o]$ if, and only if, $\left(\bigcap_{a\in A} F_a\right) \cap \left(\bigcap_{b\in B} G_b\right) = \emptyset$.

Let f denote the inversion with respect to $\mathbb{S}^{n-1}(o)$ and, for any $z \in \mathbb{S}^{n-1}(o)$, let $H^+(z)$ (respectively, $H^-(z)$) denote the closed half space $\{w \in \mathbb{E}^n : \langle w, z\rangle \le 1\}$ (respectively, $\{w \in \mathbb{E}^n : \langle w, z\rangle \ge 1\}$). Clearly, $f(F_z) = H^-(z)$ and $f(G_z) = H^+(z) \setminus \mathbf{B}^n(o)$. Hence, $\mathbb{S}^{n-1}(o)$ is the smallest sphere separating A and B such that $A \subset \mathbf{B}^n[o]$ if, and only if,

$$I = \Big(\bigcap_{a\in A} H^-(a)\Big) \cap \Big(\bigcap_{b\in B} \big(H^+(b) \setminus \mathbf{B}^n(o)\big)\Big) \tag{5.5}$$

is empty. Note that $\mathbf{B}^n(o) \cap H^-(a) = \emptyset$ for any $a \in A$. Since $A \neq \emptyset$, we have

$$I = \Big(\bigcap_{a\in A} H^-(a)\Big) \cap \Big(\bigcap_{b\in B} H^+(b)\Big). \tag{5.6}$$

As $H^-(z)$ and $H^+(z)$ are convex for any $z \in \mathbb{S}^{n-1}(o)$, Helly's theorem yields our statement. □

Remark 5.2.3. There are compact sets A and B in $\mathbb{E}^n$ with the following property: The smallest sphere separating A and B such that $A \subset \mathbf{B}^n[o]$ is $\mathbb{S}^{n-1}(o)$ and, for any finite $T \subset A \cup B$, there is a sphere $\mathbb{S}^{n-1}(q_T, r_T)$, with $r_T < 1$, that separates $T \cap A$ and $T \cap B$ such that $T \cap A \subset \mathbf{B}^n[q_T, r_T]$.

We show the following 3-dimensional example. Let us consider a circle $\mathbb{S}^1(x, r) \subset \mathbb{S}^2(o)$ with $r < 1$ and a set $A_0 \subset \mathbb{S}^1(x, r)$ that is the vertex set of a regular triangle. Let B be the image of A_0 under the reflection about x. Clearly, $\mathbb{S}^1(x, r)$ is the only circle in $\operatorname{aff} \mathbb{S}^1(x, r)$ that separates A_0 and B. Hence, every 2-sphere, that separates A_0 and B, contains $\mathbb{S}^1(x, r)$. Consider a point $a \in A_0$, another point $y \in (o, a)$, and let $A = A_0 \cup \mathbf{B}^3(y, \|a - y\|)$ (cf. Figure 5.7). Note that the smallest sphere that separates A and B, and contains A in its convex hull, is $\mathbb{S}^2(o)$. It is easy to show that for any finite set $T \subset A$, there is a sphere $\mathbb{S}^2(q_T, r_T)$, with radius $r_T < 1$, that separates T and B such that $T \subset \mathbf{B}^3[q_T, r_T]$.

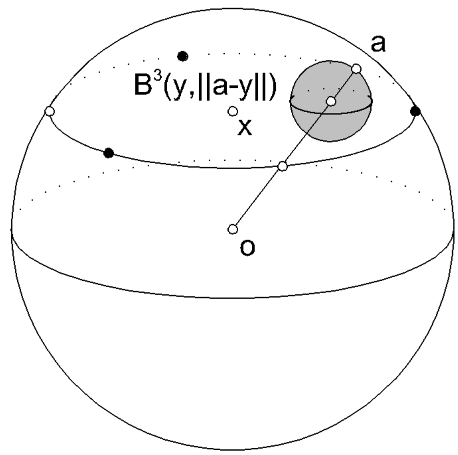

Figure 5.7: An illustration for Remark 5.2.3

Theorem 5.2.4. *Let A and B be finite sets in $\mathbb{E}^n$. Then there is a sphere $\mathbb{S}^{n-1}(q, r)$, with radius $r \leq 1$, that strictly separates A and B such that $A \subset \mathbf{B}^n(q, r)$ if, and only if, the following holds: For every $T \subset A \cup B$ with $\operatorname{card} T \leq n+2$, there is a sphere $\mathbb{S}^{n-1}(q_T, r_T)$, with $r_T \leq 1$, that strictly separates $T \cap A$ and $T \cap B$ such that $T \cap A \subset \mathbf{B}^n(q_T, r_T)$.*

Proof. We prove the "if" part of the theorem, and note that the "only if" direction is trivial. Theorem 5.2.1 guarantees the existence of a sphere $\mathbb{S}^{n-1}(q^*, r^*)$ that strictly separates A and B such that $A \subset \mathbf{B}^n[q^*, r^*]$. Hence, there is a (unique) smallest sphere $\mathbb{S}^{n-1}(q', r')$ separating A and B such that $A \subset \mathbf{B}^n[q', r']$.

By Lemma 5.2.2, there is a set $T \subset A \cup B$ with $\operatorname{card} T \leq n+1$ such that $\mathbb{S}^{n-1}(q', r')$ is the smallest sphere that separates $T \cap A$ and $T \cap B$ and whose convex hull contains

$T \cap A$. By the assumption, we have $r' < r_T \leq 1$. Since $r' < 1$, there is a sphere $\mathbb{S}^{n-1}(q, r)$ with $r \leq 1$ such that $\mathbf{B}^n[q', r'] \cap \mathbf{B}^n(q^*, r^*) \subset \mathbf{B}^n(q, r) \subset \mathbb{E}^n \setminus \left(\mathbf{B}^n(q', r') \cup \mathbf{B}^n[q^*, r^*]\right)$. This sphere clearly satisfies the conditions of Theorem 5.2.4. $\square$

5.3 Spindle convex variants of the Theorems of Carathéodory and Steinitz

Carathéodory's Theorem states that the convex hull of a set $X \subset \mathbb{E}^n$ is the union of simplices with vertices in X. Steinitz's Theorem states that if a point is in the interior of the convex hull of a set $X \subset \mathbb{E}^n$, then it is also in the interior of the convex hull of at most $2n$ points of X. This number $2n$ is minimum, as shown by the vertices of the cross-polytope and its centroid (cf. Section 2.2 and Figure 5.8). Our goal is to find spindle convex analogues of these theorems. Our main result is stated in Theorem 5.3.4.

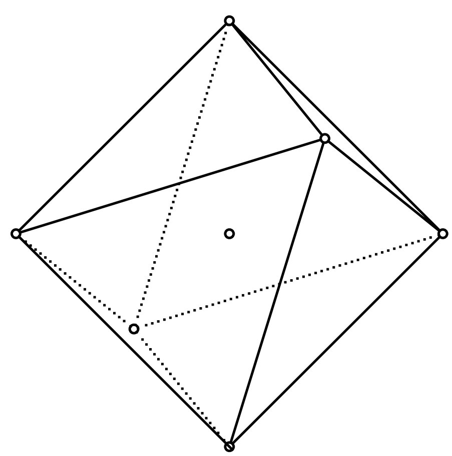

Figure 5.8: A cross-polytope with its centroid

Remark 5.3.1. Carathéodory's Theorem for the sphere states that if $X \subset \mathbb{S}^m(q,r)$ is a set in an open hemisphere of $\mathbb{S}^m(q,r)$, then every point $p \in X$ is in the spherical convex hull of at most $m+1$ points of X. The proof of this spherical equivalent uses the central projection of the open hemisphere to a hyperplane tangent to $\mathbb{S}^m(q,r)$.

Remark 5.3.2. It follows from Definition 5.1.1 that if C is a spindle convex set in $\mathbb{E}^n$ such that $C \subset \mathbf{B}^n[q]$ and $\operatorname{cr}(C) < 1$, then $C \cap \mathbb{S}^{n-1}(q)$ is spherically convex on $\mathbb{S}^{n-1}(q)$.

Lemma 5.3.3. *Let $X \subset \mathbb{E}^n$ be a closed set such that $\operatorname{cr}(X) < 1$, and let $\mathbf{B}^n[q]$ be a closed unit ball containing X. Then*

(i) $X \cap \mathbb{S}^{n-1}(q)$ *is contained in an open hemisphere of* $\mathbb{S}^{n-1}(q)$, *and*

(ii) $\operatorname{conv_s} X \cap \mathbb{S}^{n-1}(q) = \operatorname{Sconv}\left(X \cap \mathbb{S}^{n-1}(q), \mathbb{S}^{n-1}(q)\right)$ *(cf. Figure 5.9).*

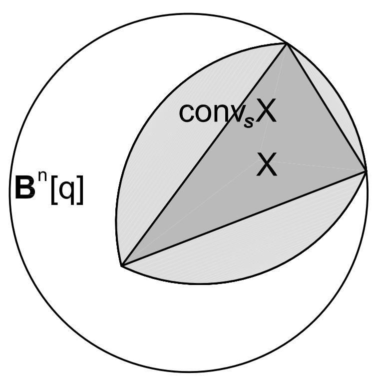

Figure 5.9: An illustration for Lemma 5.3.3

Proof. Since $\operatorname{cr}(X) < 1$, X is contained in the intersection of two distinct, closed unit balls. From this, we obtain (i) and also that $Z = \operatorname{Sconv}\left(X \cap \mathbb{S}^{n-1}(q), \mathbb{S}^{n-1}(q)\right)$ exists. Note that Remark 5.3.2 yields $Z \subset Y = \operatorname{conv_s} X \cap \mathbb{S}^{n-1}(q)$.

We suppose that there is a point $y \in Y \setminus Z$ and seek a contradiction. First, we show that there is an $(n-2)$-dimensional great-circle G of $\mathbb{S}^{n-1}(q)$ that strictly separates Z from y.

Consider an open hemisphere S of $\mathbb{S}^{n-1}(q)$ that contains Z. Since Z is compact, we may choose S in a way that y is not contained in its boundary. If y is an exterior point of S then the boundary of S strictly separates y and Z. Thus, we may assume that y is contained in S. Let f denote the central projection of S onto the hyperplane H tangent to $\mathbb{S}^{n-1}(q)$ at the spherical centre of S. Since $f(y)$ and $f(Z)$ are compact convex sets, there is an $(n-2)$-dimensional affine subspace A of H that strictly separates them in H. Thus, $G = \operatorname{aff}(A \cup \{q\}) \cap \mathbb{S}^{n-1}(q)$ strictly separates Z and y.

Let S_y (respectively, S_Z), denote the open hemisphere bounded by G that contains y (respectively, Z). Since X is compact, its distance from S_y is positive. Thus, we may move q a little towards the spherical centre of S_Z to obtain a point q' such that $X \subset B^n(q')$ and $y \notin \mathbf{B}^n[q']$. Hence, $y \notin \operatorname{conv_s} X$; a contradiction. $\square$

Theorem 5.3.4. *Let $X \subset \mathbb{E}^n$ be a closed set.*

(i) *If $p \in \operatorname{bd} \operatorname{conv_s} X$ then there is a set $Y \subset X$, with $\operatorname{card} Y \leq n$, such that $p \in \operatorname{conv_s} Y$.*

(ii) *If $p \in \operatorname{int} \operatorname{conv_s} X$ then there is a set $Y \subset X$, with $\operatorname{card} Y \leq n+1$, such that $p \in \operatorname{int} \operatorname{conv_s} Y$.*

Proof. Assume that $\operatorname{cr}(X) > 1$. We note that $\mathbf{B}[X] = \emptyset$, and thus, there is a set $Y \subset X$ of cardinality at most $n+1$ such that $\mathbf{B}[Y] = \emptyset$ by Helly's Theorem. From Corollary 5.1.14, it follows that $\operatorname{conv_s} Y = \mathbb{E}^n$. This yields our assertion.

Now we prove (i) for $\mathrm{cr}(X) < 1$. By Lemma 5.1.11, there is a closed unit ball $\mathbf{B}^n[q]$ that supports $\mathrm{conv_s}\, X$ at p. Applying Lemma 5.3.3 and the spherical version of Carathéodory's Theorem for $\mathbb{S}^{n-1}(q)$ (see Remark 5.3.1), we obtain that there is a set $Y \subset X$ of cardinality at most n such that $p \in \mathrm{conv_s}\, Y$.

We prove (i) for $\mathrm{cr}(X) = 1$ by a limit argument. Without loss of generality, we may assume that $X \subset \mathbf{B}^n[o]$. For any positive integer k, let $X_k = (1 - \frac{1}{k})X$. Let p_k be the point of $\mathrm{bd\,conv_s}(X_k)$ closest to p. Thus, $\lim_{k\to\infty} p_k = p$ and $\mathrm{cr}(X_k) < 1$. Hence, there is a set $Y_k \subset X_k$ with $\mathrm{card}\, Y_k \leq n$ such that $p_k \in \mathrm{conv_s}\, Y_k$. By compactness, there is a sequence $0 < i_1 < i_2 < \ldots$ of indices such that $\{Y_{i_k}\}$ converges to a set Y with $\mathrm{card}\, Y \leq n$. Since X is closed, $Y \subset X$ and $p \in \mathrm{conv_s}\, Y$.

In order to prove (ii) for $\mathrm{cr}(P) \leq 1$; we suppose that $p \in \mathrm{int\,conv_s}\, X$, let $x \in X \cap \mathrm{bd\,conv_s}\, X$ be arbitrary and let y be the intersection of $\mathrm{bd\,conv_s}\, X$ with the ray emanating from p and passing through x. By (i), $y \in \mathrm{conv_s}\, Y$ for some $Y \subset X$ with $\mathrm{card}\, Y \leq n$. Clearly, $p \in \mathrm{int\,conv_s}(Y \cup \{x\})$. $\square$

Another version of Carathéodory's Theorem is the "Colorful Carathéodory Theorem" (cf. [41] p. 199). The following is the spindle convex variant.

Theorem 5.3.5. *Consider $n+1$ finite point sets $X_1, X_2, \ldots, X_{n+1}$ in $\mathbb{E}^n$ such that the spindle convex hull of each contains the origin o. Then there is a set $Y \subset X_1 \cup \cdots \cup X_{n+1}$ with $\mathrm{card}\, Y = n+1$ and $\mathrm{card}(Y \cap X_i) = 1$ for $i = 1, 2, \ldots, n+1$ such that $o \in \mathrm{conv_s}\, Y$.*

5.4 Erdős–Szekeres type theorems for points in spindle convex position

In this section, we determine analogues of Erdős-Szekeres type theorems for points in spindle convex position.

Definition 5.4.1. For any $n \geq 2$ and $k \geq n+1$, let $M_n(k)$ denote the smallest positive integer such that any set of $M_n(k)$ points, in general position in $\mathbb{E}^n$, contains k points in convex position.

As we noted in Chapter 4, $M(k) = M_2(k)$ exists for any $k \geq 3$, and the best estimates are

$$2^{k-2} + 1 \leq M(k) \leq \binom{2k-5}{k-2} + 1. \quad \text{(cf. (4.1))}$$

The existence of $M_n(k)$ and the inequality $M_{n+1}(k) \leq M_n(k)$ follow from the following observation.

Remark 5.4.2. Let $2 \leq t \leq n$. For every finite set X of points in general position in $\mathbb{E}^n$, that does not contain k points in convex position, there is a projection h onto an affine subspace of dimension t such that $h(X)$ is a set of points in general position and no k of them are in convex position.

To find a spindle convex analogue of Definition 5.4.1, we observe first that some points of $\mathbb{E}^n$ are in general position if, and only if, any $n+1$ of them are in convex position.

Definition 5.4.3. For $3 \leq n+1 \leq k$, let $\widehat{M_n}(k)$ denote the least positive integer such that if X is a set of $\widehat{M_n}(k)$ points of $\mathbb{E}^n$ with the property that

(i) X is contained in a closed unit ball, and

(ii) any $n+1$ points of X are in spindle convex position,

then k points of X are in spindle convex position.

To show why we need (ii) in Definition 5.4.3 instead of the weaker condition that the points are in general position, we provide the following example: Let $X = \{p_1, p_2, \ldots, p_k\} \subset \mathbb{E}^2$, and $p_1, p_2, \ldots, p_k$ be points on an arc of radius $r > 1$ such that $\operatorname{cr}(X) < 1$. We note that any three points of X are affinely independent and X does not contain three points in spindle convex position. There are similar examples for higher dimensions.

In the remaining part of this section, we show that $M_n(k) = \widehat{M_n}(k)$ for every value of n and k. Let us assume that $X \subset \mathbb{E}^n$ is a set of points in general position that does not contain k points in convex position. Observe that for a suitably small $\varepsilon > 0$, any $n+1$ points of εX are in spindle convex position. This implies that $M_n(k) \leq \widehat{M_n}(k)$. We obtain $M_n(k) \geq \widehat{M_n}(k)$ from the following stronger version of Theorem 5.3.4.

Theorem 5.4.4. *Let $X \subset \mathbb{E}^n$ such that $\operatorname{cr}(X) \leq 1$, and let $p \in X$. Then $p \in [X]$ or $p \in \operatorname{conv_s} Y$, for some $Y \subset X$ with $\operatorname{card} Y \leq n$.*

Proof. Note that $[X]$ is the intersection of balls, of radii at least one, that contain X. We assume that there is a ball $\mathbf{B}^n[q, r]$, with $r \geq 1$, that contains X but does not contain p, and show the existence of a set $Y \subset X$ such that $p \in \operatorname{conv_s} Y$ and $\operatorname{card} Y \leq n$.

If $p \in \operatorname{bd} \operatorname{conv_s} X$, then the theorem follows from Theorem 5.3.4. Hence, we may assume that $p \in \operatorname{int} \operatorname{conv_s} X$. By Lemma 5.1.11, we have $p \in \operatorname{int} \mathbf{B}^n[q] = \mathbf{B}^n(q)$ for every unit ball $\mathbf{B}^n[q]$ that contains X. If there is a ball $\mathbf{B}^n[q_r, r]$ that contains X but does not contain p for every $r > 1$, then Blaschke's Selection Theorem yields the existence of a unit ball $\mathbf{B}^n[q]$ such that $P \subset \mathbf{B}^n[q]$ and $p \notin \mathbf{B}^n(q)$. Thus, there is an $r > 1$ such that if $X \subset \mathbf{B}^n[q, r]$ for some $q \in \mathbb{E}^n$, then $p \in \mathbf{B}^n[q, r]$. Clearly, if $1 < r_1 < r_2$ and r_2 satisfies this property then r_1 also satisfies it. Hence, there is a maximal value R satisfying this property. Let

$$X(r) = \bigcap\{\mathbf{B}^n(q, r) : P \subset \mathbf{B}^n(q, r)\}. \tag{5.7}$$

Observe that $X(r_2) \subset X(r_1)$ for every $1 < r_1 < r_2$, and that $p \in \operatorname{bd} X(R)$. Hence, by Corollary 5.1.13 and Theorem 5.3.4 for $\frac{1}{R}X$, we obtain a set $Y \subset X$, of cardinality at most n, such that any ball of radius R that contains Y contains also p. We define $Y(r)$ similarly to $X(r)$, and note that $Y(R) \subset Y(1) = \operatorname{conv_s} Y$. The theorem now follows. $\square$

If $X \subset \mathbb{E}^n$ is a set with $\operatorname{cr}(X) \leq 1$ and $\operatorname{card} X > M_n(k)$, and any $n + 1$ points of X are in spindle convex position, then X contains k points in convex position. By Theorem 5.4.4, these k points are in spindle convex position.

We note that Theorem 5.4.4 yields the spindle convex analogues of numerous other Erdős–Szekeres type results.

Chapter 6

Ball-polyhedra

In this chapter, we investigate the properties of a spindle convex analogue of polytopes.

Definition 6.0.5. Let $X \subset \mathbb{E}^n$ be a finite, nonempty set such that $\mathrm{cr}(X) \leq 1$. The set $P = \mathbf{B}[X]$ is called a *ball-polyhedron*. For any $x \in X$, we call $\mathbf{B}^n[x]$ a *generating ball* of P and $\mathbb{S}^{n-1}(x)$ a *generating sphere* of P. If $n = 2$, we also call a ball-polyhedron a *disk-polygon*.

Definition 6.0.6. Let $X \subset \mathbb{E}^n$ be a finite, nonempty set with $\mathrm{cr}(X) \leq 1$. The set $\mathrm{conv_s}\, X$ is called a *ball-polytope*.

Note that, in the theory of convex sets, ball-polyhedra correspond to bounded polyhedral domains and ball-polytopes correspond to polytopes. Though bounded polyhedral domains and polytopes are equivalent, the same does not hold for ball-polyhedra and ball-polytopes of dimensions greater than 2. Indeed, the boundary of the spindle convex hull of a finite point set X is smooth everywhere except at the points of X. Hence, no n-dimensional ball-polyhedron, $n > 2$, is the spindle convex hull of finitely many points (cf. Corollary 6.1.14).

A thorough investigation of ball-polytopes seems to require a more analytic approach. Our main goal is to describe ball-polyhedra.

6.1 The Euler-Poincaré Formula for standard ball-polyhedra

Definition 6.1.1. A *partially ordered set* or *poset* is a set S with a binary relation $\leq$ such that

1. $a \leq a$ for every $a \in S$,
2. if $a \leq b$ and $b \leq a$ then $a = b$, and
3. if $a \leq b$ and $b \leq c$ then $a \leq c$.

In a poset S, the *supremum* of the elements a and b is c, if $a \leq c$ and $b \leq c$, and $a \leq c'$ and $b \leq c'$ imply $c \leq c'$. The *infimum* of a and b is defined similarly. A poset in which every pair has a supremum and an infimum is called a *lattice*. An element of a lattice S which is less than or equal to every element of S is called the *bottom* of S. An element of S which is greater than or equal to every element of S is called the *top* of S. A lattice containing a bottom and a top is a *bounded lattice*. If there is a bottom 0 in S, then an element $a \in S$ is an *atom*, if $a \neq 0$, and $b \leq a$ implies $b = 0$ or $b = a$. If, for every $b \in S$, there is an atom $a \in S$ such that $a \leq b$, then S is called *atomic*.

It is a well-known fact that the faces of a polytope (together with the empty set and the polytope itself), ordered by containment, form a bounded atomic lattice with the vertices as atoms. This lattice is the *face-lattice* of the polytope.

Definition 6.1.2. Let X be a Hausdorff topological space, and let $\mathfrak{F} = \{F_i : i \in I\}$ be a family of finitely many subsets of X. Assume for $F_i \in \mathfrak{F}$ that there is a continuous surjective function $\sigma_i : \mathbf{B}^m[o] \to F_i$, for some $m \geq 0$, whose restriction to $\mathbf{B}^m(o)$ is a homeomorphism, and call $\sigma_i(\mathbf{B}^m(o))$ and m the *relative interior* and the *dimension* of F_i, respectively. Let $\mathfrak{F}_m$ denote the union of the elements of $\mathfrak{F}$ of dimension at most m. If the relative interiors of any two distinct elements of $\mathfrak{F}$ are disjoint, and $\sigma_i(\mathbb{S}^{m-1}(o)) \subset \mathfrak{F}_{m-1}$ for every element F_i of $\mathfrak{F}$ of positive dimension m, we say that $\mathfrak{F}$ is a *finite CW-complex*. The set $\mathfrak{F}_m$ is the *m-skeleton* of $\mathfrak{F}$. An element of $\mathfrak{F}$ is a *closed cell*, and the relative interior of a closed cell is an *open cell*. The *dimension* of $\mathfrak{F}$ is the maximum of the dimensions of the cells of $\mathfrak{F}$. The *Euler characteristic* of a finite CW-complex is defined as the number of even dimensional cells minus the number of odd dimensional cells.

It is known that the Euler characteristics of two finite CW-decompositions of the same topological space are equal. Hence, we may talk about the Euler characteristic of the space. It is also known that the Euler characteristic of a closed ball of any dimension is one. Since a convex polytope is homeomorphic to a closed ball and its faces (together with the polytope itself) form a CW-complex, we obtain the Euler-Poincaré formula for polytopes (cf. Section 2.4). For a general description of CW-complexes, the reader is referred to [31].

The main goal of this section is to prove a variant of the Euler-Poincaré formula for a certain class of ball-polyhedra. We present first an example to show that the description of the face-lattice of an arbitrary ball-polyhedron is a difficult task.

Example 6.1.3. Consider two unit spheres $\mathbb{S}^3(p)$ and $\mathbb{S}^3(-p)$ in $\mathbb{E}^4$ with $||p|| < 1$. Note that $\mathbb{S}^2(o,r) = \mathbb{S}^3(p) \cap \mathbb{S}^3(-p)$ for some $r < 1$. Let $\mathbf{B}^4[q] \subset \mathbb{E}^4$ be a closed unit ball that intersects $\mathbb{S}^2(o,r)$ in a spherical cap, greater than a hemisphere of $\mathbb{S}^2(o,r)$ and distinct from $\mathbb{S}^2(o,r)$. Observe that $F = \mathbb{S}^2(o,r) \cap \mathbf{B}^4[q] \cap \mathbf{B}^4[-q]$ is homeomorphic to a 2-dimensional band (cf. Figure 6.1).

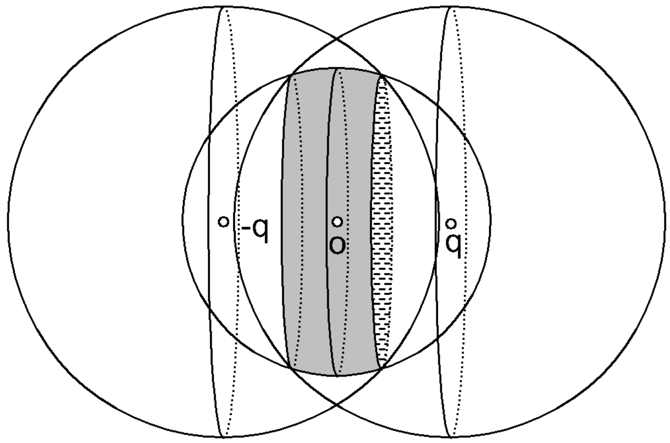

Figure 6.1: $F = \mathbb{S}^2(o,r) \cap \mathbf{B}^4[-q] \cap \mathbf{B}^4[q]$

Our common sense says that F "deserves" the name of a 2-face of the ball-polyhedron $P = \mathbf{B}^4[p] \cap \mathbf{B}^4[-p] \cap \mathbf{B}^4[q] \cap \mathbf{B}^4[-q]$. Hence, Example 6.1.3 demonstrates that even a satisfactory definition for the face-lattice of a ball-polyhedron, one that models the face-lattice of a convex polytope, does not lead to a CW-decomposition of the boundary of ball-polyhedra.

Now we generalize the definition of great-sphere (cf. Section 3.4) for arbitrary spheres.

Definition 6.1.4. Let $\mathbb{S}^m(q,r)$ be a sphere of $\mathbb{E}^n$. The intersection of $\mathbb{S}^m(q,r)$ with an affine subspace of $\mathbb{E}^n$ that passes through q is called a *great-sphere* of $\mathbb{S}^m(q,r)$. In particular, $\mathbb{S}^m(q,r)$ is a great-sphere of itself.

Definition 6.1.5. Let $P = \mathbf{B}[X] \subset \mathbb{E}^n$ be a ball-polyhedron. The family $\{\mathbf{B}^n[x] : x \in X\}$ is called *reduced*, if $\mathbf{B}[X] \neq \mathbf{B}[X \setminus \{x\}]$ for any $x \in X$ (cf. Figure 6.2).

If P contains more than one point, then it has a unique reduced family. If P is a singleton, say $P = \{p\}$, then there are infinitely many reduced families generating P; for instance, $P = \mathbf{B}^n[p - u] \cap \mathbf{B}^n[p + u]$ for any $u \in \mathbb{E}^n$ with $\|u\| = 1$.

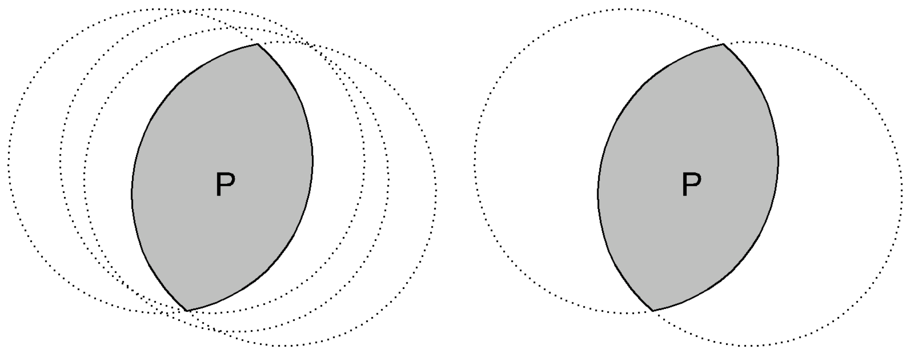

Figure 6.2: Not reduced and reduced families of generating balls

Definition 6.1.6. Let $P \subset \mathbb{E}^n$ be a ball-polyhedron containing more than one point, and let $\mathbb{S}^m(q, r)$, where $0 \leq m \leq n - 1$, be a sphere such that $P \cap \mathbb{S}^m(q, r) \neq \emptyset$. If $\mathbb{S}^m(q, r)$ is the intersection of some of the generating spheres of P from the reduced family, then we say that $\mathbb{S}^m(q, r)$ is a *supporting sphere* of P.

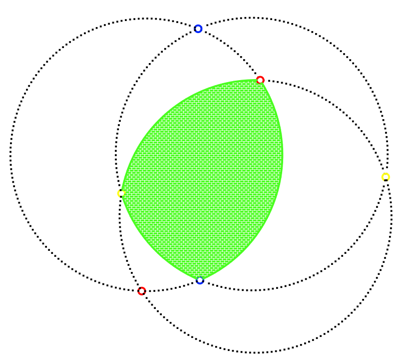

Figure 6.3: Supporting spheres of a disk-polygon

Note that the intersection of finitely many spheres in $\mathbb{E}^n$ is either empty, or a sphere, or a point. In particular, if $n = 2$, then the supporting spheres of P are the generating spheres, (which are unit circles) and the intersections of two generating spheres (which are pairs of distinct points). If $n = 2$, no point belongs to more than two generating spheres in the reduced family (cf. Figure 6.3).

In the same way as the faces of a convex polytope are described in terms of supporting hyperplanes, we describe the faces of a certain class of ball-polyhedra in terms of supporting spheres.

Definition 6.1.7. An n-dimensional ball-polyhedron $P \subset \mathbb{E}^n$, containing more than one point, is *standard* if, for any supporting sphere $\mathbb{S}^l(p, r)$ of P, the intersection $F = P \cap \mathbb{S}^l(p, r)$ is homeomorphic to a closed Euclidean ball of some dimension. We call F a *face* of P. The *dimension* of F, denoted by $\dim F$, is the dimension of the ball that is homeomorphic to F. If $\dim F = 0$, $\dim F = 1$ or $\dim F = n - 1$, then we say that F is a *vertex*, an *edge* or a *facet*, respectively. We regard P as a face of itself.

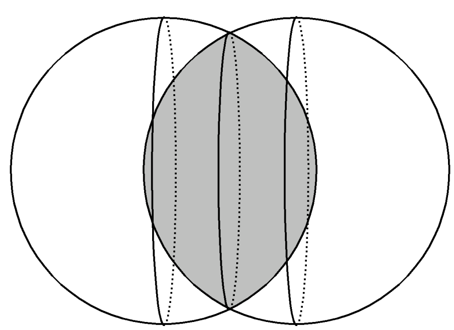

Figure 6.4: A non-standard ball-polyhedron

Note that the dimension of F is independent of the choice of the supporting sphere containing F.

A disk-polygon is standard if, and only if, its reduced family of generating circles has at least three elements. Furthermore, if $n > 2$, and $X \subset \mathbb{E}^n$ is the vertex set of an n-dimensional regular polyhedron with $\operatorname{cr}(X) < 1$, then $\mathbf{B}[X]$ is a standard ball-polyhedron. For non-standard ball-polyhedra, we give the example of the intersection of two unit balls (cf. Figure 6.4), and Example 6.1.3.

In Section 6.3, we present reasons why standard ball-polyhedra are natural, relevant objects of study in $\mathbb{E}^3$.

Definition 6.1.8. Let C be a convex body in $\mathbb{E}^n$ and $p \in \operatorname{bd} C$. Then the *Gauss image* of p with respect to C is the set of outward unit normal vectors of hyperplanes that support C at p.

Note that the Gauss image of a point is a spherically convex subset of $\mathbb{S}^{n-1}(o)$.

Theorem 6.1.9. *The faces of a standard ball-polyhedron P are the closed cells of a finite CW-decomposition of* $\operatorname{bd} P$.

Proof. Let $\{\mathbb{S}^{n-1}(p_1), \ldots, \mathbb{S}^{n-1}(p_k)\}$ be the reduced family of generating spheres of P. We define the *relative interior* (resp., the *relative boundary*) of an m-dimensional face F of P as the set consisting of the points of F that are mapped to $\mathbf{B}^m(o)$ (resp., $\mathbb{S}^{m-1}(o)$) under any homeomorphism between F and $\mathbf{B}^m[o]$.

Let $p \in \operatorname{bd} P$, and consider the sphere

$$\mathbb{S}^t(q, r) = \bigcap \{\mathbb{S}^{n-1}(p_i) : p_i \in \mathbb{S}^{n-1}(p), i \in \{1, \ldots, k\}\}. \tag{6.1}$$

Clearly, $\mathbb{S}^t(q,r)$ is a supporting sphere of P. Moreover, as p has a t-dimensional neighbourhood in $\mathbb{S}^t(q,r)$, contained in the face $F = \mathbb{S}^t(q,r) \cap P$, F is t-dimensional. This shows that p belongs to the relative interior of F. Hence, the relative interiors of the faces of P cover $\operatorname{bd} P$.

Assume that p is in the relative interior of a face F' of P. Clearly, $F \subset F'$ by the definitions of F and $\mathbb{S}^t(q,r)$. Note that Gauss image of p with respect to $\bigcap\{\mathbf{B}^n[p_i] : p_i \in \mathbb{S}^{n-1}(b), i \in \{1,\ldots,k\}\} \supseteq P$ is $(n-m-1)$-dimensional, which implies that the Gauss-image of p with respect to P is at least $(n-m-1)$-dimensional. From this, it follows that the dimension of F' is at most m, which yields $F' = F$.

A similar consideration shows that if p is in the relative boundary of a face F' of P, then it is in the relative interior of a smaller dimensional face. This concludes the proof. □

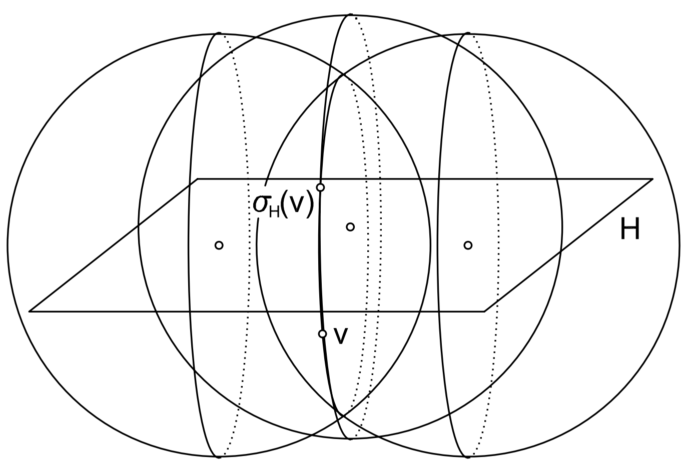

Figure 6.5: An illustration for the proof of Corollary 6.1.10

Corollary 6.1.10. *The reduced family of the generating balls of a standard ball-polyhedron P in $\mathbb{E}^n$ consists of at least $n+1$ unit balls.*

Proof. Since the faces of P form a CW-decomposition of $\operatorname{bd} P$, P has a vertex v. Note that at least n generating spheres of the reduced family contain v. We denote the centres of n such spheres by $q_1, q_2, \ldots, q_n$. Let $H = \operatorname{aff}\{q_1, q_2, \ldots, q_n\}$. Then $\mathbf{B}[\{q_1, q_2, \ldots, q_n\}]$ is symmetric about H and, since v is a vertex of P, $v \notin H$. Let σ_H be the reflection of $\mathbb{E}^n$ about H (cf. Figure 6.5). Then $S = \bigcap_{i=1}^{n} \mathbb{S}^{n-1}(x_i)$ contains v and $\sigma_H(v)$, which implies that S contains more than one point of P. Since P is a standard ball-polyhedron, there is a unit ball $\mathbf{B}^n[q_{n+1}]$ in the reduced family of generating balls of P that does not contain S. □

Corollary 6.1.11. *Let Λ be the family consisting of the empty set and all faces of a standard ball-polyhedron $P \subset \mathbb{E}^n$. Then Λ is a finite bounded lattice with respect to ordering by inclusion. The atoms of Λ are the vertices of P and Λ is atomic.*

Proof. First, we show that the intersection of two faces F_1 and F_2 is a face (or the empty set). The intersection of the two supporting spheres that intersect P in F_1 and F_2 is another supporting sphere $\mathbb{S}^m(q, r)$ of P. Then $\mathbb{S}^m(q, r) \cap P = F_1 \cap F_2$ is a face of P. From this, the existence of the infimum of F_1 and F_2 follows.

Next, by the finiteness of Λ, the existence of infimum yields the existence of supremum for any two elements F_1 and F_2 of Λ. The supremum of F_1 and F_2 is the infimum of all the (finitely many) elements of Λ that are above F_1 and F_2.

Vertices of P are clearly atoms of Λ. Using Theorem 6.1.9 and induction on the dimension of the face, it is easy to show that every face is the supremum of its vertices. □

Corollary 6.1.12. *A standard ball-polyhedron P in $\mathbb{E}^n$ has m-dimensional faces for every $0 \leq m \leq n-1$.*

Proof. We use an inductive argument on m, where we go from $m = n-1$ to $m = 0$. Clearly, P has facets. An m-face F of P is homeomorphic to $\mathbf{B}^m[o]$. Hence, if $m > 0$, the relative boundary of F is homeomorphic to $\mathbb{S}^{m-1}(o)$. As the $(m-1)$-skeleton of P covers the relative boundary of F, P has $(m-1)$-faces. □

Corollary 6.1.13 (Euler-Poincaré Formula). *For any standard n-dimensional ball-polyhedron P,*

$$1 + (-1)^{n+1} = \sum_{i=0}^{n-1} (-1)^i f_i(P),$$

with $f_i(P)$ denoting the number of i-dimensional faces of P.

Proof. Note that a ball-polyhedron in $\mathbb{E}^n$ is a convex body, and hence, it is homeomorphic to $\mathbf{B}^n[o]$. Thus, our statement immediately follows from Theorem 6.1.9. □

Corollary 6.1.14. *Let P be an n-dimensional standard ball-polyhedron, $n \geq 3$. Then P is the spindle convex hull of its $(n-2)$-dimensional faces. Furthermore, P is not the spindle convex hull of its $(n-3)$-dimensional faces.*

Proof. Let p be a point on the facet $F = P \cap \mathbb{S}^{n-1}(q)$ of P, and let C be a 2-dimensional great-circle of $\mathbb{S}^{n-1}(q)$ that contains p. Since F is spherically convex on $\mathbb{S}^{n-1}(q)$, $C \cap F$ is a unit circle arc of length less than π. Let $a, b \in \mathbb{S}^{n-1}(q)$ be the two endpoints of $C \cap F$. Then a and b belong to the relative boundary of F. Hence, by Theorem 6.1.9, a and b belong to an $(n-2)$-face. Clearly, $p \in \mathrm{conv}_{\mathrm{s}}\{a, b\}$. Thus, the facets of P are contained in the spindle convex hull of the $(n-2)$-dimensional faces of P, which yields our first statement.

Next, note that by Corollary 6.1.12, P has an $(n-2)$-dimensional face F. Then $F = P \cap \mathbb{S}^{n-1}(q_1) \cap \mathbb{S}^{n-1}(q_1)$, where $\mathbb{S}^{n-1}(q_1)$ and $\mathbb{S}^{n-1}(q_2)$ are generating spheres of P from the reduced family. Let p be a point in the relative interior of F. Clearly, $p \notin \mathrm{conv_s}((\mathbf{B}^n[q_1] \cap \mathbf{B}^n[q_2]) \setminus \{p\})$. On the other hand, $\mathrm{conv_s}(P \setminus \{p\}) \subset \mathrm{conv_s}((\mathbf{B}^n[q_1] \cap \mathbf{B}^n[q_2]) \setminus \{p\})$. □

6.2 Monotonicity of the inradius, the minimal width and the diameter of a ball-polyhedron under a contraction of the centres

We begin with a definition.

Definition 6.2.1. Let X and Y be finite subsets of $\mathbb{E}^n$. If there is a surjection $f : X \to Y$ such that $\mathrm{dist}(f(p), f(q)) \leq \mathrm{dist}(p, q)$ for any $p, q \in X$, then we say that Y is a *contraction* of X. Let x, x' be points of X. If for each x and x', there are continuous curves $\gamma_x : [0, 1] \to \mathbb{E}^n$ such that $\gamma_x(0) = x$, $Y = \{\gamma_x(1) : x \in X\}$ and $\mathrm{dist}(\gamma_x(t), \gamma_{x'}(t))$ is a nonincreasing function of $t \in [0, 1]$, then we say that Y is a continuous contraction of X.

The Kneser-Poulsen conjecture, one of the famous open problems of discrete geometry, states that under a contraction of the centres, the volume of the union (resp., intersection) of finitely many balls in $\mathbb{E}^n$ does not increase (resp., decrease). Recently, the conjecture has been proven in the plane by K. Bezdek and Connelly in [4], and it has been proven for continuous contractions for $n \geq 3$ by Csikós in [15]. The interested reader is referred to [5], [14] and [16] for further information.

In this section, we apply a contraction to the centres of the generating balls of a ball-polyhedron, and ask whether the inradius, the circumradius, the diameter or the minimum width of the ball-polyhedron may decrease. We denote the inradius of a convex set $C \subset \mathbb{E}^n$ by $\mathrm{ir}(C)$.

Proposition 6.2.2. *Let $X \subset \mathbb{E}^n$ be a finite point set contained in a closed unit ball and let Y be a contraction of X. Then* $\mathrm{ir}(\mathbf{B}[Y]) \geq \mathrm{ir}(\mathbf{B}[X])$.

Proof. Let c and C denote, respectively, the incentre of $B(X)$ and the circumcentre of X. Note that $X \subset \mathbf{B}^n[c, 1 - \mathrm{ir}(\mathbf{B}[X])]$ and $\mathbf{B}^n[C, 1 - \mathrm{cr}(X)] \subset \mathbf{B}[X]$. Thus, $\mathrm{ir}(\mathbf{B}[X]) + \mathrm{cr}(X) = 1$. We obtain $\mathrm{ir}(\mathbf{B}[Y]) + \mathrm{cr}(Y) = 1$ similarly. Hence, our proposition is an immediate consequence of the inequality $\mathrm{cr}(X) \geq \mathrm{cr}(Y)$. This inequality has been proven, for example, in [1]. □

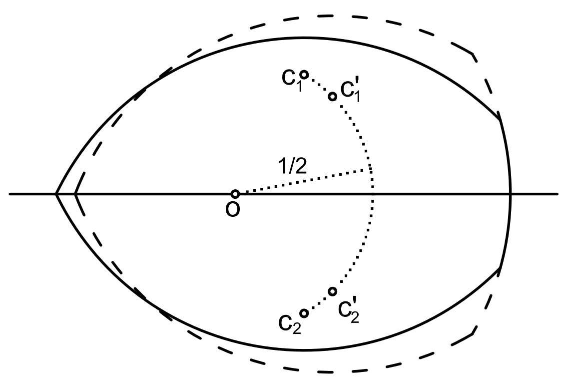

Figure 6.6: The diameter and the circumradius of a ball-polyhedron may decrease under contraction

The following construction (cf. Figure 6.6) shows that both the diameter and the circumradius of an intersection of unit disks in the plane may decrease under a contraction of the centres. We define the points with their coordinates in a Descartes coordinate system.

Let $X = \{o, c_1, c_2\}$ and $Y = \{o, c_1', c_2'\}$, where o is the origin and

$$c_1 = \left(\tfrac{1}{2}\cos\tfrac{\pi}{3}, \tfrac{1}{2}\sin\tfrac{\pi}{3}\right), \qquad c_2 = \left(\tfrac{1}{2}\cos\tfrac{\pi}{3}, -\tfrac{1}{2}\sin\tfrac{\pi}{3}\right),$$
$$c_1' = \left(\tfrac{1}{2}\cos\tfrac{\pi}{4}, \tfrac{1}{2}\sin\tfrac{\pi}{4}\right), \qquad c_2' = \left(\tfrac{1}{2}\cos\tfrac{\pi}{4}, -\tfrac{1}{2}\sin\tfrac{\pi}{4}\right).$$

The existence of the curves

$$\gamma_1(t) = \left(\tfrac{1}{2}\cos\left(\tfrac{\pi}{3} - \tfrac{\pi}{12}t\right), \tfrac{1}{2}\sin\left(\tfrac{\pi}{3} - \tfrac{\pi}{12}t\right)\right),$$
$$\gamma_2(t) = \left(\tfrac{1}{2}\cos\left(\tfrac{\pi}{3} - \tfrac{\pi}{12}t\right), -\tfrac{1}{2}\sin\left(\tfrac{\pi}{3} - \tfrac{\pi}{12}t\right)\right),$$

where $t \in [0,1]$, shows that Y is a continuous contraction of X. We show that $\mathrm{diam}(\mathbf{B}[Y]) < \mathrm{diam}(\mathbf{B}[X])$ and $\mathrm{cr}(\mathbf{B}[Y]) < \mathrm{cr}(\mathbf{B}[Y])$.

It is a well-known fact that the two lines, passing through the endpoints of a diameter D of a plane convex body and perpendicular to D, are supporting lines of the body. The only two chords of $\mathbf{B}[X]$ (respectively, $\mathbf{B}[Y]$) satisfying this property are the intersections of $\mathbf{B}[X]$ with the x-axis and the line passing through the points c_1 and c_2 (respectively, c_1' and c_2'). Thus, it is easy to see that the diameter of $\mathbf{B}[X]$ (respectively, $\mathbf{B}[Y]$) is the length of the intersection of $\mathbf{B}[X]$ (respectively, $\mathbf{B}[Y]$) with the x-axis. This yields that

$$\mathrm{diam}(\mathbf{B}[Y]) = 1 + \frac{\sqrt{7}-1}{2\sqrt{2}} < 1 + \frac{\sqrt{15}-\sqrt{3}}{4} = \mathrm{diam}(\mathbf{B}[X]).$$

We note that the circumcircle of $\mathbf{B}[X]$ coincides with the circumcircle of the convex hull of the vertices of $\mathbf{B}[X]$. Hence, it follows from a straightforward calculation that

$$\mathrm{cr}(\mathbf{B}[Y]) = 0.74645\ldots < 0.82963\ldots = \mathrm{cr}(\mathbf{B}[X]).$$

A similar example shows that the minimal width of an intersection of unit disks on the plane may decrease under a continuous contraction of the centres. We set $X = \{o, c_1, c_2\}$, and $Y = \{o, c_1', c_2'\}$, where

$$c_1 = \left(\tfrac{4}{5}\cos\tfrac{\pi}{10}, \tfrac{4}{5}\sin\tfrac{\pi}{10}\right), \qquad c_2 = \left(\tfrac{4}{5}\cos\tfrac{\pi}{10}, -\tfrac{4}{5}\sin\tfrac{\pi}{10}\right),$$

$$c_1' = c_2' = \left(\tfrac{4}{5}, 0\right).$$

Now

$$\gamma_1(t) = \left(\tfrac{4}{5}\cos\left(\tfrac{\pi}{10} - \tfrac{\pi}{10}t\right), \tfrac{4}{5}\sin\left(\tfrac{\pi}{10} - \tfrac{\pi}{10}t\right)\right),$$
$$\gamma_2(t) = \left(\tfrac{4}{5}\cos\left(\tfrac{\pi}{10} - \tfrac{\pi}{10}t\right), -\tfrac{4}{5}\sin\left(\tfrac{\pi}{10} - \tfrac{\pi}{10}t\right)\right),$$

where $t \in [0, 1]$.

By an argument similar to the one for diameter, we obtain that $\mathrm{w}(\mathbf{B}[X])$ (respectively, $\mathrm{w}(\mathbf{B}[y])$ is the length of the intersection of $\mathbf{B}[X]$ (respectively, $\mathbf{B}[Y]$) with the x-axis. Thus,

$$\mathrm{w}(\mathbf{B}[X]) = \sqrt{1 - \frac{4}{5}\sin^2\frac{\pi}{10}} - \frac{4}{5}\cos\frac{\pi}{10} + 1 = 1.200199\ldots > 1.2 = \mathrm{w}(\mathbf{B}[Y]).$$

6.3 Finding an analogue of a theorem of Steinitz for ball-polyhedra in $\mathbb{E}^3$

K. Bezdek and Naszódi [8] defined the vertices, edges and faces of a non-standard 3-dimensional ball-polyhedron in the following way. Let P be a ball-polyhedron in $\mathbb{E}^3$ with at least three generating balls in the reduced family, let $\mathbb{S}^k(x, r)$ be a supporting sphere of P, and let $F = \mathbb{S}^k(x, r) \cap P$. If the dimension of F is 2 (respectively, 0), then F is called a *face* (respectively, *vertex*) of P. If $\dim F = 1$, then the connected components of F are *edges* of P. Note that a face of P is spherically convex on $\mathbb{S}^k(x,r)$, the endpoints of an edge are vertices of P, and every vertex is adjacent to at least three edges and three faces of P.

Consider the intersection of two distinct unit balls $\mathbf{B}^3(q_1)$ and $\mathbf{B}^3(q_2)$ such that $\mathrm{dist}(q_1,q_2) < 2$. If we "cut off" little disjoint pieces of $\mathbb{S}^2(q_1) \cap \mathbb{S}^2(q_2)$ by unit balls, we may construct a 3-dimensional ball-polyhedron P with two faces that meet along a series of edges (cf. [8]). It is easy to show that the family of vertices, edges and faces of P (together with the empty set and P itself) do not form a lattice with respect to containment. The following remark shows how face structure plays a role in the standardness of 3-dimensional ball-polyhedra.

Remark 6.3.1. A ball-polyhedron P in $\mathbb{E}^3$ is standard if, and only if, the vertices, edges and faces of P (together with $\emptyset$ and P) form a lattice with respect to containment. Furthermore, the intersection of any two distinct faces of a standard ball-polyhedron $P \subset \mathbb{E}^3$ is either empty, or one vertex or one edge of P.

Before introducing the main topic of this section, we recall a few elementary notions from graph theory. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite set, called the *set of vertices* of G, and $E(G)$ is a finite multiset consisting of some undirected pairs of (not necessarily distinct) vertices of G. An element $\{p,q\}$ of $E(G)$ is an *edge* of G *with endpoints* p *and* q. Two vertices are *adjacent* if they belong to the same edge. Two edges are *adjacent* if they share a vertex.

A graph is *simple* if it contains no loop (an edge with identical endpoints) and no parallel edges (two edges with the same two endpoints). A graph G is connected if, for any two vertices $p,q \in V(G)$, there is a finite sequence of vertices

$p = x_0, x_1, x_2, \ldots, x_k = q$ such that x_{i-1} and x_i are adjacent for $i = 1, 2, \ldots, k$. Such a sequence is called a *path* from p to q. A path from p to p, with pairwise distinct edges and with $k > 1$, is called a *cycle*. A graph G is k-connected, if $\operatorname{card} V(G) \geq k$, and G remains connected after deleting any at most $k-1$ of its vertices.

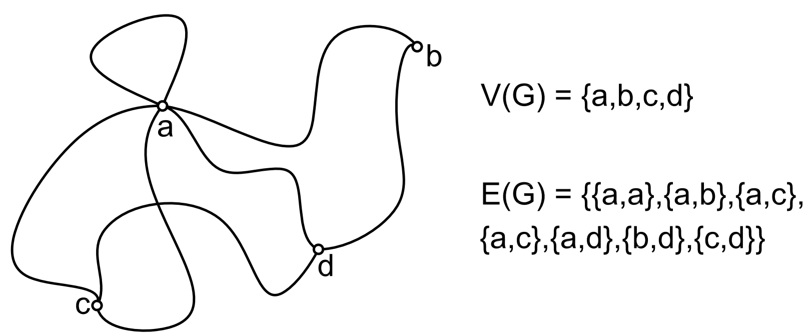

Figure 6.7: A realization of the graph G

A *realization* of a graph G is a pair of mappings $\xi_v : V(G) \to \mathbb{E}^n$ and $\xi_e : E(G) \to C^n[0,1]$, where the image of an edge E with endpoints p and q is a simple closed continuous curve in $\mathbb{E}^n$ with endpoints $\xi_v(p)$ and $\xi_v(q)$ (cf. Figure 6.7). A graph G is *planar* if it has a realization in $\mathbb{E}^2$ where the images of two distinct edges E_1 and E_2 intersect only at the images of vertices that belong to both E_1 and E_2.

The *edge-graph* G of a 3-dimensional ball-polyhedron or polytope P is defined as follows. $V(G)$ is the set of vertices of P, and $E(G)$ contains the edge $\{p, q\}$, where $p, q \in V(G)$, exactly k times if there are exactly k edges of P that connect p and q. The edge-graph of a ball-polyhedron P contains no loops, but may contain parallel edges. Moreover, it is 2-connected and planar.

A famous theorem of Steinitz (cf., for example [50], pp. 103-126) states that a graph is the edge-graph of a convex polyhedron if, and only if, it is simple, planar and

3-connected. In what follows, we investigate whether an analogue of this theorem holds for standard ball-polyhedra in $\mathbb{E}^3$.

Proposition 6.3.2. *Let $\bar{P}$ be a convex polyhedron in $\mathbb{E}^3$ such that every face of $\bar{P}$ is inscribed in a circle. Let Λ denote the face-lattice of $\bar{P}$. Then there is a sequence $\{P_1, P_2, P_3, \dots\}$ of standard ball-polyhedra in $\mathbb{E}^3$ with face-lattices isomorphic to Λ such that $\lim_{k\to\infty} kP_k = \bar{P}$ in the Hausdorff metric.*

Proof. Let $\mathcal{F}$ be the family of the (two-dimensional) faces of $\bar{P}$. For a face $F \in \mathcal{F}$, let c_F denote the circumcenter, r_F denote the circumradius, and n_F denote the inner unit normal vector of F. We define P'_k as the following intersection of closed balls of radius k:

$$P'_k = \bigcap_{F\in\mathcal{F}} \mathbf{B}\left[c_F + \left(\sqrt{k^2 - r_F^2}\right) n_F, k\right]. \tag{6.2}$$

Clearly, $P_k = \frac{1}{k}P'_k$ is a ball-polyhedron in $\mathbb{E}^3$. Note that $\frac{1}{k}p$ is a vertex of P_k for every $p \in V(\bar{P})$. Moreover, a simple approximation argument shows that if K is sufficiently large, then, for every $k \geq K$, P'_k is a standard ball-polyhedron in $\mathbb{E}^3$ with a face-lattice isomorphic to Λ. Observe that $\lim_{k\to\infty} P'_k = \bar{P}$. Now, for $k = 1, 2, \dots, K-1$, replace P_k by P_K. The sequence of ball-polyhedra obtained in this way satisfies the requirements of the proposition. $\square$

Corollary 6.3.3. *If G is an edge-graph of a convex polyhedron $\bar{P}$ in $\mathbb{E}^3$, such that every face of $\bar{P}$ is inscribed in a circle, then G is the edge-graph of a standard ball-polyhedron in $\mathbb{E}^3$.*

We note that not every 3-connected, simple, planar graph is the edge-graph of a convex polyhedron in $\mathbb{E}^3$ with all faces inscribed in a circle. (cf. [30], pp. 286-287).

Proposition 6.3.4. *The edge-graph of a standard ball-polyhedron P in $\mathbb{E}^3$ is simple, planar and 3-connected.*

Proof. Let G be the edge-graph of P. It is clearly planar, and it is easy to show that G has at least four vertices. First, we show that G contains no parallel edges, from which it follows that G is simple.

Assume that two vertices v and w are connected by at least two edges, say E_1 and E_2. From the reduced family of generating spheres of P, let Q be the intersection of those that contain E_1 or E_2. Clearly, $Q = \{v, w\}$, and this contradicts Remark 6.3.1.

We show that G is 3-connected. Let v and w be two distinct vertices of G. Consider two vertices s and t of G, both different from v and w. We need to show that there is a path from s to t that avoids v and w.

We define a graph G_v as follows. Let $V(G_v)$ be the set of vertices of P that lie on the same face as v and are distinct from v. Two vertices of G_v are connected with an edge if, and only if, there is an edge of G connecting them that lies on a face of G containing v. We define G_w similarly. By Remark 6.3.1, G_v and G_w are cycles (cf. Figure 6.8). Moreover, v and w are incident with at most two faces in common.

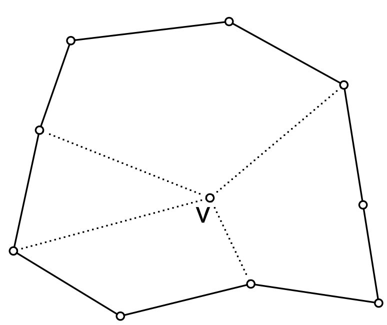

Figure 6.8: G_v

Case 1. v and w are not incident to any common face; that is, $v \notin G_w$ and $w \notin G_v$.

Since G is connected, there is a path connecting s and t. We may assume that this path does not pass through any vertex twice. Assume that this path includes v by passing through two edges, say $E_1 = \{v_1, v\}$ and $E_2 = \{v, v_2\}$. Clearly, $v_1 \neq w \neq v_2$, and v_1 and v_2 are contained in G_v, which is a cycle. Thus, the edges E_1 and E_2 in the path may be replaced by a sequence of edges of G_v that connects v_1 and v_2. If the path passes through w then it may be modified in the same manner to avoid w. Thus, we obtain the desired path.

Case 2. v and w are incident to one or two common faces.

Let $\bar{G}$ be the subgraph of $G_v \cup G_w$ spanned by $V(G_v) \cup V(G_w) \setminus \{v, w\}$. Since P is standard, $\bar{G}$ is a cycle. Similarly to the preceding argument, any path from s to t may be modified, using edges of $\bar{G}$, such that it does not pass through v and w. $\square$

6.4 Ball-polyhedra in $\mathbb{E}^3$ with symmetric sections

The following conjecture is due to K. Bezdek (cf. [29]).

Conjecture 6.4.1. *Let C be a convex body in $\mathbb{E}^3$ such that any planar section of C is axially symmetric. Then C is either a body of revolution or an ellipsoid.*

A remarkable result related to this conjecture is due to Montejano [43], who proved that if $C \subset \mathbb{E}^3$ is a convex body and $p \in \operatorname{int} C$ such that every planar section of C through p is axially symmetric, then there is a planar section of C through p which is a disk. Our main goal in this section is to show that Conjecture 6.4.1 holds for the class of ball-polyhedra with the weaker condition in Montejano's theorem.

Theorem 6.4.2. *Let P be a ball-polyhedron in $\mathbb{E}^3$, and let $p \in \operatorname{int} P$ such that any planar section of P passing through p is axially symmetric. Then P is either a point, or a unit ball or the intersection of two unit balls.*

Proof. Let $\mathbb{S}^{n-1}(q_1), \mathbb{S}^{n-1}(q_2), \dots, \mathbb{S}^{n-1}(q_k)$ denote the generating spheres of P in the reduced family. Suppose that $k \geq 3$, and let p be any point of $\operatorname{int} P$. We show that there is a plane H passing through p such that $P \cap H$ is not axially symmetric.

Since $k \geq 3$, P has an edge E. Let u_1 be a point in the relative interior of E, and let u_2 be a point in the relative interior of a face F of P that does not contain E. By slightly moving u_1 on E and u_2 on F, we may assume that the plane H, spanned by p, u_1 and u_2, does not contain any vertex of P and is neither parallel nor perpendicular to the line passing through q_i and q_j, for any $1 \leq i < j \leq k$.

Since F does not contain E, H intersects at least three edges of P. Thus, $H \cap P$ is a plane convex body in H bounded by the union of at least three circular arcs (cf. Figure 6.9). Moreover, since H is neither parallel nor perpendicular to the line passing through q_i and q_j, for any $1 \leq i < j \leq k$, it follows that the radii of these arcs are pairwise distinct. Hence, $H \cap P$ is not axially symmetric. $\square$

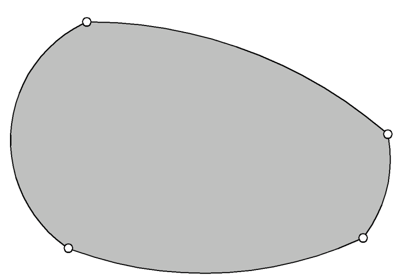

Figure 6.9: $H \cap P$

6.5 Isoperimetric inequalities for spindle convex sets

The Isoperimetric Inequality states that, among n-dimensional convex bodies of a given surface area, the Euclidean ball has maximal n-dimensional volume (cf. for example [26]).

Remark 6.5.1. Let $0 < r \leq 1$. Among the spindle convex sets of surface area equal to that of $\mathbf{B}^n[o, r]$; the spindle convex set $\mathbf{B}^n[o, r]$ has maximal n-dimensional volume.

The discrete version of the Isoperimetric Inequality states the following (cf. [24]). If $P \subset \mathbb{E}^2$ is a convex polygon with at most k vertices and with a given perimeter ℓ, then $\text{area}(P)$ is equal to or less than the area $\text{area}(P_k)$ of a regular k-gon P_k. Furthermore, $\text{area}(P) = \text{area}(P_k)$ if, and only if, P is a regular k-gon. In this section, we prove an analogue of this statement for disk-polygons.

Note that a disk-polygon is standard if, and only if, its reduced family of generating circles has at least two members.

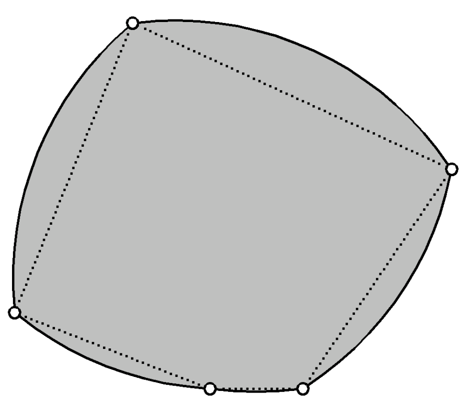

Figure 6.10: A disk-polygon and its underlying polygon

Definition 6.5.2. Let $P \subset \mathbb{E}^2$ be a standard disk-polygon with k edges, $k \geq 3$. Then P is called a *k-sided disk-polygon*. Two distinct vertices of P are *consecutive*, if they are contained in the same edge of P. Two distinct edges of P are *consecutive*, if they share a common vertex. The convex hull of the set of vertices of P is the *underlying polygon* of P (cf. Figure 6.10). The disk-polygon P is *regular* if its underlying polygon is a regular polygon.

Theorem 6.5.3. *Let $k \geq 3$ and $P \subset \mathbb{E}^2$ be a disk-polygon with at most k edges and with perimeter ℓ. Then* $\operatorname{area}(P) \leq \operatorname{area}(P_k)$*, the area of a regular k-sided disk-polygon P_k of perimeter ℓ. Furthermore,* $\operatorname{area}(P) = \operatorname{area}(P_k)$ *if, and only if, P is a regular k-sided disk-polygon of perimeter ℓ.*

Proof. Note that $\operatorname{area}(P) \leq \operatorname{area} \mathbf{B}^2[o]$ and that the limit of a sequence of disk-polygons, with perimeter ℓ and with at most k edges, is a disk-polygon with perimeter ℓ and with at most k edges. From this, it easily follows that there is a disk-polygon, with perimeter ℓ and with at most k edges, that has maximal area among such disk-polygons. Hence, it is sufficient to show that if P is not a regular k-sided disk-polygon, then its area is not maximal. Note that if we allow more than two vertices of P to be contained in the same unit circle arc, then P may be regarded as a (possibly degenerate) k-sided disk-polygon.

We show that if P is not equilateral, then its area is not maximal. Then the equality of the angles of a disk-polygon, with maximal area, follows immediately from the classical discrete isoperimetric inequality.

Let P be not equilateral. Then we may assume that P has two consecutive edges E_a and E_b of different lengths which are not contained in the same unit circle arc. Let us denote the arc lengths of E_a and E_b by α and β, respectively. We may assume that $0 < \alpha < \beta$. Let a and p (respectively, b and p) denote the endpoints of E_a (respectively, E_b). Set $d = \|a - b\|$. Let F denote the convex domain bounded by E_a, E_b and $[a, b]$ (cf. Figure 6.11).

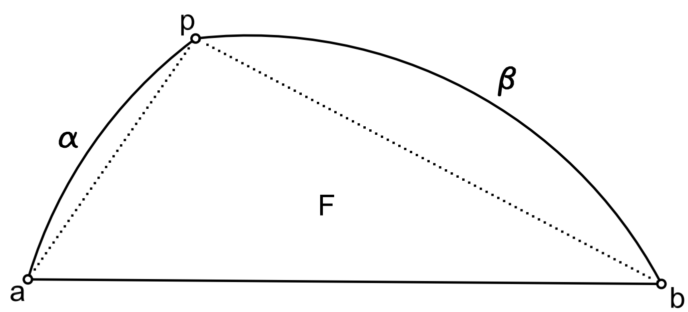

Figure 6.11: Illustration for the proof of Theorem 6.5.3

We consider $A = \mathrm{area}(F)$ as a function of α with parameters d and $\rho = \alpha + \beta$. Using Heron's formula for the area T of the triangle $[a, b, p]$, we obtain

$$A(\alpha) = \frac{\alpha - \sin\alpha}{2} + \frac{\beta - \sin\beta}{2} + T =$$

$$\frac{\alpha + \beta}{2} - \frac{\sin\alpha + \sin\beta}{2} + \sqrt{2d^2 - d^4 - \frac{(\cos\alpha - \cos\beta)^2}{4} - (\cos\alpha + \cos\beta)d^2}.$$

The derivative of A is

$$A'(\alpha) =$$

$$-\frac{\cos\alpha - \cos\beta}{2} + \frac{1}{2T}\left[\frac{(\cos\alpha - \cos\beta)(\sin\alpha + \sin\beta)}{2} + (\sin\alpha - \sin\beta)d^2\right].$$

By trigonometric identities, we obtain that

$$A'(\alpha) = -\frac{\cos\alpha - \cos\beta}{2T}\left[T - \frac{\sin\alpha + \sin\beta}{2} + d^2 \cot\frac{\rho}{2}\right].$$

Let

$$B = \frac{\sin\alpha + \sin\beta}{2} - d^2 \cot\frac{\rho}{2}.$$

As $\cos\alpha > \cos\beta$ and $T > 0$, it follows from $T - B < 0$ that $A'(\alpha) > 0$. We show that $T - B < 0$. Observe that

$$T^2 - B^2 = \frac{-2\left(d^2 - \sin^2\frac{\rho}{2}\right)^2}{1 - \cos\rho}.$$

This expression is clearly nonpositive, and it is zero only if $d = \sin\frac{\rho}{2}$; that is, if the points a, p and b are on the same unit circle arc. Thus, $T^2 - B^2 < 0$.

As T is clearly positive, the assertion follows from $B > 0$. We regard B as a function of d with parameters α and β. Obviously, $B(d)$ is continuous on $\mathbb{R}$, and $B(0) > 0$. As $B^2 \geq T^2 > 0$, B does not change sign on the interval $(0, \sin\frac{\alpha}{2} + \sin\frac{\beta}{2})$, the domain of T. Thus, $B(d) > 0$ for $d \in (0, \sin\frac{\alpha}{2} + \sin\frac{\beta}{2})$. $\square$

In Chapter 7, we prove a more general version of Theorem 6.5.3.

Chapter 7

Isoperimetric inqualities for k_g-polygons

7.1 Introduction and preliminaries

The discrete isoperimetric problem is to determine the maximal area polygon with at most k vertices and with a given perimeter. It is a classical result that the unique optimal polygon in $\mathbb{E}^2$ is the regular one. The same result for $\mathbb{H}^2$ was proven by K. Bezdek [3], and for $\mathbb{S}^2$ by L. Fejes Tóth [24]. We refer to these results as the *classical (discrete) isoperimetric inequalities*. For an overview of results on isoperimetric problems, the reader is referred to [13].

In Theorem 6.5.3, we proved that, among disk-polygons of a given perimeter and with at most k edges, a regular k-sided disk-polygon has the largest area. Now we extend our investigation to larger families of geometric figures.

In this chapter, $\mathbf{M}$ denotes any of the following three geometries of constant sectional curvature $K \in \{0, -1, 1\}$: the Euclidean plane ($K = 0$), the hyperbolic plane ($K = -1$), or the sphere ($K = 1$). If a and b are two points of $\mathbf{M}$ (if $\mathbf{M} = \mathbb{S}^2$, we assume that they are not antipodal), then $\overline{ab}$ denotes the shortest geodesic segment connecting them.

If $\gamma : [0, 1] \to \mathbb{E}^2$ is a simple closed continuous curve then, by the Jordan Curve Theorem (cf. [48]), $\mathbb{E}^2 \setminus \gamma([0, 1])$ consists of two connected components, exactly one of which is bounded. The bounded component is the *interior*, and the other one is the *exterior* of γ. This theorem and the notions of interior and exterior may easily be modified for the hyperbolic plane. In the sphere, we have two bounded components, and we define the interior (respectively, exterior) of γ as the one with smaller (respectively, larger) area if it exists.

Definition 7.1.1. Let Γ be a simple closed polygonal curve in $\mathbf{M}$ and let k_g be a non-negative constant. If $\mathbf{M} = \mathbb{S}^2$, we assume that Γ is contained in an open hemisphere. Consider the closed curve P obtained by joining consecutive vertices of Γ by curves of constant geodesic curvature k_g facing outward (resp. inward); that is, each curve lies in the closed half plane, bounded by the geodesic containing the corresponding side of Γ, on the side of the outer (resp. inner) normal vector of the side. If k_g is the geodesic curvature of a circle of radius r, we assume also that Γ has sides of length at most $2r$ and that the smooth arcs of P, connecting two consecutive vertices, are shorter than or equal to a semicircle. We call P an *outer* (resp. *inner*) *k_g-polygon*, and say that its *underlying polygon* is Γ (cf. Figure 7.1).

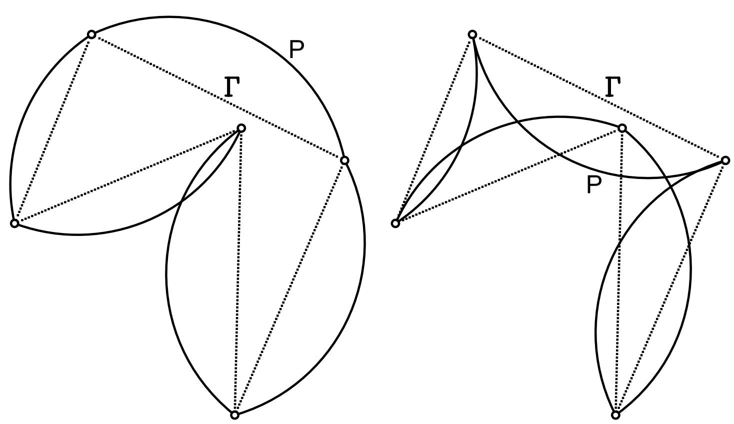

Figure 7.1: An outer and an inner k_g-polygon

We note that the boundary of a disk-polygon is an outer k_g-polygon in $\mathbb{E}^2$. As a

k_g-polygon may even be self-intersecting, its converse is false.

Definition 7.1.2. Let P be an (outer or inner) k_g-polygon. The *vertices* of P are the vertices of Γ. Two vertices of P are *consecutive* if they are consecutive vertices of Γ. Let a and b be consecutive vertices of P. The arc of P, that connects a and b and does not contain other vertices of P, is a *side* of P, and we denote it by $\widehat{ab}$. The convex region bounded by $\overline{ab}$ and $\widehat{ab}$ is an *ear* of P (cf. Figure 7.2). If Γ is a regular polygon, we say that P is *regular*. The *area* of an outer k_g-polygon P is the sum of the area of the interior of the underlying polygon Γ and the areas of the ears. Similarly, the area of an inner k_g-polygon P is the area of the interior of Γ minus the sum of the areas of the ears. The *perimeter* $\mathrm{perim}(P)$ of P is the arc length of P. A (k_g, ℓ)*-polygon* is a k_g-polygon with perimeter ℓ.

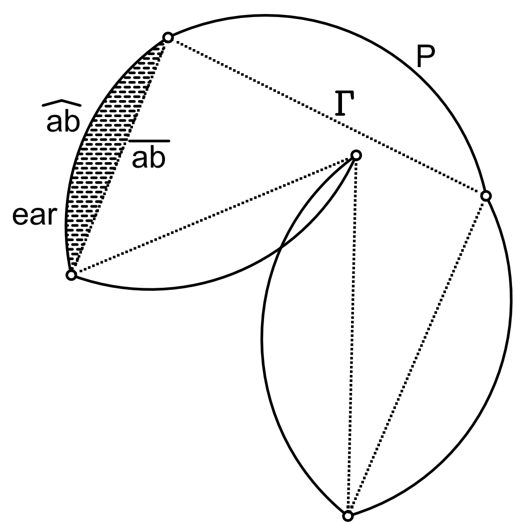

Figure 7.2: An ear of an outer k_g-polygon

Note that if a region is covered by more than one ear, its area is counted with multiplicity.

Remark 7.1.3. By Theorem 3.5.1, curves of constant geodesic curvature k_g in $\mathbf{M} = \mathbb{E}^2$ are straight line segments if $k_g = 0$, and Euclidean circle arcs of radius $r = \frac{1}{k_g}$ if $k_g > 0$. In $\mathbf{M} = \mathbb{S}^2$, these curves are circle arcs of spherical radius r with $k_g = \cot r$. In $\mathbf{M} = \mathbb{H}^2$, they are hyperbolic straight line segments if $k_g = 0$, hypercycle arcs with distance r from a line such that $k_g = \tanh r$ if $0 < k_g < 1$, horocycle arcs if $k_g = 1$ and circle arcs of hyperbolic radius r with $k_g = \coth r$ if $1 < k_g$.

The discrete isoperimetric inequality for the family of (k_g, ℓ)-polygons with at most k vertices makes sense only if the parameters k_g, ℓ and k satisfy the following restrictions, which we assume throughout Chapter 7.

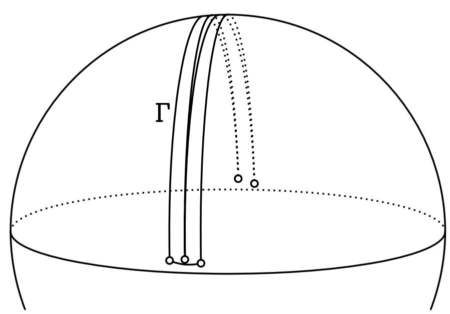

Figure 7.3: A (k_g, ℓ)-polygon whose class does not contain a regular (k_g, ℓ)-polygon

Remark 7.1.4. We assume that if k_g is the geodesic curvature of a circle of radius r, then

(7.1.4.1) ℓ is not greater than k times the length of a semicircle of radius r;

(7.1.4.2) if $\mathbf{M} = \mathbb{S}^2$ and $rk \geq \pi$, then ℓ is less than the perimeter of the k_g-polygon with the regular k-gon inscribed in a great-circle of $\mathbb{S}^2$ as its underlying polygon.

The assumption in (7.1.4.1) implies that the family of (k_g, ℓ)-polygons with at most k vertices is not empty. Furthermore, if $\mathbf{M} \neq \mathbb{S}^2$ or $rk < \pi$, then it yields the existence of a regular (k_g, ℓ)-polygon with k vertices. If $\mathbf{M} = \mathbb{S}^2$ and $rk \geq \pi$, then the existence of this (k_g, ℓ)-polygon follows from (7.1.4.2) (cf. Figure 7.3).

Definition 7.1.5. An outer (resp. inner) (k_g, ℓ)-polygon with at most k vertices is *optimal*, if its area is at least the areas of outer (resp. inner) (k_g, ℓ)-polygons with at most k vertices.

Theorem 7.1.6. *Let $\ell > 0$, $k_g \geq 0$ and $k \in \mathbb{Z}_+$ satisfy the conditions in Remark 7.1.4. Then the optimal inner (k_g, ℓ)-polygons in $\mathbf{M}$ are the regular ones.*

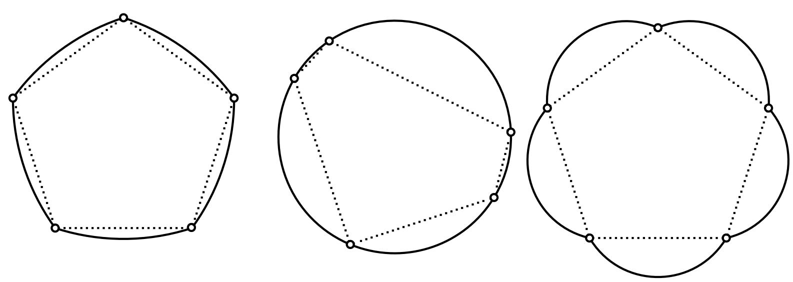

Figure 7.4: Optimal outer (k_g, ℓ)-pentagons

The main result of Chapter 7 is the following theorem.

Theorem 7.1.7. *Let $k_g \geq 0$, $\ell > 0$ and k satisfy the conditions in Remark 7.1.4. If ℓ is not the perimeter of a circle of geodesic curvature k_g, then the optimal outer (k_g, ℓ)-polygons in $\mathbf{M}$ are the regular ones. If ℓ is the perimeter of a circle of geodesic curvature k_g, then a (k_g, ℓ)-polygon is optimal if, and only if, its underlying polygon Γ is inscribed in a circle of geodesic curvature k_g (cf. Figure 7.4).*

We note that the proof of Theorem 6.5.3 proves without any change also Theorem 7.1.7 for $\mathbf{M} = \mathbb{E}^2$. In Section 7.3, we show that, by a similar method, the proof of Theorem 7.1.7 may be squeezed out for hyperbolic circles. In Section 7.4, we give a different, differential geometric proof of Theorem 7.1.7 that holds for any curve of constant geodesic curvature in $\mathbb{H}^2$. In Section 7.5, we show how to modify the proof in Section 7.4 for the sphere.

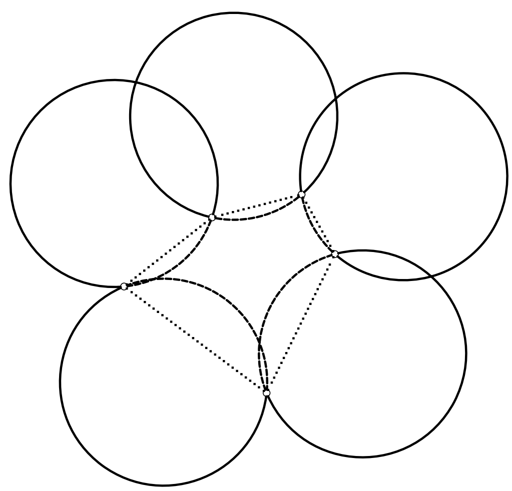

Figure 7.5: A "big eared" outer k_g-pentagon

Remark 7.1.8. If k_g is the geodesic curvature of a circle, then we may consider the discrete isoperimetric problem for outer k_g-polygons with *no* side shorter than a semicircle (cf. Figure 7.5). Since a big eared k_g-polygon is optimal if, and only if, the small eared inner k_g-polygon built around the same underlying polygon is optimal, it follows that the isoperimetric problem for "big eared" outer k_g-polygons may be reduced to Theorem 7.1.6.

7.2 Proof of Theorem 7.1.6

Recall that the area of P is the area of the interior of Γ minus the sum of the areas of the ears of P. By the classical isoperimetric inequalities, it is sufficient to show that if, in a family of (k_g, ℓ)-polygons different only in a single vertex, the sum of the areas of the two consecutive ears meeting at that vertex attains its minimum only if the ears are congruent.

Consider an arc-length parametrized curve $\phi : [0, \hat{\ell}] \to \mathbf{M}$, of constant geodesic curvature k_g, with endpoints $a = \phi(0)$ and $b = \phi(\hat{\ell})$, and select a third point p on $\phi([0, \hat{\ell}])$. For $x \in \{a, b\}$, let E_x denote the ear bounded by $\overline{xp}$ and the corresponding arc $\widehat{ab}$ of ϕ. We show that area(E_a)+area(E_b), as a function of p, attains its minimum only if p is the midpoint of $\phi([0, \hat{\ell}])$. The area of the domain bounded by $\overline{ab}$ and $\phi([0, \hat{\ell}])$ is fixed and hence, we want to maximize the area A of the triangle $[a, b, p]$. In $\mathbb{E}^2$, A is clearly maximal only if p is the midpoint.

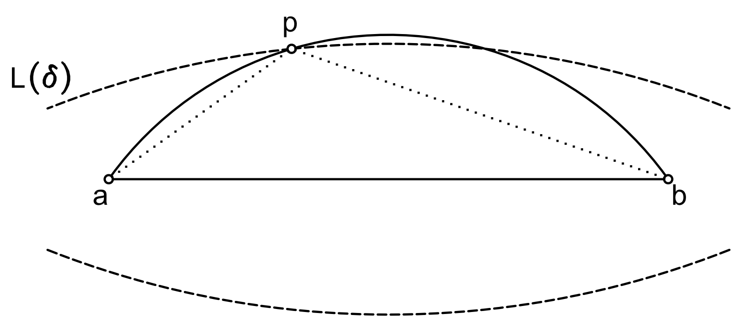

Figure 7.6: A Lexell figure in $\mathbb{H}^2$

In the case $\mathbf{M} \neq \mathbb{E}^2$, we let $\delta > 0$ and consider the set of points $L(\delta) = \{x \in \mathbf{M} :$ area$([a, x, b]) = \delta\}$. We call $L(\delta)$ a *Lexell figure*. In [24] p. 91, it is shown that $L(\delta)$

is a pair of circular arcs in $\mathbb{S}^2$, and a pair of hypercycles in $\mathbb{H}^2$. The Lexell figure is symmetric about both the geodesic segment $\overline{ab}$ and its perpendicular bisector.

If A is maximal with the constraint $p \in \phi([0, \hat{\ell}])$, then the corresponding Lexell figure is tangent to ϕ at p. Note that a curve of constant geodesic curvature $k_g > 0$ corresponds to a Euclidean circle arc or a straight line segment in our models of $\mathbb{H}^2$ and $\mathbb{S}^2$. Hence, if A is maximal, the p is the only common point of the Lexell figure and $\phi([0, \hat{\ell}])$. By the symmetry of the Lexell figure, p is the midpoint of $\phi([0, \hat{\ell}])$.

Remark 7.2.1. This proof shows that the discrete isoperimetric inequality for outer k_g-polygons is not a straightforward corollary of the classical discrete isoperimetric inequalities: as equal sides of Γ maximize the area of the region bounded by Γ, they also minimize the sum of the areas of the ears.

7.3 Proof of Theorem 7.1.7 for k_g-polygons consisting of hyperbolic circle arcs

Consider a k-sided outer k_g-polygon P, bounded by hyperbolic circle arcs of radius r, and note that $\coth r = k_g$.

If k_g is the geodesic curvature of a circle of perimeter ℓ, then by the classical isoperimetric inequalities, a (k_g, ℓ)-polygon is optimal if, and only if, its vertices are on a circle of geodesic curvature k_g. Thus, we may assume that ℓ is not the perimeter of a circle of geodesic curvature k_g.

A standard compactness argument for outer (k_g, ℓ)-polygons shows the existence of an optimal polygon. Hence, we assume that P is not a regular (k_g, ℓ)-polygon, and show that P is not optimal.

The areas of the ears are constant if the shape of the underlying polygon Γ is altered without changing the lengths of the sides. In $\mathbb{E}^2$ and $\mathbb{H}^2$, a polygon with given side lengths has maximal area if, and only if, its vertices lie on a curve of constant geodesic curvature. In particular, such a polygon with maximal area is strictly convex; that is, it is convex and its angle at each vertex is strictly less than π. Thus, we may assume that Γ is strictly convex and that P is not equilateral.

Let us choose two consecutive sides $\widehat{ap}$ and $\widehat{bp}$ of P such that $\mathrm{arclength}(\widehat{ap}) < \mathrm{arclength}(\widehat{bp})$, and a, p and b are not on the same curve of constant geodesic curvature k_g. Let $d = \cosh(\mathrm{dist}_H(a,b)) - 1$ and $u = \sinh r = 1/\sqrt{k_g^2 - 1}$. Let α and β denote the angles of the sections belonging to the circle arcs $\widehat{ap}$ and $\widehat{bp}$, respectively. First, we compute the area A of the convex figure bounded by $\widehat{ap}$, $\widehat{pb}$ and $\overline{ab}$. Then, fixing d and $\lambda = \alpha + \beta$, we show that the derivate of A with respect to α is positive under the condition $0 < \alpha < \beta$.

It is easy to see (cf. Section 3.2) that the area of a hyperbolic disk of radius r is

$$\mathrm{area}_H(\mathrm{disk}) = 4\pi \sinh^2\left(\frac{r}{2}\right).$$

By this formula and Proposition 3.6.2, we have

$$A = \lambda(\sqrt{u^2+1} - 1) - 2\arctan\frac{\sin\alpha}{\frac{\sqrt{u^2+1}+1}{\sqrt{u^2+1}-1} - \cos\alpha} - 2\arctan\frac{\sin\beta}{\frac{\sqrt{u^2+1}+1}{\sqrt{u^2+1}-1} - \cos\beta} +$$

$$+2\arctan\frac{\Delta}{1+x+y+z},$$

where $\Delta = \sqrt{1 - x^2 - y^2 - z^2 + 2xyz}$, and $x = 1+u^2-u^2\cos\alpha$, $y = 1+u^2-u^2\cos\beta$ and $z = d+1$ are the cosine hyperbolics of the side lengths of the hyperbolic triangle $[a,b,p]$.

After differentiation and simplification, we have

$$A' = \frac{-u^2(2\sqrt{u^2+1}(\cos\alpha - \cos\beta)\Delta - E)}{\Delta(2+u^2-\cos\alpha)(2+u^2-\cos\beta)},$$

where

$$E = du^2(\sin(\beta-\alpha) + \sin\alpha - \sin\beta) + 2d(\sin\beta - \sin\alpha) + \tag{7.1}$$

$$+u^2(\cos\alpha - \cos\beta)\left[(\sin\alpha + \sin\beta - \sin(\alpha+\beta))u^2 + 2(\sin\alpha + \sin\beta)\right].$$

We omit a tedious calculation that shows that

$$A' \cdot (2\sqrt{u^2+1}(\cos\alpha - \cos\beta)\Delta + E) = \frac{2u^2(\cos\alpha-\cos\beta)^2(\sin^2\frac{\alpha+\beta}{2}u^2 - d)^2}{\Delta(1-\cos(\alpha+\beta))}. \tag{7.2}$$

Note that the right-hand side of (7.2) is nonnegative. Furthermore, this expression is zero if, and only if, $\sin^2\frac{\alpha+\beta}{2}u^2 = d$; that is, a, p and b are on the same arc of constant geodesic curvature k_g.

Thus, $A' > 0$ follows from $E > 0$. As in the proof of Theorem 6.5.3, we regard E as a function of d. Note that since $\sin\alpha + \sin\beta > \sin(\alpha+\beta)$ for any $0 < \alpha < \beta \leq \pi$,

$$E(0) = u^2(\cos\alpha - \cos\beta)\left[(\sin\alpha + \sin\beta - \sin(\alpha+\beta))u^2 + 2(\sin\alpha + \sin\beta)\right] > 0.$$

To show that $E(d) > 0$ for every value of d in its domain, we may use an argument similar to that in the proof of Theorem 6.5.3.

7.4 Proof of Theorem 7.1.7 for the hyperbolic plane

As in the previous section, it is sufficient to show that if ℓ is different from the perimeter of a circle of geodesic curvature k_g, Γ is strictly convex and P is not equilateral, then P is not optimal. We may also assume that $k_g > 0$, and that there are three consecutive vertices a, p and b of P such that $\widehat{ap}$ is shorter than $\widehat{bp}$, and a, p and b are not on the same curve of constant geodesic curvature k_g.

We show that, under our conditions, p does not maximize the area $A(p)$ of the figure $F(p)$, bounded by the geodesic segment $\overline{ab}$ and the two sides $\widehat{ap}$ and $\widehat{bp}$, under the constraint that the arc length $L(p)$ of $\widehat{ap} \cup \widehat{bp}$ is fixed. The constraint lets p move along a continuous curve τ, which may or may not degenerate to a single point. The first case happens only if both $\widehat{ap}$ and $\widehat{bp}$ are semicircles, which contradicts the unequality of these sides. Thus, τ does not degenerate to a single point, and $\widehat{ap}$ is not a semicircle. Note that p is an endpoint of τ if, and only if, the larger side $\widehat{bp}$ is a semicircle, and in that case, p moves only in one direction.

We show that if $\widehat{bp}$ is not a semicircle and p moves towards the symmetric position, then the derivative of $A(p)$ in the direction of τ is positive. Since τ is continuous, the assertion follows also in the case that $\widehat{bp}$ is a semicircle.

Definition 7.4.1. Let $p, q \in \mathbb{H}^2$ and $k_g \geq 0$. If there is a curve of constant geodesic curvature k_g connecting p and q, then let $f_q(p)$ denote the arc length of a shortest such curve. The function f_q is the *k_g-arc-length function* belonging to q.

Note that the domain of f_q is either $\mathbb{H}^2$ or a closed disk.

To prove the assertion, we show the existence of a vector $v \in T_p\mathbb{H}^2$ such that $v(L) = 0$, $v(A) > 0$ and $v(f_a) > 0$. In other words, we show that the area $A(p)$ increases as p approaches the symmetric position on τ.

Definition 7.4.2. Let $t_a, t_b \in T_p\mathbb{H}^2$ be the unit tangent vectors of the oriented sides $\widehat{pa}$ and $\widehat{pb}$ (directed from p to a and from p to b), respectively. Let $t_e = \frac{t_b - t_a}{\|t_b - t_a\|}$; that is, t_e is the unit vector in $T_p\mathbb{H}^2$ in the direction of the angular bisector of $-t_a$ and t_b.

If the angle of P at p is less than π, let $t_i = \frac{t_a+t_b}{\|t_a+t_b\|}$. If the angle of P at p is greater than π, let $t_i = -\frac{t_a+t_b}{\|t_a+t_b\|}$. Thus, t_i is the unit vector in $T_p\mathbb{H}^2$ in the direction of the internal angular bisector of t_a and t_b.

Let $0 < \gamma < \pi$ be the angle between t_a and t_i, and let $0 < \sigma_a < \frac{\pi}{2}$ (resp. $0 < \sigma_b < \frac{\pi}{2}$) be the angle between the geodesic segment $\overline{pa}$ (resp. $\overline{pb}$) and the curve $\widehat{pa}$ (resp. $\widehat{pb}$) (cf. Figure 7.7.)

We note that since $\widehat{ap}$ and $\widehat{bp}$ are not contained in the same curve of geodesic curvature k_g, we have $\gamma \neq \frac{\pi}{2}$.

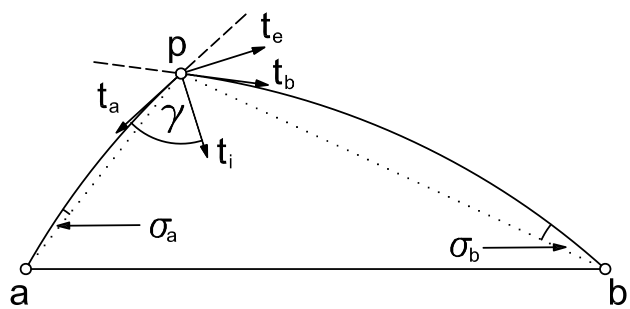

Figure 7.7: An illustration for Definition 7.4.2

Lemma 7.4.3. *Let $t \in T_p\mathbb{H}^2$ be a unit vector such that the oriented angle between t_a and t is ϕ and the orientation is given by the ordered basis (t_a, t_i). Then the derivative of f_a in the direction of t is*

$$t(f_a) = -\cos\phi - \sin\phi \tan\sigma_a. \tag{7.3}$$

Proof. Let $u \in T_p\mathbb{H}^2$ be orthogonal to the geodesic segment $\overline{pa}$. Set $d_a : \mathbb{H}^2 \to \mathbb{R}$, $d_a(x) = \mathrm{dist}(a, x)$. Clearly, $u(d_a) = 0$, and hence

$$u(f_a) = 0. \tag{7.4}$$

Next, let $c : (-\varepsilon, \varepsilon) \to \mathbb{H}^2$ be the arc length parametrization of a curve of constant geodesic curvature k_g such that $c(0) = p$ and $c([0, \varepsilon]) \subset \widehat{pa}$. The vector t_a is represented as the derivative along the curve c at 0, and thus,

$$t_a(f_a) = \frac{\mathrm{d}}{\mathrm{d}t} f_a(c(t))|_{t=0} = -1. \tag{7.5}$$

Since $\widehat{pa}$ is not a semicircle, it follows that u and t_a are not parallel and that we may decompose t as a linear combination $t = \mu t_a + \lambda u$. From (7.4), (7.5) and this decomposition, we have

$$t(f_a) = \mu t_a(f_a) = -\mu = -\frac{\langle t, v\rangle}{\langle t_a, v\rangle} = -\frac{\cos(\phi - \sigma_a)}{\cos \sigma_a}.$$

Now, the addition formula for the cosine function yields the assertion. $\square$

Lemma 7.4.4. *With reference to Definition 7.4.2 and Lemma 7.4.3,*

$$L_e = t_e(L) = \cos\gamma\big(\tan\sigma_b - \tan\sigma_a\big), \ \textit{and} \tag{7.6}$$

$$L_i = t_i(L) = -2\cos\gamma - \sin\gamma\big(\tan\sigma_a + \tan\sigma_b\big). \tag{7.7}$$

Moreover, if $x = -L_i t_e + L_e t_i$*, then*

$$x(L) = 0 \quad \textit{and} \quad x(f_a) > 0. \tag{7.8}$$

Proof. Note that our conditions imply that $\sigma_a, \sigma_b < \frac{\pi}{2}$, the oriented angles between t_a and t_i as well as between t_b and t_i are γ, the oriented angle between t_a and t_e is $\gamma + \frac{\pi}{2}$ and the one between t_b and t_i is $\gamma - \frac{\pi}{2}$. Now (7.6) and (7.7) follow from Lemma 7.4.3.

The first formula in (7.8) is obvious. By substituting (7.6) and (7.7) into the definition of x, applying Lemma 7.4.3 and simplifying, we obtain

$$x(f_a) = \frac{\sin\big(2\gamma - \sigma_a - \sigma_b\big)}{\cos\sigma_a \cos\sigma_b} > 0.$$

The inequality follows from that facts that $2\gamma - \sigma_a - \sigma_b$ is the angle of the underlying polygon Γ at p, $0 < 2\gamma - \sigma_a - \sigma_b < \pi$, and $0 < \sigma_a, \sigma_b < \frac{\pi}{2}$. $\square$

According to Lemma 7.4.4, we need only show that $x(A) > 0$. For this purpose, we compute the differential of A explicitly.

Lemma 7.4.5. *The derivatives of the area $A(p)$ in the directions t_e and t_i are*

$$A_e = t_e(A) = \frac{\cos\gamma}{\sqrt{k_g^2 - 1}} \left(\tan \frac{f_b(p)\sqrt{k_g^2 - 1}}{2} - \tan \frac{f_a(p)\sqrt{k_g^2 - 1}}{2} \right) \quad \textit{and} \tag{7.9}$$

$$A_i = t_i(A) = -\frac{\sin\gamma}{\sqrt{k_g^2 - 1}} \left(\tan \frac{f_b(p)\sqrt{k_g^2 - 1}}{2} + \tan \frac{f_a(p)\sqrt{k_g^2 - 1}}{2} \right). \tag{7.10}$$

Proof. For $w \in T_p\mathbb{H}^2$, we compute the derivative $w(A)$.

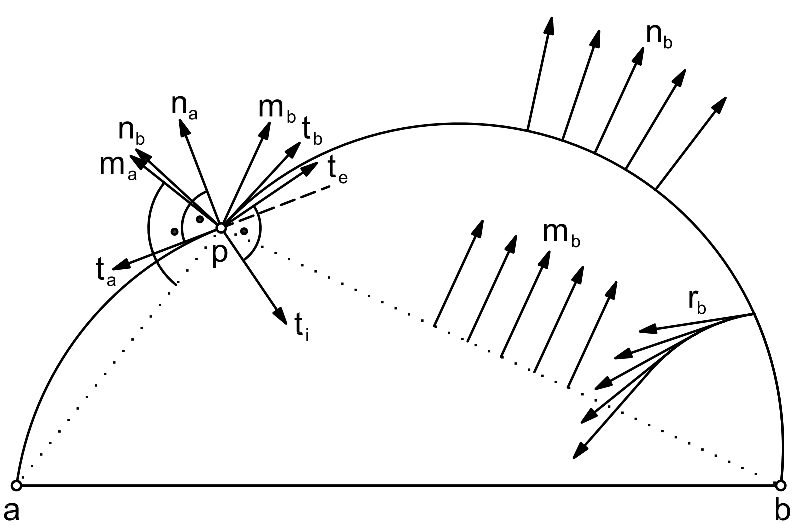

Figure 7.8: An illustration for the proof of Lemma 7.4.5

Choose a curve $\eta : (-\epsilon, \epsilon) \to \mathbb{H}^2$ describing a motion of $\eta(0) = p$ with initial speed vector $\eta'(0) = w$. Let $u \in \{a, b\}$. Consider the arc length parametrization $\zeta_u : \mathbb{R} \to \mathbb{H}^2$ of the curve, of constant geodesic curvature k_g, containing the arc $\widehat{up}$

such that $\zeta_u(0) = u$ and $\zeta_u(f_u(p)) = p$. Let $R_u(q, \theta)$ denote the rotation of $q \in \mathbb{H}^2$ about u with angle θ with respect to a fixed orientation of $\mathbb{H}^2$. Rotations about u form a one parameter group of isometries generated by the Killing field r_u, where $r_u(q) = \partial_\theta R_u(q, 0)$ (see Figure 7.8). There is a smooth function $\theta_u : (-\epsilon, \epsilon) \to \mathbb{R}$ such that the map $[0, 1] \to \mathbb{H}^2$, $\lambda \mapsto R_u(\zeta_u(\lambda f_u(\eta(t))), \theta_u(t))$ is a parametrization of the arc $\widehat{u\eta(t)}$. The initial speed vector field of this variation of the side $\widehat{up}$ is the vector field v_u along the curve $\widehat{up}$, the value of which at the point $q = \zeta_u(\lambda f_u(p))$ is

$$v_u(q) = \frac{\partial}{\partial t} R_u(\zeta_u(\lambda f_u(\eta(t))), \theta_u(t))\bigg|_{t=0} = \lambda w(f_u)\zeta_u'(\lambda f_u(p)) + \theta_u'(0) r_u(q).$$

The speed vector $v_u(p)$ coincides with the speed vector w of p:

$$w = -w(f_u)t_u + \theta_u'(0) r_u(p). \tag{7.11}$$

Let n_u be the outer unit vector field along, and orthogonal to, the arc $\widehat{up}$. The derivative of A with respect to w is

$$w(A) = \sum_{u \in \{a,b\}} \int_{\widehat{up}} < n_u, v_u > ds = \sum_{u \in \{a,b\}} \theta_u'(0) \int_{\widehat{up}} < n_u, r_u > ds. \tag{7.12}$$

Note that since r_u is a Killing field, it is divergence free. Hence,

$$\int_{\widehat{up}} < n_u, r_u > ds = \int_{\overline{up}} < m_u, r_u > ds, \tag{7.13}$$

with m_u denoting the outer unit normal of the polygon Γ along the side $\overline{up}$.

From (3.7), it is easy to compute that the perimeter of a circle C of radius r in the hyperbolic plane is

$$\mathrm{perim}_H(C) = 2\pi \sinh r. \tag{7.14}$$

From (7.14), we obtain that

$$r_u = \pm \sinh(d_u(q)) m_u,$$

with $d_u(q) = \mathrm{dist}_H(u,q)$ for $q \in \mathbb{H}^2$. Since the coefficient of m_u does not change sign on $\widehat{up}$,

$$\begin{aligned}\int_{\overline{up}} < m_u, r_u > ds &= \pm(\cosh d_u(p) - 1) =< r_u(p), m_u(p) > \frac{\cosh d_u(p) - 1}{\sinh d_u(p)} \\ &=< r_u(p), m_u(p) > \tanh \frac{d_u(p)}{2}.\end{aligned} \tag{7.15}$$

From (7.11), (7.12) and (7.15), we obtain that

$$\begin{aligned} w(A) &= \sum_{u\in\{a,b\}} < \theta'_u(0) r_u(p), m_u(p) > \tanh \frac{d_u(p)}{2} \\ &= \sum_{u\in\{a,b\}} (< w, m_u(p) > + w(f_u) < t_u, m_u(p) >) \tanh \frac{d_u(p)}{2}.\end{aligned} \tag{7.16}$$

If $w = t_i$ or $w = t_e$, then the angles between the unit vectors w, $m_u(p)$ and t_u are known explicitly, and an explicit expression for $w(f_u)$ is also given in Lemma 7.4.3. Substituting these values into (7.16), we obtain

$$\begin{aligned} t_i(A) &= \sum_{u\in\{a,b\}} \left(\cos\left(\frac{\pi}{2} + \gamma - \sigma_u\right) - \frac{\cos(\gamma - \sigma_u)}{\cos\sigma_u} \cos\left(\frac{\pi}{2} - \sigma_u\right)\right) \tanh \frac{d_u(p)}{2} \\ &= - \sum_{u\in\{a,b\}} \frac{\sin\gamma}{\cos\sigma_u} \tanh \frac{d_u(p)}{2}\end{aligned} \tag{7.17}$$

and,

$$t_e(A) = -\frac{\cos\gamma}{\cos\sigma_a} \tanh \frac{d_a(p)}{2} + \frac{\cos\gamma}{\cos\sigma_b} \tanh \frac{d_b(p)}{2}. \tag{7.18}$$

Using the identity $\tan(\mathrm{i}t)/\mathrm{i} = \tanh t$, (3.25) in Proposition 3.6.4 for $\mathbb{H}^2$ may be written as

$$\frac{\tanh \frac{d(s)}{2}}{\cos d(s)} = \frac{\tan \frac{s\sqrt{k_g^2 - 1}}{2}}{\sqrt{k_g^2 - 1}}.$$

Substituting this into (7.17) and (7.18) yields the assertion. $\square$

Finally, we show that $x(A) > 0$ by direct calculation. Using the definition of x in Lemma 7.4.4 and Formulae (3.23), (7.6), (7.7), (7.9) and (7.10), we have that

$$x(A) = -L_i A_e + L_e A_i = \frac{2\cos^2\gamma}{\sqrt{k_g^2 - 1}} \left(\tan \frac{f_b(p)\sqrt{k_g^2 - 1}}{2} - \tan \frac{f_a(p)\sqrt{k_g^2 - 1}}{2} \right).$$

Note that, by an algebraic identity and continuity, we have

$$x(A) = \begin{cases} \frac{2\cos^2\gamma}{\sqrt{1-k_g^2}} \left(\tanh \frac{f_b(p)\sqrt{1-k_g^2}}{2} - \tanh \frac{f_a(p)\sqrt{1-k_g^2}}{2} \right) & \text{if } 0 < k_g < 1, \\ \cos^2\gamma \big(f_b(p) - f_a(p) \big) & \text{if } k_g = 1, \\ \frac{2\cos^2\gamma}{\sqrt{k_g^2-1}} \left(\tan \frac{f_b(p)\sqrt{k_g^2-1}}{2} - \tan \frac{f_a(p)\sqrt{k_g^2-1}}{2} \right) & \text{if } 1 < k_g. \end{cases}$$

Thus, $x(A) > 0$ follows from the monotonicity of the functions $t \mapsto \tan t$ and $t \mapsto \tanh t$.

7.5 Proof of Theorem 7.1.7 for the sphere

We follow the proof in Section 7.4.

By Remark 7.1.4, Jensen's inequality and the observation that the length of a chord of a circle is a concave function of the length of the corresponding arc, we obtain that the perimeter of P is less than that of a great circle. Note that a spherical polygon, with given side lengths and with perimeter less than that of a great-circle, has maximal area if, and only if, its vertices lie on a curve of constant geodesic curvature. Thus, we may assume that Γ is strictly convex. Furthermore, we may assume that $k_g > 0$, and there are three consecutive vertices a, p and b of P such that $\widehat{ap}$ is shorter than $\widehat{bp}$, and a, p and b are not on the same curve of constant geodesic curvature k_g. We show that in that case $A(p)$ is not maximal.

We may assume also that $\widehat{bp}$ is not a semicircle, and show that when p moves towards the symmetric position, under the constraint that the arc length $L(p)$ of the curve $\widehat{ap} \cup \widehat{bp}$ does not change, then the derivative of $A(p)$ is positive.

We use the notations of Definitions 7.4.1 and 7.4.2. We note that the proofs of Lemmas 7.4.3 and 7.4.4 hold for $\mathbb{S}^2$ without any change, and verify the following variant of Lemma 7.4.5.

Lemma 7.5.1. *The derivatives of the area $A(p)$ in the directions t_e and t_i are*

$$A_e = t_e(A) = \frac{\cos\gamma}{\sqrt{k_g^2+1}} \left(\tan \frac{f_b(p)\sqrt{k_g^2+1}}{2} - \tan \frac{f_a(p)\sqrt{k_g^2+1}}{2} \right) \quad \textit{and} \tag{7.19}$$

$$A_i = t_i(A) = -\frac{\sin\gamma}{\sqrt{k_g^2+1}} \left(\tan \frac{f_b(p)\sqrt{k_g^2+1}}{2} + \tan \frac{f_a(p)\sqrt{k_g^2+1}}{2} \right). \tag{7.20}$$

Proof. We use the notations and the argument of the proof of Lemma 7.4.5. Since the perimeter of a spherical circle C of radius r is

$$\operatorname{perim}_S(C) = 2\pi \sin r,$$

it follows that the inital speed vector field r_u of the rotation about the point $u \in \{a, b\}$ is

$$r_u = \pm \sin(d_u(q)) m_u,$$

with $d_u(q) = \operatorname{dist}_S(u, p)$ for $q \in \mathbb{S}^2$, and m_u denoting the outer unit normal of the polygonal curve Γ along the geodesic segment $\overline{up}$. The assertion now follows from an argument similar to that in the proof of Lemma 7.4.5. □

Finally, the definition of x in Lemma 7.4.4 and Formulae (3.23), (7.6), (7.7), (7.19) and (7.20)yield that

$$x(A) = -L_i A_e + L_e A_i = \frac{2\cos^2\gamma}{\sqrt{k_g^2+1}} \left(\tan \frac{f_b(p)\sqrt{k_g^2+1}}{2} - \tan \frac{f_a(p)\sqrt{k_g^2+1}}{2} \right),$$

and hence, $x(A) > 0$.

Chapter 8

On a conjecture of Maehara

8.1 A counterexample to a conjecture of Maehara

Maehara [40] proved the following Helly-type theorem for spheres.

Theorem 8.1.1 (Maehara). *Let $\mathfrak{F}$ be a family of at least $n+3$ distinct $(n-1)$-spheres in $\mathbb{E}^n$. If any $n+1$ spheres in $\mathfrak{F}$ have a point in common, then all of them have a point in common.*

Maehara points out that the assertion does not hold if we replace either $n+3$ or $n+1$ by a smaller number. First, we prove a variant of Theorem 8.1.1.

Theorem 8.1.2. *Let $\mathfrak{F}$ be a family of $(n-1)$-spheres in $\mathbb{E}^n$, $0 \leq k \leq n-1$. Suppose that $\operatorname{card} \mathfrak{F} \geq n-k$ and that any $n-k$ spheres in $\mathfrak{F}$ intersect in a sphere of dimension at least $k+1$. Then they all intersect in a sphere of dimension at least $k+1$. Furthermore, $k+1$ may not be reduced to k.*

Proof. Consider

$$S = \bigcap_{i=1}^{n-k} \mathbb{S}^{n-1}(c_i, r_i), \quad \text{where} \quad \mathbb{S}^{n-1}(c_i, r_i) \in \mathfrak{F} \quad \text{for} \quad i = 1, 2, \ldots, n-k$$

such that $\dim S$ is minimal amongst all intersections of $n-k$ spheres from $\mathfrak{F}$. By assumption, S is a sphere with $\dim S \geq k+1$ (cf. Figure 8.1).

By induction on $n-k$, it is easy to show that there is some $1 \leq t \leq n-k$ such that

$$S = \bigcap_{1 \le i \le n-k, i \ne t} \mathbb{S}^{n-1}(c_i, r_i).$$

Let us denote the preceding subfamily of $\mathfrak{F}$ by $\mathfrak{F}'$. By the minimality of $\dim S$, we

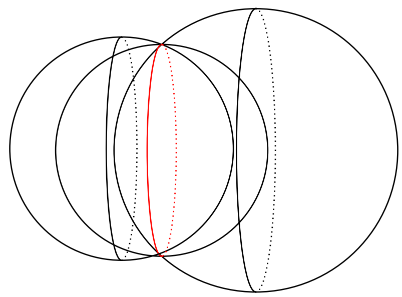

Figure 8.1: An illustration for the proof of Theorem 8.1.2 with $n = 3$ and $k = 0$

have

$$S = \mathbb{S}^{n-1}(c, r) \cap \left(\bigcap_{\mathbb{S}^{n-1}(c', r') \in \mathfrak{F}'} \mathbb{S}^{n-1}(c', r') \right)$$

for any $\mathbb{S}^{n-1}(c, r) \in \mathfrak{F}$. Hence, S is the intersection of all the spheres from $\mathfrak{F}$.

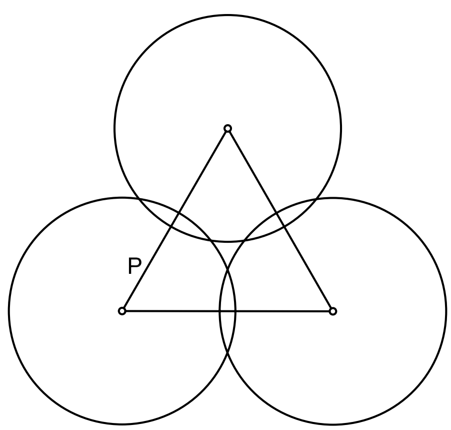

Figure 8.2: An illustration for the second part of the proof of Theorem 8.1.2 with $n = 2$ and $k = 0$

To show that $k+1$ may not be replaced by k, consider a regular n-simplex P in $\mathbb{E}^n$ with $\mathrm{cr}(P) = 1 + \epsilon$, where $\epsilon > 0$ is sufficiently small. Let $\mathfrak{F}$ consist of the $n+1$ unit spheres with the vertices of P as centres (cf. Figure 8.2). Then the intersection of any $n-k$ spheres from $\mathfrak{F}$ is a sphere of dimension k, but the intersection of all the spheres is empty. $\square$

Maehara [40] conjectured the following stronger version of Theorem 8.1.1.

Conjecture 8.1.3 (Maehara). *Let $n \geq 3$ and $\mathfrak{F}$ be a family of at least $n+2$ distinct $(n-1)$-dimensional unit spheres in $\mathbb{E}^n$. Suppose that any $n+1$ spheres in $\mathfrak{F}$ have a point in common. Then all the spheres in $\mathfrak{F}$ have a point in common.*

After Proposition 3 in [40], Maehara points out the importance of the condition $n \geq 3$ by showing the following statement, also known as *Ţiţeica's theorem* (sometimes called *Johnson's theorem*). This theorem was proved by the Romanian mathematician, G. Ţiţeica in 1908 (for historical details, see [2] or [34], p. 75).

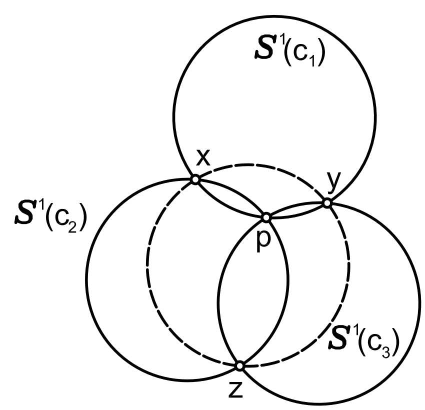

Figure 8.3: Ţiţeica's theorem for four circles

Theorem 8.1.4 (Ţiţeica). *Let $\mathbb{S}^1(c_1)$, $\mathbb{S}^1(c_2)$ and $\mathbb{S}^1(c_3)$ be unit circles in $\mathbb{E}^2$ that intersect in a point p (see Figure 8.3). Let $\{x, p\} = \mathbb{S}^1(c_1) \cap \mathbb{S}^1(c_2)$, $\{y, p\} = \mathbb{S}^1(c_1) \cap \mathbb{S}^1(c_3)$ and $\{z, p\} = \mathbb{S}^1(c_2) \cap \mathbb{S}^1(c_3)$. Then x, y and z lie on a unit circle.*

In the remaining part of Section 8.1, we show that Conjecture 8.1.3 is false for $n \geq 4$. To construct a suitable family $\mathfrak{F}$ of unit spheres, we need the following lemma. Note that the sphere circumscribed about a simplex P is the unique sphere that contains each vertex of P. The circumscribed sphere of P does not necessarily coincide with the circumsphere of P.

Lemma 8.1.5. *The following are equivalent.*

8.1.5.1 *There is an n-simplex $P \subset \mathbb{E}^n$ with $\mathbb{S}^{n-1}(o, R)$ circumscribed about P and a sphere $\mathbb{S}^{n-1}(x_1, r)$, tangent to all facet-hyperplanes of P, such that either $R^2 - 2rR = d^2$ or $R^2 + 2rR = d^2$ holds, with $d = \|x_1 - o\|$.*

8.1.5.2 *There is a family of $n+2$ distinct $(n-1)$-dimensional unit spheres in $\mathbb{E}^n$ such that any $n+1$, but not all, of the spheres have a non-empty intersection.*

Proof. First, we show that 8.1.5.2 follows from 8.1.5.1. Observe that $R^2 - 2rR = d^2$ yields $R > d$, from which we obtain that $x_1 \in \mathbf{B}^n(o, R)$. Similarly, if $R^2 + 2rR = d^2$, then $x_1 \notin \mathbf{B}^n[o, R]$. Thus, $x_1 \notin \mathbb{S}^{n-1}(o, R)$. Since $\mathbb{S}^{n-1}(x_1, r)$ is tangent to every facet-hyperplane of P, x_1 is not contained in any of these hyperplanes.

Consider the inversion f in the sphere $\mathbb{S}^{n-1}(x_1, r)$. Let $V(P) = \{a_i : i = 2, 3, \ldots, n+2\}$, and let H_i denote the facet-hyperplane of P that does not contain a_i. Let $\mathbb{S}^{n-1}(c_1, r_1) = f(\mathbb{S}^{n-1}(o, R))$, and, for $i = 2, 3, \ldots, n+2$, let $\mathbb{S}^{n-1}(c_i, r_i) = f(H_i)$ and $x_i = f(a_i)$.

Since H_i is tangent to $\mathbb{S}^{n-1}(x_1, r)$, $\mathbb{S}^{n-1}(c_i, r_i)$ is a sphere that is tangent to $\mathbb{S}^{n-1}(x_1, r)$ and contains x_1. Hence $r_i = \frac{r}{2}$ for $i = 2, 3, \ldots, n+2$. We show that $r_1 = \frac{r}{2}$. If $x_1 \in \mathbf{B}^n(o, R)$ then, using the definition of inversion and the equations in 8.1.5.1, we have (cf. Figure 8.4)

$$2r_1 = \operatorname{diam} \mathbb{S}^{n-1}(c_1, r_1) = \frac{r^2}{R+d} + \frac{r^2}{R-d} = \frac{2r^2 R}{R^2 - d^2} = r. \tag{8.1}$$

If $x_1 \notin \mathbf{B}^n[o, R]$, then

$$2r_1 = \operatorname{diam} \mathbb{S}^{n-1}(c_1, r_1) = \frac{r^2}{d-R} - \frac{r^2}{d+R} = \frac{2r^2 R}{d^2 - R^2} = r. \tag{8.2}$$

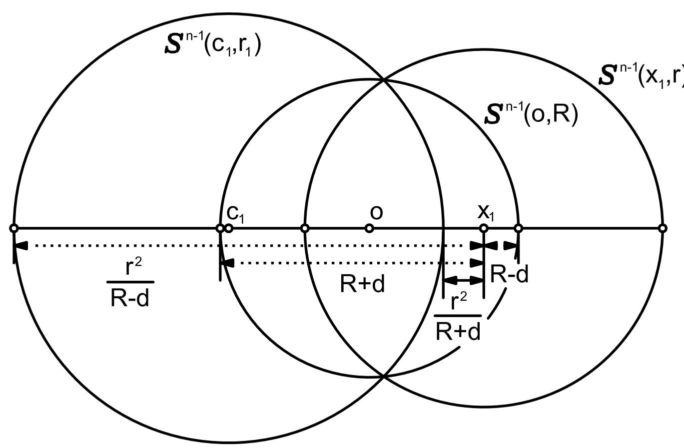

Figure 8.4: An illustration for (8.1)

Let $\mathfrak{F} = \{\mathbb{S}^{n-1}(c_i, \frac{r}{2}) : i = 1, \ldots, n+2\}$. Observe that for every $i \neq 1$, we have that $x_1 \in \mathbb{S}^{n-1}(c_i, \frac{r}{2})$, and for every $j \neq i$, we have $x_i \in \mathbb{S}^{n-1}(c_1, \frac{r}{2}) \cap \mathbb{S}^{n-1}(c_j, \frac{r}{2})$. Thus, $\mathfrak{F}$ is a family of $n+2$ spheres of radius $\frac{r}{2}$, any $n+1$ of which have a nonempty intersection.

Assume that there is a point $y \in \bigcap \mathfrak{F}$. Since $x_1 \notin \mathbb{S}^{n-1}(o, R)$, it follows that $y \neq x_1$, and thus, $x_1 \notin \mathbb{S}^{n-1}(c_1, \frac{r}{2}) = f(\mathbb{S}^{n-1}(o, R))$. Hence, $z = f(y) = f^{-1}(y)$

exists. Observe that z is contained in every facet-hyperplane of P, and also, in its circumscribed sphere; a contradiction. Thus, $\mathfrak{F}' = \{\mathbb{S}^{n-1}(\frac{2}{r} \cdot c_i) : i = 1, \ldots, n+2\}$ is a family of unit spheres that satisfies 8.1.5.2.

A similar argument shows that 8.1.5.1 follows from 8.1.5.2. □

Theorem 8.1.6. *For any $n \geq 4$, there is a family of $n+2$ distinct $(n-1)$-dimensional unit spheres in $\mathbb{E}^n$ such that any $n+1$, but not all, of them have a common point.*

Proof. We apply Lemma 8.1.5 and construct a simplex P and a sphere $\mathbb{S}^{n-1}(x_1, r)$ such that they satisfy 8.1.5.1. We set $m = n - 1$.

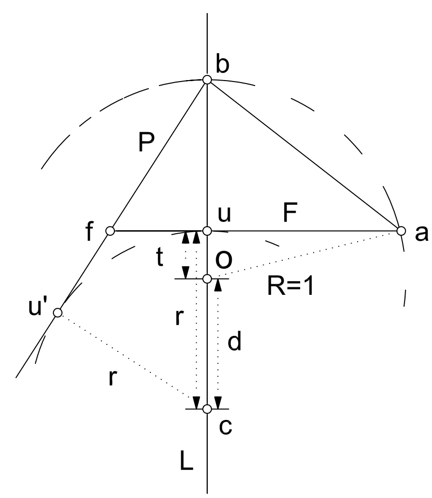

Figure 8.5: An illustration for the proof of Theorem 8.1.6

Consider a line L passing through o, and a hyperplane H which is orthogonal to L and is at a given distance $t \in (0,1)$ from o. Let u denote the intersection point of L and H. We observe that $t = \|u\|$ and let $b = \frac{1}{t}u$. Then $b \in \mathbb{S}^{n-1}(o,1)$. Let F be a regular m-simplex in H whose circumsphere in H is $S^{n-1}(o,1) \cap H$. Thus, u is the centroid of F and the sphere circumscribed about $P = [F, b]$ is $\mathbb{S}^{n-1}(o,1)$. Clearly, there is a unique sphere $\mathbb{S}^{n-1}(c,r)$, tangent to every facet-hyperplane of P, such that $c \in L$ and $c \notin P$. We set $d = \|c\|$ (cf. Figure 8.5).

Our aim is to prove that, with a suitable choice of t, P and $\mathbb{S}^{n-1}(c,r)$ satisfy the conditions in 8.1.5.1. To prove this, we calculate $h_m(t) = d(t)^2 + 2r(t) - 1$, and show that this function has a root on the interval $(0,1)$ for $m \geq 3$. We note that if $h_m(t) = 0$ for some value of t, then P and $\mathbb{S}^{n-1}(c,r)$ satisfy the first equality in 8.1.5.1 for $R = 1$ and $x_1 = c$.

Consider a vertex a of F and the centroid f of the facet of F that does not contain a. Then $\|b-u\| = 1-t$ and $\|a-u\| = \sqrt{1-t^2}$. Note that, in an m-dimensional regular simplex, the distance of the centroid from a vertex of the simplex is m times as large as its distance from a facet-hyperplane. Thus, we have $\|u-f\| = \frac{\sqrt{1-t^2}}{m}$. We observe that $\mathbb{S}^{n-1}(c,r)$ is tangent to the facet-hyperplane H_a of P that does not contain a. Let u' denote the intersection point of $\mathbb{S}^{n-1}(c,r)$ and H_a. Clearly, u', f and b are collinear and $\|u-c\| = \|u'-c\| = r$. Furthermore, the two triangles $[c,u',b]$ and $[f,u,b]$ are co-planar and similar. Hence,

$$\frac{\|b-f\|}{\|b-c\|} = \frac{\|u-f\|}{\|u'-c\|}. \tag{8.3}$$

We have that $\|b-f\| = \sqrt{(1-t)^2 + \frac{1-t^2}{m^2}}$, $\|b-c\| = 1+r-t$, $\|u'-c\| = r$ and $\|u-f\| = \frac{\sqrt{1-t^2}}{m}$. Solving (8.3) for r, we obtain

$$r = \frac{\sqrt{1+t}}{m^2}\left(\sqrt{m^2+1-(m^2-1)t} + \sqrt{1+t}\right). \tag{8.4}$$

Note that $d = |r-t|$. From this and (8.4), we have

$$h_m(t) = \left(\frac{\sqrt{1+t}}{m^2}\left(\sqrt{m^2+1-(m^2-1)t} + \sqrt{1+t}\right) - t\right)^2 + \tag{8.5}$$

$$+\frac{2\sqrt{1+t}}{m^2}\left(\sqrt{m^2+1-(m^2-1)t}+\sqrt{1+t}\right)-1.$$

We observe that $h_3\left(\frac{1}{2}\right)=0$. Let $m>3$. Then $h_m(0)<0$, $h_m(1)>0$ and h_m is continuous on $[0,1]$. Thus, h_m has a root on the interval $(0,1)$, and we obtain a simplex P and a sphere $\mathbb{S}^{n-1}(c,r)$ that satisfy the conditions in 8.1.5.1. $\square$

8.2 Maehara-type problems in the hyperbolic space and on the sphere

In this section we investigate whether we can extend the results and problems from Section 8.1 to the hyperbolic and spherical spaces. Note that in our models of hyperbolic and spherical spaces, hyperbolic and spherical balls correspond to Euclidean balls. Let $\mathbf{B}^n_H[y,r]$, $\mathbf{B}^n_H(y,r)$ and $\mathbb{S}^{n-1}_H(y,r)$ denote the hyperbolic closed ball, open ball and sphere, with radius r and centre y, and denote the corresponding spherical objects by $\mathbf{B}^n_S[y,r]$, $\mathbf{B}^n_S(y,r)$ and $\mathbb{S}^{n-1}_S(y,r)$.

Proposition 8.2.1. *Let $\mathfrak{F}$ be a family of at least $n+3$ distinct $(n-1)$-spheres in $\mathbf{M}$, and $\mathbf{M}=\mathbb{H}^n$ or $\mathbf{M}=\mathbb{S}^n$. If any $n+1$ spheres in $\mathfrak{F}$ have a non-empty intersection then there is a point common to every sphere in $\mathfrak{F}$.*

Proof. If $\mathbf{M}=\mathbb{H}^n$, we may immediately apply Theorem 8.1.1. If $\mathbf{M}=\mathbb{S}^n$, we consider $\mathfrak{F}'=\{\mathbb{S}^n(o)\}\cup\{\mathbb{S}^n(c_i):i=1,2,\ldots,\operatorname{card}\mathfrak{F}\}$, where $\mathfrak{F}=\{\mathbb{S}^n(o)\cap\mathbb{S}^n(c_i):i=1,2,\ldots,\operatorname{card}\mathfrak{F}\}$. Now, Theorem 8.1.1 yields the required statement. $\square$

In the next part, we examine variants of Conjecture 8.1.3 for $\mathbb{H}^2$ and $\mathbb{S}^2$.

Theorem 8.2.2. *Let $\mathfrak{F}$ be a family of at least four congruent hyperbolic circles (resp.*

horocycles, resp. congruent hypercycles) in $\mathbb{H}^2$ *such that any three members of* $\mathfrak{F}$ *have a non-empty intersection. Then there is a point common to every member of* $\mathfrak{F}$.

Proof. By Proposition 8.2.1, we need only consider the case card $\mathfrak{F} = 4$. Suppose that any three members of $\mathfrak{F}$ intersect, but not all four do. The four elements of $\mathfrak{F}$ correspond to the intersections of $\mathbf{B}^2(o)$ with four Euclidean circles, which we denote by $\mathbb{S}^1(c_i, R_i)$, for each $i = 1, 2, 3, 4$. Note that the congruence of the elements of $\mathfrak{F}$ does not imply the congruence of these Euclidean circles.

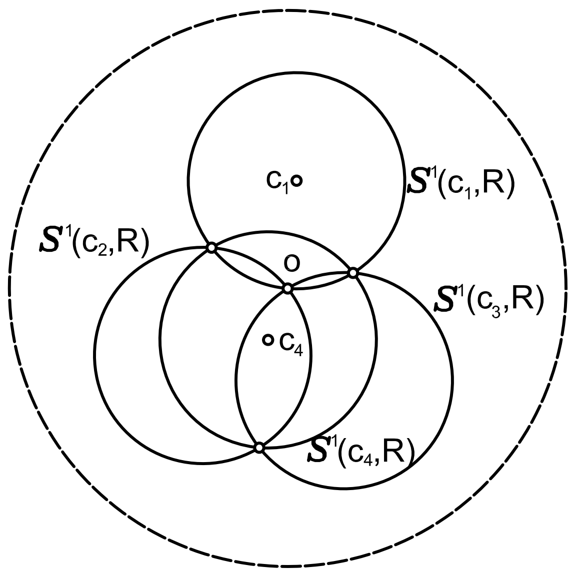

Figure 8.6: Four congruent hyperbolic circles in $\mathbb{H}^2$

It is easy to see that $\bigcap_{i=1}^{3} \mathbb{S}^1(c_i, R_i)$ contains only one point, which we may assume to be o. By an appropriate choice of indices, we may assume also that $o \in \mathbf{B}^2(c_4, R_4)$. Now, $\mathbb{S}^1(c_1, R_1), \mathbb{S}^1(c_2, R_2)$ and $\mathbb{S}^1(c_3, R_3)$ are congruent, and thus, Theorem 8.1.4 yields that $\mathbb{S}^1(c_4, R_4)$ is congruent to the other three Euclidean circles.

In the case of horocycles, $\mathbb{S}^1(c_4, R_4) \subset \mathbf{B}^2(o, 1)$ and hence it does not correspond to a horocycle; a contradiction. In the case of hypercycles, $\mathbb{S}^1(c_4, R_4)$ does not intersect the circle at infinity in the same angle as $\mathbb{S}^1(c_1, R_1), \mathbb{S}^1(c_2, R_2)$ and $\mathbb{S}^1(c_3, R_3)$, and hence, $\mathfrak{F}$ is not a family of congruent hypercycles; a contradiction.

We may assume that $\mathfrak{F}$ consists of four congruent hyperbolic circles. Note that if d is the diameter of the hyperbolic circle corresponding to a Euclidean circle $\mathbb{S}^1(c, R)$, then

$$\cosh d = 1 + \frac{2\|z - x\|^2}{(1 - \|x\|^2)(1 - \|z\|^2)},$$

with x and z as the points of $\mathbb{S}^2(x_1, r_1)$ closest to and farthest from o, respectively. Thus, it is easy to show that the radius of the hyperbolic circle corresponding to $\mathbb{S}^1(c_4, R_4)$ is strictly smaller than the radii of the other three hyperbolic circles; a contradiction. $\square$

Theorem 8.2.3. *Let $\mathfrak{F}$ be a family of at least four circles of radius $r < \frac{\pi}{2}$ in $\mathbb{S}^2$. If any three circles in $\mathfrak{F}$ have a non-empty intersection, then there is a point common to every circle in $\mathfrak{F}$.*

Proof. We may assume that $\operatorname{card} \mathfrak{F} = 4$. Suppose that $\mathfrak{F} = \{\mathbb{S}^1_S(y_i, r) : i = 1, \ldots, 4\}$ is a family of spherical circles of radius r, $0 < r < \frac{\pi}{2}$, such that any three circles have a non-empty intersection, but there is no point common to each circle in $\mathfrak{F}$.

We choose points $x_i \in \cap(\mathfrak{F} \setminus \{\mathbb{S}^1_S(y_i, r)\})$, for $i = 1, 2, 3, 4$, and let $X = \{x_1, \ldots, x_4\}$. We set $d = d_S(y_1, x_1)$. Clearly, we may choose the indices of the circles in a way that $x_1 \in \mathbf{B}^1_S(y_1, r)$. In other words, we may assume that $d < r$.

Let H be the tangent plane of $\mathbb{S}^2$ at x_1. Consider the stereographic projection p, from $\mathbb{S}^2$ onto H (cf. Figure 8.7). For $i = 1, 2, 3, 4$, let $\mathbb{S}^1(c_i, R_i) = p(\mathbb{S}^1_S(c_i, r))$. An easy computation yields that $2R_1 = \tan \frac{r+d}{2} + \tan \frac{r-d}{2}$, and $2R_i = \tan r$ for

$i = 2, 3, 4$. Observe that the function $f : [0, r] \to \mathbb{R}$, $f(d) = \tan \frac{r+d}{2} + \tan \frac{r-d}{2}$ is strictly increasing and bijective. From Theorem 8.1.4, it follows that $f(d) = \tan r$ and $d = r$, a contradiction. $\square$

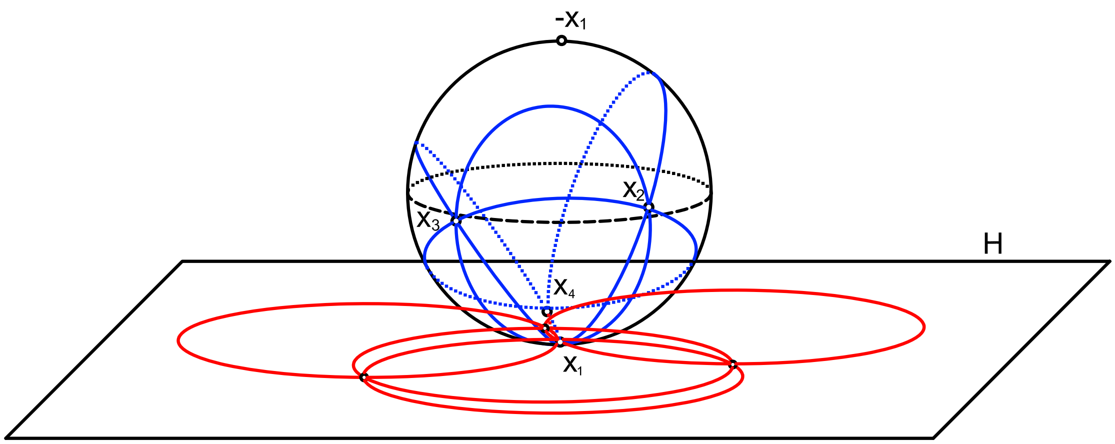

Figure 8.7: Stereographic projection from $\mathbb{S}^2$ onto H

In the remaining part, we show that the statement in Conjecture 8.1.3 does not hold for $n \geq 4$ if we replace $\mathbb{E}^n$ by $\mathbb{H}^n$, or for $n \geq 3$ if we replace $\mathbb{E}^n$ by $\mathbb{S}^n$.

To construct an example in $\mathbb{H}^n$, we recall the Euclidean construction described in Lemma 8.1.5 and Theorem 8.1.6, and use the notations established there. Theorem 8.1.6 yields a simplex P and a sphere $\mathbb{S}^{n-1}(x_1, r)$ that satisfies $R^2 - 2rR = d^2$, which implies $x_1 \in \mathbf{B}^n(o, R)$. In other words, our construction yields a family $\mathfrak{F} = \{\mathbb{S}^{n-1}(c_i, \frac{r}{2}) : i = 1, 2, \ldots, n+2\}$ of congruent spheres such that $\|c_1 - x_1\| < \frac{r}{2}$.

In the proof of Theorem 8.1.6, h_m is a continuous function of t, and has a root in the interval $(0, 1)$, for $m \geq 3$. It is easy to check that, for a suitably small $\delta > 0$, the interval $[0, \delta]$ is contained in the range of h_m. Thus, for any $\tau \in [0, \delta]$, there is a simplex P, with $\mathbb{S}^{n-1}(o,R)$ circumscribed about it, and a sphere $\mathbb{S}^{n-1}(c,r)$ tangent to any facet-hyperplane of P, that satisfies $(1+\tau)R^2-2rR=\|c-o\|^2$. Applying the inversion f of Lemma 8.1.5 to such a simplex and the sphere circumscribed about it, we obtain a family of spheres that satisfies the conditions in 8.1.5.2 with the exception that the radius of $\frac{2}{r}f(\mathbb{S}^{n-1}(o,R))$ is not one, but greater than or equal to one. These observations are summarized in the following statement.

Proposition 8.2.4. *Let $n \geq 4$. Then there is an $\varepsilon > 0$, depending only on the value of n, such that, for every $\lambda \in (1, 1+\varepsilon)$, there is a family $\mathfrak{F} = \{\mathbb{S}^{n-1}(c_1, \lambda R)\} \cup \{\mathbb{S}^{n-1}(c_i, R) : i = 2, 3, \ldots, n+2\}$ of spheres in $\mathbb{E}^n$ that satisfies the following:*

(i) *any $n+1$ spheres of $\mathfrak{F}$ have a non-empty intersection, but there is no point common to every sphere in $\mathfrak{F}$,*

(ii) *$\|c_1 - x_1\| < R$, where $\{x_1\} = \bigcap(\mathfrak{F} \setminus \{\mathbb{S}^{n-1}(c_1, \lambda R)\})$.*

We use Proposition 8.2.4 to construct families of congruent hyperbolic spheres.

Theorem 8.2.5. *For every $n \geq 4$, there is an $r > 0$ and a family $\mathfrak{F} = \{\mathbb{S}_H^{n-1}(y_i, r) : i = 1, 2, \ldots, n+2\}$ of congruent spheres in $\mathbb{H}^n$ such that any $n+1$ spheres of $\mathfrak{F}$ have a non-empty intersection, but there is no point common to every sphere in $\mathfrak{F}$.*

Proof. We use the notations of Proposition 8.2.4, with $\lambda \in (1, 1+\varepsilon)$ and $\mathfrak{F} = \{\mathbb{S}^{n-1}(c_1, \lambda R)\} \cup \{\mathbb{S}^{n-1}(c_i, R) : i = 2, \ldots, n+2\}$.

We may assume that $x_1 = o$. For every $R \in (0, 1/2)$ and $i = 2, 3, \ldots, n+2$, $\mathbb{S}^{n-1}(c_i, R)$ represents a hyperbolic sphere $\mathbb{S}_H^{n-1}(y_i, h(R))$. We note that $h(R)$ does not depend on i, but on R. If $R \in (0, 1/2)$, then $\mathbb{S}^{n-1}(c_1, \lambda R)$ represents a hyperbolic sphere, which we denote by $\mathbb{S}_H^{n-1}(y_1, k(R))$. Clearly, if $h(R) = k(R)$, then $\mathfrak{F}$ satisfies the conditions of our theorem.

We observe that for $i = 2, 3, \ldots, n+2$, $\mathbb{S}^{n-1}(c_i, 1/2) \cap \mathbf{B}^n(o, 1)$ represents a horosphere, and $\mathbb{S}^{n-1}(c_1, \lambda/2)$ represents a hyperbolic sphere. Thus, for a sufficiently small $\delta > 0$, $h(1/2-\delta) > k(1/2-\delta)$. Clearly, $k(R)$ and $h(R)$ are continuous functions of R. Thus, the assertion follows from

$$\lim_{R \to 0^+} \frac{k(R)}{h(R)} = \lambda, \tag{8.6}$$

and $\lambda > 1$.

Observe that the diameter of $\mathbb{S}^{n-1}(c_2, R)$, passing through o, is a diameter of also $\mathbb{S}_H^{n-1}(y_2, h(R))$. Thus, $h(R) = \operatorname{arctanh}(2R)$. We note that if a hyperbolic sphere $\mathbb{S}_H^{n-1}(y, r)$ corresponds to the Euclidean sphere $\mathbb{S}^{n-1}(c, R)$ then, for any fixed R, r is a strictly increasing function of $d_H(c, o)$. Hence $2\operatorname{arctanh}(\lambda R) \leq k(R) < \operatorname{arctanh}(2\lambda R)$. From this, we obtain

$$\frac{2\operatorname{arctanh}(\lambda R)}{\operatorname{arctanh}(2R)} \leq \frac{k(R)}{h(R)} < \frac{\operatorname{arctanh}(2\lambda R)}{\operatorname{arctanh}(2R)}. \tag{8.7}$$

The required limit is the consequence of L'Hospital's Rule. □

Proposition 8.2.6. *For every $n \geq 3$, there is an $r > 0$ and a family $\mathfrak{F}$ of $n+2$ distinct $(n-1)$-spheres in $\mathbb{S}^n$, of radius r, such that any $n+1$ spheres of $\mathfrak{F}$ have a non-empty intersection, but there is no point common to every sphere in $\mathfrak{F}$.*

Proof. Let $\{y_i : i = 2, 3, \ldots, n+2\}$ be the vertex set of a regular spherical simplex P. Let the radius and the centre of the sphere circumscribed about P be r and y_1, respectively. We may choose P in a way that $r = 2\arctan\sqrt{1 - \frac{2}{n}}$. An elementary calculation shows that $\mathfrak{F} = \{\mathbb{S}_S^{n-1}(y_i, r) : i = 1, 2, \ldots, n+2\}$ has the desired property.

$\square$

Chapter 9

Packing by seven homothetic copies

9.1 Preliminaries

The focus of this chapter is systems of "far" points in normed spaces in general, and normed planes in particular. Given $k \geq 3$, we look for sets of k points in an oval C with the minimum pairwise C-distance as large as possible. This is equivalent to packing k congruent homothetic copies of C into C.

More specifically, let C be an oval and $k \geq 2$. A compactness argument yields that there is a maximal value $f_k(C)$ such that C contains k points at pairwise C-distances at least $f_k(C)$. Let

$$f_k = \min\{f_k(C) : C \in \mathfrak{C}\} \quad \text{and} \quad F_k = \max\{f_k(C) : C \in \mathfrak{C}\}.$$

Recall that $\mathfrak{C}$ and $\mathfrak{M}$ denote the family of ovals, and the family of o-symmetric ovals, respectively (cf. Section 2.3). By Blaschke's Selection Theorem, f_k and F_k exist.

Similarly, we may define $g_k(C)$ as the greatest value such that k congruent homothetic copies of C pack into C. We may set

$$g_k = \min\{g_k(C) : C \in \mathfrak{C}\} \quad \text{and} \quad G_k = \max\{g_k(C) : C \in \mathfrak{C}\}.$$

Clearly, these values also exist. The following theorem establishes a straightforward connection between $f_k(C)$ and $g_k(C)$. This connection was stated directly in [38] in 2003. Before 2003, the idea had already appeared in the literature a few times (see, for example, [21] and [39]).

Theorem 9.1.1 (Lassak, Lángi). *Let C be a convex body in $\mathbb{E}^n$ and $k \geq 2$. Then the following are equivalent.*

(i) *C contains k points at pairwise C-distances at least d,*

(ii) *C is packed by k homothetic copies of ratio $\frac{d}{2+d}$.*

We list a few results about the values of $f_k(C)$ and $F_k(C)$ (cf. [20] and [11]). If a and b are consecutive vertices of a convex k-gon P, and $\operatorname{dist}_P(a,b)$ is at least as large (respectively, as small) as the relative length of the edges of a regular k-gon, we say that $[a,b]$ is a *relatively long* (respectively, *relatively small*) edge of P.

Theorem 9.1.2 (Doliwka, Lassak). *Every convex pentagon has a relatively short side and a relatively long side.*

Since the relative length of the edges of a regular pentagon is equal to $\sqrt{5}-1 \approx 1.236$, this result yields that, among any five boundary points of an oval, there are two at a relative distance at most $\sqrt{5}-1$. Clearly, the value $\sqrt{5}-1$ is the least possible.

Theorem 9.1.3 (Böröczky, Lángi). *Among five arbitrary points of an oval, there is a pair at a relative distance at most $\sqrt{5}-1$.*

Theorem 9.1.4 (Böröczky, Lángi). *Among six arbitrary points of an oval, there is a pair at a relative distance at most $2-\frac{2\sqrt{5}}{5} \approx 1.106$.*

In other words, $F_5 = \sqrt{5}-1$ and $F_6 = 2-\frac{2\sqrt{5}}{5}$. Böröczky and Lángi in [11] conjectured that $F_7 = 1$. We verify their conjecture.

Theorem 9.1.5. *Let $C \in \mathfrak{C}$ and let $a_1, a_2, \ldots, a_7$ be points in C. Then $\operatorname{dist}_C(a_i, a_j) \leq 1$ for some $i \neq j$.*

Let us call an oval C *optimal* if it contains seven points at the minimum pairwise relative distance one. In this case, we say that the points *fit* C. The problem is to determine the optimal ovals and the set of points fitting them. We present the following three examples.

A result of Gołąb [27] states that there is an affine regular hexagon H inscribed in C for every $C \in \mathfrak{M}$. The vertices and the centre of H fit C, and hence, C is optimal. Next, any parallelogram P contains many sets of seven points at pairwise P-distances at least one. Any oval $C \subset P$ containing such a set is optimal.

Finally, let $H = [a_1, a_2, \ldots, a_6]$ be a regular hexagon and $S = [b_1, b_2, b_3, b_4]$ be a rectangle circumscribed about H such that $[a_1, a_2] \subset [b_1, b_2]$ and $a_1 \in [b_1, a_2]$. Let c be the centre of H and $m = (b_3 + b_4)/2$. Let $a'_4 \in (b_3, a_4)$ and $a'_5 \in (a_5, b_4)$ such that $|a_4 a'_4| = |a_5 a'_5|$ and let $p \in (c, m)$ (cf. Figure 9.1). Finally, let $C = [a_1, a_2, a_3, a'_4, a'_5, a_6]$. If p is close enough to c, all the pairwise C-distances of the vertices of C and p are at least one.

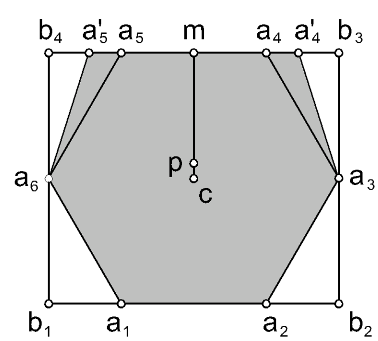

Figure 9.1: An optimal oval

We collect our results about optimal ovals and fitting sets of points.

Theorem 9.1.6. *Let $C \in \mathfrak{C}$ such that $Q = [a_1, a_2, \ldots, a_7] \subset C$ and $\mathrm{dist}_C(a_i, a_j) \geq 1$ for all $i \neq j$.*

9.1.6.1 *If C is strictly convex, then Q is an affine regular hexagon with some a_i as centre.*

9.1.6.2 *If $\mathrm{card}\,(\mathrm{bd}\, Q \cap \{a_1, a_2, \ldots, a_7\}) \neq 6$, then there is a parallelogram P such that $C \subset P$ and $\mathrm{dist}_P(a_i, a_j) \geq 1$ for all $i \neq j$.*

Using Theorem 9.1.1, we may immediately reformulate Theorems 9.1.5 and 9.1.6.

Corollary 9.1.7. *No oval is packed by seven homothetic copies of ratio greater than* $1/3$.

Corollary 9.1.8. *Let $C \in \mathfrak{C}$ be packed by seven homothetic copies of ratio $1/3$ with points $a_1, a_2, \ldots, a_7$ as centres. Let $Q = [a_1, a_2, \ldots, a_7]$.*

9.1.8.1 *If C is strictly convex, then Q is an affine regular hexagon with some a_i as centre.*

9.1.8.2 *If $\mathrm{card}\,(\mathrm{bd}\, Q \cap \{a_1, a_2, \ldots, a_7\}) \neq 6$ then there is a parallelogram P containing C such that P is packed by seven homothetic copies of ratio $1/3$ with $a_1, a_2, \ldots, a_7$ as centres.*

The following lemma is applied in the proofs of Theorems 9.1.5 and 9.1.6 in Sections 9.2 and 9.3. We note that analogous form of 9.1.9.1 has been verified in [37]. Theorem 9.1.5 when $Q = [a_1, a_2 \ldots, a_7]$ is not a hexagon is a consequence of [37] and Lemma 3 of [11]. In that case we prove Theorem 9.1.5 for the sake of Theorem 9.1.6.

Lemma 9.1.9. *Let $C \in \mathfrak{C}$, $k \geq 6$, $Q = [a_1, a_2, \ldots, a_k] \subset C$ be a (possibly degenerate) convex k-gon and $T \subset Q$ be an inscribed triangle of largest area with a side coinciding* *with a side of Q.*

9.1.9.1 *Q has a side of C-length at most one.*

9.1.9.2 *If the C-lengths of the sides of Q are at least one then C is not strictly convex, and there is a parallelogram P such that $C \subset P$ and the sides of Q are of P-length at least one.*

Proof. We may assume that $T = [a_1, a_2, a_i]$ for a suitable value of i. Observe that $(\mathrm{bd}\, Q) \setminus T$ has a component W with at least three edges. We assume that $\{a_2, a_i\} \subset \mathrm{cl}\, W$; that is, $i \geq 5$. As relative distance and area ratio do not change under an affine transformation, we may assume that T is an isosceles triangle with a right angle at a_1. Let b be the point such that $S = [a_1, a_2, b, a_i]$ is a square. Since T is a triangle of maximal area inscribed in Q, we have $a_j \in [a_2, b, a_i]$ for $j = 3, \ldots, i-1$.

Let $m_1 = (a_2 + b)/2$, $m_2 = (b + a_i)/2$ and $m = (a_i + a_2)/2$. If $a_3 \in [a_2, m_1, m] \setminus [m, m_1]$ then $\mathrm{dist}_C(a_2, a_3) \leq \mathrm{dist}_T(a_2, a_3) < 1$. If $a_{i-1} \in [a_i, m, m_2] \setminus [m, m_2]$ then $\mathrm{dist}_C(a_{i-1}, a_i) < 1$. Thus, we may assume that $a_j \in S_0 = [m, m_2, b, m_3]$ for $3 \leq j \leq i - 1$. Then $\mathrm{dist}_C(a_j, a_{j+1}) \leq \mathrm{dist}_T(a_j, a_{j+1}) \leq 1$, and 9.1.9.1 follows.

If for some $3 \leq j \leq i-2$, the points a_j and a_{j+1} are not on parallel sides of S_0 then $\mathrm{dist}_C(a_j, a_{j+1}) \leq \mathrm{dist}_T(a_j, a_{j+1}) < 1$. Let a_j and a_{j+1} be on parallel sides of S_0. Then $i \in \{5, 6\}$.

If $i = 6$, then $a_3 = m_1$, $a_4 = b$ and $a_5 = m_2$, whence $S \subset C$. If $S \neq C$, then $\mathrm{dist}_C(a_i, a_j) < 1$ for some $i \neq j$. If $S = C$, then S satisfies the conditions of (9.1.9.2) for P.

Let $i = 5$ and, say, $a_3 \in [m_1, m]$ and $a_4 \in [m_2, b]$. Let M denote the closed infinite strip containing S and bounded by the lines passing through $[a_1, a_2]$ and

$[b, a_5]$ (cf. Figure 9.2). From $\mathrm{dist}_C(a_2, a_3) \geq 1$, we obtain that $C \subset M$, and hence $\mathrm{dist}_C(a_3, a_4) = 1$. Thus, every parallelogram P circumscribed about C, that has a pair of opposite edges contained in $\mathrm{bd}\, M$, satisfies the conditions in 9.1.9.2. We observe also that C is not strictly convex. $\square$

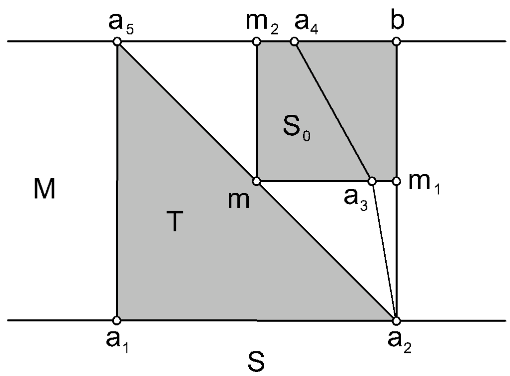

Figure 9.2: An illustration for the proof of Lemma 9.1.9

9.2 Proof of Theorems 9.1.5 and 9.1.6 when $Q = [a_1, a_2, \ldots, a_7]$ is a hexagon

Assume that $Q = [a_1, a_2, \ldots, a_6]$ and $a_7 \in \mathrm{int}\, Q$. Let $a_i = q_i$ for $i = 1, 2, \ldots, 6$, $q_7 = q_1$ and $q_0 = q_6$.

We use the following terms and notations. For any i, j, k, l with $1 \leq i, j, k, l \leq 6$ and $\{i, j\} \neq \{k, l\}$, α_i denotes the angle of Q at q_i, $q_{ij} = (q_i + q_j)/2$, and $L_{ij,kl}$ denotes the straight line passing through $[q_{ij}, q_{kl}]$. We note that $q_i = q_{ii}$, and set $L_{i,kl} = L_{ii,kl}$ and $L_{i,k} = L_{ii,kk}$. In addition, $S_i = [q_i, q_{i+1}]$ for $i = 1, 2, \ldots, 6$ and M_i denotes the maximal chord of Q that is parallel to S_i and with minimal Euclidean distance from S_i.

If $\alpha_{i-1} + \alpha_i + \alpha_{i+1}$ is greater than 2π, equal to 2π or less than 2π, we say that a_i is a *large, normal* or *small vertex of* Q, respectively. Observe that q_i and q_{i+3} are either both normal, or one of them is large and the other one is small.

Note that $\alpha_i + \alpha_{i+1} \leq \pi$ implies that Q is contained in a parallelogram with S_i as a side. From this, it readily follows that there is a triangle T_i inscribed in Q with the property: S_i is a side of T_i and T_i has maximum area of all triangles inscribed in Q. In this case, Theorems 9.1.5 and 9.1.6 follow from Lemma 9.1.9. Accordingly, we assume that the sum of every two consecutive angles of Q is greater than π.

Next, it is a simple matter to check that Q has one of the following properties:

Case 1. Every second vertex of Q *is large.*

Case 2. Q *has three consecutive vertices such that the second one is large and the two other ones are not small.*

Case 3. Q *has three consecutive vertices such that the second one is normal and the two other ones are not small.*

Case 1. Let the large vertices be q_1, q_3 and q_5, and $b_i = q_1 + q_3 + q_5 - 2q_i$ for $i = 1, 3, 5$ (cf. Figure 9.3). Then $Q \subset [b_1, b_3, b_5]$ and every maximal chord of Q passes through q_1, q_3 or q_5. Let Q_i denote the homothetic copy of $\operatorname{int} Q$, with ratio 1/2 and with q_i as centre. Let $P_i = [q_i, q_{(i-1)i}, q_{(i-1)(i+1)}, q_{(i+1)i}]$ for $i = 2, 4, 6$, $T_2 = [q_{13}, q_{14}, q_{36}]$, $T_4 = [q_{35}, q_{36}, q_{25}]$, $T_6 = [q_{15}, q_{25}, q_{14}]$ and $T = \operatorname{int}[q_{14}, q_{25}, q_{36}]$.

We assume that $\operatorname{dist}_Q(q_i, q_{i+1}) \geq 1$ for each i. Then we need only to show that for any $p \in \operatorname{int} Q$,

$$(*)_i \qquad \operatorname{dist}_Q(p, q_i) < 1$$

for some i. Let $p \in \operatorname{int} Q$. We consider the position of p with respect to certain polygons. By symmetry, we assume that $p \in Q_1 \cup P_2 \cup T_2 \cup T$.

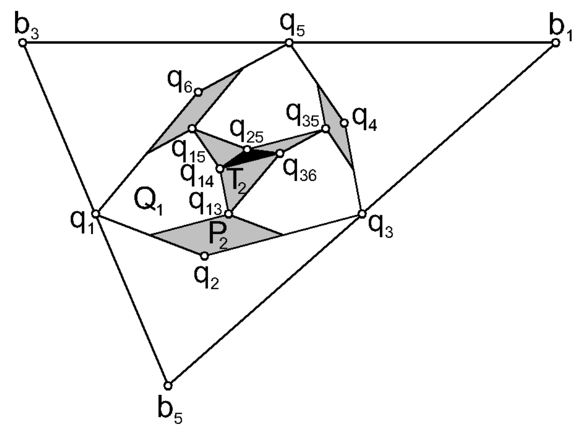

Figure 9.3: Dissecting Q: the orientation of q_{14}, q_{25} and q_{36} is clockwise

We claim that

(1) $(*)_1$ *for* $p \in Q_1$,

(2) $(*)_2$ *for* $p \in P_2$,

(3) $(*)_2$ *or* $(*)_4$ *or* $(*)_6$ *for* $p \in T_2$, and

(4) $(*)_2$ *or* $(*)_4$ *or* $(*)_6$ *for* $p \in T$.

The statement in (1) is trivial. Note that $P_2 \cap \operatorname{int} Q$ is covered by the homothetic copy of $\operatorname{int} Q$, with ratio $1/2$ and with q_2 as centre; whence (2). If $\operatorname{dist}_Q(q_2, q_{14}) < 1$ and $\operatorname{dist}_Q(q_2, q_{36}) < 1$, then (3) is immediate.

Let $\operatorname{dist}_Q(q_2, q_{14}) \geq 1$ and $\operatorname{dist}_Q(q_2, q_{36}) \geq 1$, and set $\{s_1\} = L_{35,25} \cap [q_{13}, q_{15}]$ and $\{s_2\} = L_{15,25} \cap [q_{13}, q_{35}]$. From $\operatorname{dist}_Q(q_1, q_2) \geq 1$ and $\operatorname{dist}_Q(q_2, q_3) \geq 1$, we have that q_2 is in the interior of the parallelogram $[q_{13}, (q_1 + b_5)/2, b_5, (q_3 + b_5)/2]$. Thus the set of points in $[q_{13}, q_{35}, q_{15}]$, at a Q-distance less than one from q_2, is

$[q_{13}, s_1, q_{25}, s_2] \setminus ([s_1, q_{25}] \cup [q_{25}, s_2])$. Similar statements are obtained for q_4 and q_6. Let $\{w_1\} = L_{35,36} \cap [q_{13}, q_{14}]$, $\{w_2\} = L_{15,14} \cap [q_{13}, q_{36}]$ and $\{w\} = [q_{14}, w_2] \cap [q_{36}, w_1]$. As $\text{dist}_Q(q_2, q_{14}) \geq 1$ and $\text{dist}_Q(q_2, q_{36}) \geq 1$, it follows that w_1, w_2 and w exist. Note that if $p \in [q_{13}, w_2, w, w_1]$, or $p \in [q_{14}, w_2, q_{36}] \setminus [w_2, q_{14}]$ or $p \in [q_{14}, w_1, q_{36}] \setminus [w_1, q_{36}]$, then $(*)_2$, $(*)_4$ and $(*)_6$ follow, respectively.

A slight modification of this argument yields our theorems when exactly one of $\text{dist}_Q(q_2, q_{14})$ and $\text{dist}_Q(q_2, q_{36})$ is less than one.

Finally, we verify (4). If $T \cap (T_2 \cup T_4 \cup T_6) \neq \emptyset$ then $T \subset T_2 \cup T_4 \cup T_6$, and our theorems follow from (3). Let $T \cap (T_2 \cup T_4 \cup T_6) = \emptyset$ (cf. Figures 9.3 and 9.4).

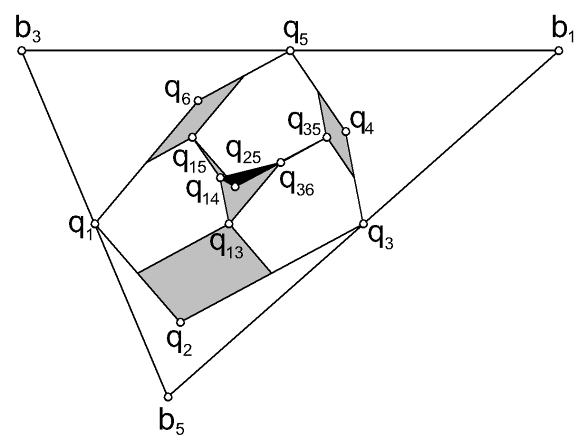

Figure 9.4: Dissecting Q: the orientation of q_{14}, q_{25} and q_{36} is counterclockwise

We distinguish positions of lines, that contain a vertex of T and a side of some T_i. If $L_{15,25} \cap T = L_{35,25} \cap T = \emptyset$, then $(*)_2$. If $L_{15,25} \cap T \neq \emptyset \neq L_{35,25} \cap T$ then $L_{13,14} \cap T = L_{15,14} \cap T = \emptyset$ and $(*)_4$. Accordingly, let $L_{15,25} \cap T \neq \emptyset$ and $L_{35,25} \cap T = \emptyset$, and similarly, $L_{13,14} \cap T \neq \emptyset$, $L_{15,14} \cap Q = \emptyset$, $L_{35,36} \cap T \neq \emptyset$ and $L_{13,36} \cap T = \emptyset$. We show that in this case $\text{dist}_Q(q_i, q_{i+1}) < 1$ for some i; a contradiction.

Recall that the vertex q_i is called "small", if $\alpha_{i-1} + \alpha_i + \alpha_{i+1} < 2\pi$. Observe that for any vector $v \neq 0$, there is a "small" vertex q_i of Q such that the half line $\{q_i + tv : t > 0\}$ intersects Q. For $v = \frac{1}{2}(q_1 + q_3 + q_5 - q_2 - q_4 - q_6)$, we may assume that $\{q_2 + tv : t > 0\}$ intersects Q.

Let H_i denote the open supporting half plane of Q_3 that contains $[q_{3i}, q_{3(i+1)}]$ Let $u = q_{23} + v + (q_{14} - q_{36})$. Since the translates of q_{23}, by v or $q_{14} - q_{36}$, are in $H_1 \cap H_2$, we obtain $u \in H_1 \cap H_2$. Observe that $u = q_{35} + (q_1 - q_6)$, and hence, $u \in H_4 \cap H_5 \cap H_6$. Since q_5 is a large vertex, $u \in H_3$ and $u \in \operatorname{int} Q_5$. It now follows that $\operatorname{dist}_Q(q_1, q_6) < 1$.

Case 2. Let q_2 be large, and q_1 and q_3 be not small. We show that $\operatorname{dist}_C(a_i, a_j) \leq 1$, and if C is strictly convex then $\operatorname{dist}_C(a_i, a_j) < 1$ for some $i \neq j$.

Recall that $S_i = [q_i, q_{i+1}]$ and M_i is the maximal chord of Q that is parallel to S_i and with the minimal Euclidean distance from S_i. Note that, as the sum of any two consecutive angles of Q is greater than π, every maximal chord of Q intersects S_j and S_{j+3} for some $j \in \{1, 2, 3\}$. If M_i intersects S_j and S_{j+3}, we say that M_i is a *j-type maximal chord*. Observe that M_j is not j-type and M_6 is not 3-type. If M_6 and M_3 are 1-type and 2-type, respectively, then we observe that q_2 is not a small vertex; a contradiction. Hence, we have twelve possibilities depending on the types of M_1, M_2, M_3 and M_6. Let $\{d_1\} = L_{5,6} \cap L_{1,2}$, $\{d_2\} = L_{6,1} \cap L_{2,3}$, $\{d_3\} = L_{1,2} \cap L_{3,4}$ and $\{d_4\} = L_{2,3} \cap L_{4,5}$.

i) M_3 and M_6 are 1-type, M_1 is 2-type and M_2 is 3-type.

If $\|q_1 - d_2\| < \|q_1 - q_6\|$ then it follows from the type of M_1 that $\operatorname{dist}_Q(q_1, q_2) < 1$. Similarly, $\|q_2 - d_3\| < \|q_1 - q_2\|$ implies $\operatorname{dist}_Q(q_2, q_3) < 1$, and $\|q_3 - d_4\| < \|q_2 - q_3\|$ implies $\operatorname{dist}_Q(q_3, q_4) < 1$. Let $\|q_1 - d_2\| \geq \|q_1 - q_6\|$, $\|q_2 - d_3\| \geq \|q_1 - q_2\|$ and

$\|q_3 - d_4\| \geq \|q_2 - q_3\|$. Let f_1 be the intersection of $L_{1,2}$ and the line through q_6 that is parallel to $L_{3,4}$, f_2 be the intersection of $L_{2,3}$ and the line through q_1 that is parallel to $L_{3,4}$, and g be the intersection of $L_{1,2}$ and the line through q_4 that is parallel to $L_{1,6}$ (cf. Figure 9.5). Since q_3 is not small and $\|q_3 - d_4\| \geq \|q_2 - q_3\|$, we have $\|q_3 - q_4\| \geq \|q_3 - d_3\|$. From $\|q_2 - d_3\| \geq \|q_1 - q_2\|$, we obtain that $\|q_3 - d_3\| \geq \|q_1 - f_2\|$. As $\|q_1 - d_2\| \geq \|q_1 - q_6\|$ and Q is nondegenerate, we have also $\|q_1 - f_2\| > \|q_6 - f_1\|$. Note that $\|q_6 - f_1\| < \|q_1 - f_2\| \leq \|q_3 - d_3\| \leq \|q_4 - q_3\|$. Hence, $2\|q_6 - f_1\| < \|q_4 - d_3\|$; whence $2\|q_6 - q_1\| < \|q_4 - g\|$. Since M_6 is 1-type, we obtain that $\mathrm{dist}_Q(q_1, q_6) < 1$.

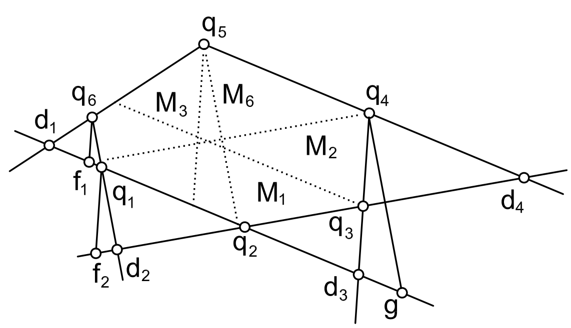

Figure 9.5: An illustration for i) of Case 2 in Section 9.2

If M_3 and M_6 are 2-type, M_1 is 3-type and M_2 is 1-type, then a similar argument yields $\mathrm{dist}_Q(q_{i-1}, q_i) < 1$ for some $i \in \{1, 2, 3, 4\}$.

ii) M_6 and M_1 are 2-type.

Let e_1 denote the intersection of S_5 and the line through q_2 that is parallel to S_6, and e_2 denote the intersection of S_2 and the line through q_6 that is parallel to S_1 (cf. Figure 9.6). Since M_6 and M_1 are 2-type, it follows that e_1 and e_2 exist. Observe that $\|d_1 - q_1\| \leq \|q_1 - q_2\|$ or $\|d_2 - q_1\| \leq \|q_1 - q_6\|$.

Assume that $\|d_1 - q_1\| \leq \|q_1 - q_2\|$. From this, we obtain that $2\|q_1 - q_6\| \leq \|q_2 - e_1\|$, and thus, $\text{dist}_C(q_1, q_6) \leq \text{dist}_Q(q_1, q_6) \leq 1$. We may assume that $\text{dist}_C(q_1, q_6) = 1$, which yields that S_5 and S_2 are parallel, and q_1 is on the line L that divides the infinite strip, bounded by $L_{5,6}$ and $L_{2,3}$, into two congruent strips.

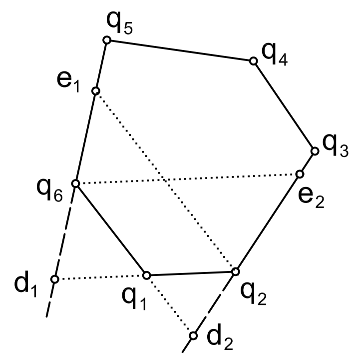

Figure 9.6: An illustration for ii) of Case 2 in Section 9.2

Let L' denote the line passing through q_5 and parallel to S_1. Since q_3 is not a small vertex, L' is a supporting line of Q. Hence, if L does not separate S_5 and q_4, then $\text{dist}_Q(q_4, q_5) < 1$. Accordingly, we may assume that q_4 is in the closed strip bounded by L and $L_{2,3}$. Note that if $q_4 \notin L$, then $\text{dist}_Q(q_2, q_3) < 1$ or $\text{dist}_Q(q_3, q_4) < 1$. Assume that $q_4 \in L$. If $e_1 = q_5$, then $e_2 = q_3$, and q_1, q_2 and q_3 are all normal; a contradiction. If $e_1 \neq q_5$, then consider the lines L_1 and L_2 that support C at the endpoints of a maximal chord of C parallel to S_5, and observe that the parallelogram P with two sides containing S_5 and S_2, and with sidelines L_1 and L_2 satisfies the conditions in Theorem 9.1.5.

If M_1 and M_2 are 3-type, or M_2 and M_3 are 1-type then a similar argument shows our theorems. Hence, we have examined all the possibilities for the types of M_1, M_2, M_3 and M_6.

Case 3. Let q_2 be normal and q_1, q_3 be not small. As $\alpha_1+\alpha_2+\alpha_3=2\pi$, it follows that $L_{6,1}$ and $L_{3,4}$ are parallel. Since relative distance is an affine invariant, we may assume that $[q_1,q_3,q_5]$ is a regular triangle, and also that $\sphericalangle(q_6-q_1,q_3-q_1)\leq\frac{\pi}{2}$.

Let $b_i=a_1+a_3+a_5-2a_i$ for $i=1,3,5$. Since Q is convex and q_1 and q_5 are not small, it follows that $q_2,q_4,q_6\in[b_1,b_3,b_5]$. Let $f=(q_3+b_1)/2$ and L be the line containing q_{13} and that is parallel to $L_{1,6}$.

Subcase 3.1, $q_2\notin L$. Let $d=(q_3+b_5)/2$. We show that if $\mathrm{dist}_Q(q_i,q_{i+1})\geq 1$ for every i then $q_2\in[q_1,b_5,d]\setminus[q_1,d]$, $q_4\in[q_5,f,b_1]$ and $q_6\in[q_{15},q_5,b_3]$. This yields that $\frac{\pi}{6}<\sphericalangle(q_3-q_1,q_2-q_1)$, and $\frac{\pi}{6}\leq\sphericalangle(q_4-q_5,q_3-q_5)$; that is, q_3 is not a large vertex (a contradiction).

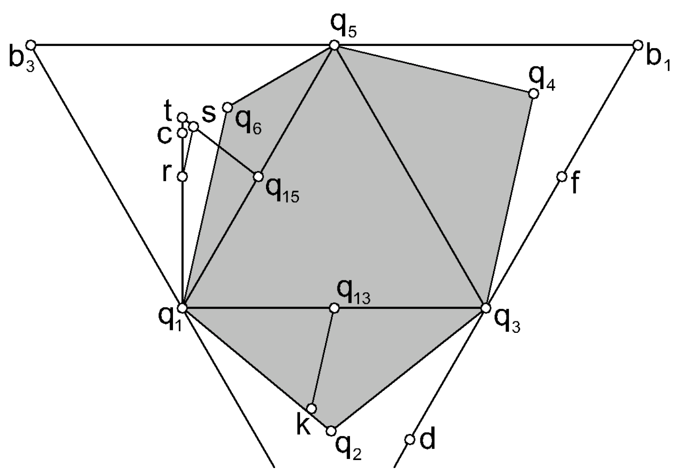

Figure 9.7: An illustration for Subcase 3.1 in Section 9.2

Assume that $\|q_6-q_1\|\leq\|q_4-q_3\|$. Note that, for $i=1,3$, the points of $[q_i,q_{13},k]\setminus[q_{13},k]$ are at Q-distances less than one from q_i. Thus, we may assume that $k\in\operatorname{int}Q$. Let c be the centroid of $[q_1,q_5,b_3]$, $r=q_1+\frac{1}{2}(q_5-q_{13})$ and $s=q_1+\frac{1}{2}(k-q_5)$.

Let t be the intersection point of the line, through q_{15} and s, and the line through q_1 and r; cf. Figure 9.7. Observe that $[q_1, r, s, q_{15}]$ is a homothetic image of $[q_5, q_{13}, k, q_1]$ of ratio $-1/2$, and this implies that $\|q_6 - q_1\| = 4\|s - r\|$. If $[q_1, q_6] \cap [q_{15}, s] = \emptyset$ or $t \in [q_1, c]$, then $\mathrm{dist}_Q(q_1, q_6) < 1$. Let $[q_1, q_6] \cap [q_{15}, s] \neq \emptyset$ and $t \notin [q_1, c]$. Then $q_6 \in [b_3, q_{15}, q_5]$ and $k \in [q_1, d, b_5]$. From this, we obtain that $q_2 \in [q_1, d, b_5] \setminus [q_1, d]$. Since $\|q_4 - q_3\| \geq \|q_6 - q_1\|$, it follows that $q_4 \in [q_5, f, b_1]$.

We argue similarly if $\|q_6 - q_1\| \geq \|q_4 - q_3\|$.

Subcase 3.2, $q_2 \in L$. Observe that there is a maximal chord of Q, passing through q_3 and parallel to S_1. We denote it by M_1'. Similarly, we define M_2' as the maximal chord of Q, passing through q_1 and parallel to S_2. Note that $q_3 \in M_1'$ and $M_1' \cap ((q_1, q_6) \cup S_5) \neq \emptyset$. If $M_1' \cap (q_1, q_6) \neq \emptyset$ then $\mathrm{dist}_C(q_1, q_2) \leq \mathrm{dist}_Q(q_1, q_2) = 1$. Moreover, if $\mathrm{dist}_C(q_1, q_2) = 1$ then M_1' is maximal also in C, and C is not strictly convex. Similarly, if $M_2' \cap (q_3, q_4) \neq \emptyset$ then $\mathrm{dist}_C(q_2, q_3) \leq 1$, and $\mathrm{dist}_C(q_2, q_3) < 1$ or C is not strictly convex. Let $M_1 \cap S_5 \neq \emptyset \neq M_2 \cap S_3$.

Let w be the intersection of $L_{1,6}$ and the line containing M_1, and let $f = (q_3 + b_1)/2$ (cf. Figure 9.8). Observe that $[q_1, q_3, w]$ is a homothetic copy of $[q_1, q_2, q_{13}]$ of ratio -2, and that $2|q_{13}q_2| \geq |q_1q_6|$. Similarly, we obtain that $2|q_{13}q_2| \geq |q_3q_4|$. As in Subcase 3.1, this and $\mathrm{dist}_Q(q_1, q_6) \geq 1$ imply that $q_6 \in [q_{15}, q_5, b_3]$, $q_4 \in [q_5, f, b_1]$ and $\frac{\pi}{6} \leq \sphericalangle(q_2 - q_1, q_3 - q_1)$. Since q_1 is not a small vertex, it follows that $\sphericalangle(q_2 - q_1, q_3 - q_1) = \frac{\pi}{6}$, $q_4 \in [q_5, f]$, $q_6 \in [b_3, q_{15}]$ and $M_1' = M_1 = [q_3, q_6]$.

Let $\{x\} = L_{1,3} \cap L_{4,5}$. Notice that $[q_3, q_4, x]$ is a homothetic copy of $[q_1, q_2, q_{13}]$ of ratio -2, and thus, $\|q_4 - q_3\| = 2\|q_2 - q_{13}\|$. Similarly, $\|q_6 - q_1\| = 2\|q_2 - q_{13}\|$. Observe that $q_1 \in M_5$, $\mathrm{dist}_Q(q_5, q_6) = 1$ and $M_5 \cap S_4 \neq \emptyset$. Let $\{y\} = M_5 \cap S_4$. As $\mathrm{dist}_C(q_5, q_6) \leq \mathrm{dist}_Q(q_5, q_6)$, we may assume that $\mathrm{dist}_C(q_5, q_6) = 1$. In this case,

$[q_1, y]$ is a maximal chord of C. If $y \neq q_4$ then $y \in (q_4, q_5)$ and C is not strictly convex. If $y = q_4$, then Q is a regular hexagon.

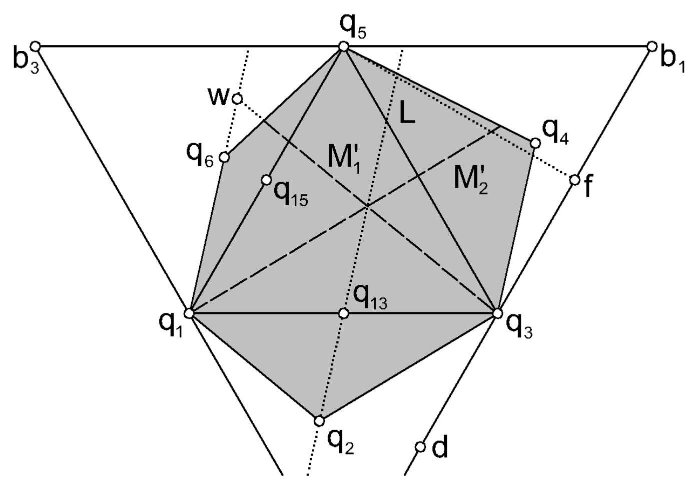

Figure 9.8: An illustration for Subcase 3.2 in Section 9.2

Let c be the centre of Q. If $p \neq c$ is a point of $[q_i, q_{i+1}, c]$ then $\mathrm{dist}_Q(q_i, p) < 1$ or $\mathrm{dist}_Q(q_{i+1}, p) < 1$. Hence, the only point of Q, at a Q-distance at least one from every vertex of Q, is the centre of Q.

The last case is $a_7 \in \mathrm{bd}\, Q$. We regard Q as a degenerate heptagon and prove Theorems 9.1.5 and 9.1.6 in Section 9.3.

9.3 Proof of Theorems 9.1.5 and 9.1.6 when $Q = [a_1, a_2, \dots, a_7]$ is not a hexagon

Let $A = [a_1, a_2, \dots, a_7]$. Note that if Q is a triangle, then Theorems 9.1.5 and 9.1.6 are valid, and if Q is a pentagon with $(\mathrm{bd}\, Q \cap A) = 6$, then Theorem 9.1.6 follows immediately. Thus, we are left with the following possibilities:

Case 1. Q is a (possibly degenerate) heptagon.

Case 2. Q is a quadrilateral.

Case 3. Q is a pentagon with $\mathrm{card}(\mathrm{int}\, Q \cap A) = 2$.

Case 1. We assume that no triangle, of the largest possible area inscribed in Q, has a side that coincides with a side of Q; otherwise, Theorems 9.1.5 and 9.1.6 follow from Lemma 9.1.9.

Let T be a triangle of the largest possible area inscribed in Q with $V(T) \subset V(Q)$. A suitable labelling of the points of A yields that the vertices of Q are $a_1, a_2, \ldots, a_7$ in counterclockwise cyclic order, and $T = [a_1, a_3, a_6]$. Since relative distance is an affine invariant, we may assume that T is a regular triangle. Let $b_i = a_1 + a_3 + a_6 - 2a_i$ for $i = 1, 3, 6$. As T is a triangle of the largest area and Q is convex, we have $a_2 \in [a_1, b_6, a_3]$, $\{a_4, a_5\} \subset [a_3, b_1, a_6]$ and $a_7 \in [a_6, b_3, a_1]$.

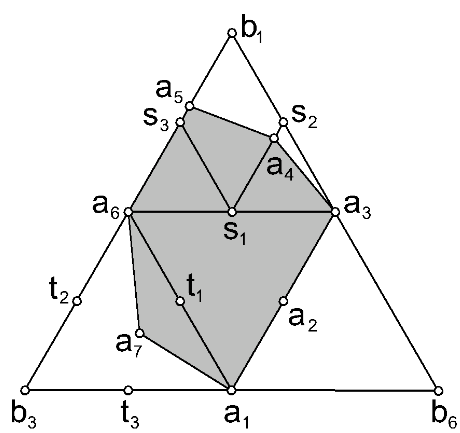

Figure 9.9: Q is a heptagon

Let $s_1 = (a_3 + a_6)/2$, $s_2 = (a_3 + b_1)/2$, $s_3 = (b_1 + a_6)/2$, $t_1 = (a_6 + a_1)/2$, $t_2 = (a_6 + b_3)/2$ and $t_3 = (b_3 + a_1)/2$. Our assertion follows if $\mathrm{dist}_Q(a_3, a_4) < 1$ or

$\mathrm{dist}_Q(a_5, a_6) < 1$, hence we may assume that $\{a_4, a_5\} \subset [s_1, s_2, b_1, s_3]$. Note that the convexity of Q implies $\mathrm{dist}_Q(a_4, a_5) \leq 1$, and thus, Theorem 9.1.5 (cf. Figure 9.9). To prove Theorem 9.1.6, we assume that $\mathrm{dist}_Q(a_i, a_{i+1}) \geq \mathrm{dist}_C(a_i, a_{i+1}) \geq 1$ for every i. Then $\mathrm{dist}_C(a_4, a_5) = 1$, and a_4, a_5 are on parallel sides of the rhombus $[s_1, s_2, b_1, s_3]$. We may assume that $a_4 \in [s_1, s_2]$ and $a_5 \in [b_1, s_3]$.

Let L_1 be the line through a_1 and a_3, and L_2 be the line through b_1 and b_3. Let H_1 and H_2 be the open half planes, containing $\mathrm{int}\, Q$, and bounded by the lines L_1 and L_2, respectively. Observe that there are points $u \in (a_5, a_6)$ and $v \in (a_1, a_3)$ such that $v - u = 2(a_3 - a_4)$. As $\mathrm{dist}_C(a_3, a_4) \geq 1$, it follows that $[u, v]$ is a maximal chord of C, and so, $C \subset H_1 \cap H_2$.

Since $C \subset H_1$, we have $a_2 \in [a_1, a_3]$. Thus, a_2 is the midpoint of $[a_1, a_3]$ and $\mathrm{dist}_C(a_1, a_2) = \mathrm{dist}_C(a_2, a_3) = 1$. As $\mathrm{dist}_C(a_1, a_3) = 2$, there are parallel supporting lines L_3 and L_4 of C passing through a_1 and a_3, respectively. Let $a_1 \in L_3$ and $a_3 \in L_4$. Let P be the parallelogram bounded by L_1, L_2, L_3 and L_4. Clearly, $C \subset P$.

We show that the P-length of every side of Q is at least one. We verify that $\mathrm{dist}_P(a_7, a_1) \geq 1$ and $\mathrm{dist}_P(a_6, a_7) \geq 1$, and note that the other inequalities are immediate.

From $\mathrm{dist}_Q(a_6, a_7) \geq 1$ and $\mathrm{dist}_Q(a_7, a_1) \geq 1$, we have $a_7 \in [t_1, t_2, b_3, t_3]$. This yields $\mathrm{dist}_P(a_7, a_1) \geq 1$. Let x be the vertex of P on $[t_2, a_6]$, and $t = (a_1 + t_2)/2$. Observe that $a_7 \in [t_1, t, t_2]$. If $a_7 \notin [t_1, t] \cup [t, t_2]$ then $\mathrm{dist}_Q(a_6, a_7) < 1$; a contradiction. If $a_7 \in [t_1, t] \cup [t, t_2]$ then $\mathrm{dist}_P(a_6, a_7) = 1$.

Case 2. We may assume that the counterclockwise order of the vertices of Q is a_1, a_2, a_3, a_4. Note that there is a parallelogram S such that $Q \subset S$ and two consecutive sides of S are sides of Q. We may assume that $S = [a_1, a_2, b, a_4]$ for some

$b \in \mathbb{E}^2$.

Let $m_1 = (a_1+a_2)/2$, $m_2 = (a_2+b)/2$, $m_3 = (b+a_4)/2$, $m_4 = (a_4+a_1)/2$ and $m = (a_1 + b)/2$. Since Q is convex, we have $a_3 \notin [a_1, a_2, a_4]$. If $a_3 \in [a_2, m_2, m] \setminus [m_2, m]$ or $a_3 \in [a_4, m, m_3] \setminus [m, m_3]$ then $\mathrm{dist}_C(a_2, a_3) \leq \mathrm{dist}_Q(a_2, a_3) < 1$ or $\mathrm{dist}_C(a_3, a_4) \leq \mathrm{dist}_Q(a_3, a_4) < 1$, respectively. Let $a_3 \in [m, m_2, b, m_3]$.

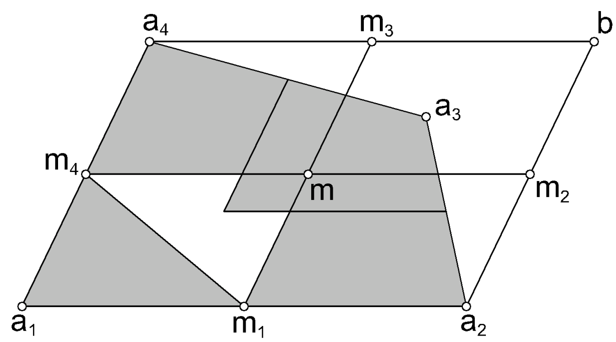

Figure 9.10: An illustration for Case 2 in Section 9.3

Observe that every point of $[a_1, m_1, m_4]$, $[m_1, a_2, m_2, m]$, $[m_2, m, m_3, b] \cap Q$ and $[m_3, a_4, m_4, m]$ are at Q-distances at most one from a_1, a_2, a_3 and a_4, respectively. Moreover, $[m_1, m, m_4]$ does not contain two points at a Q-distance greater than one (cf. Figure 9.10). This proves Theorem 9.1.5. To prove Theorem 9.1.6, we assume that $\mathrm{dist}_C(a_i, a_j) \geq 1$ for any $i \neq j$.

Subcase 2.1, $a_3 \notin [m_2, b] \cup [b, m_3]$. Since $\mathrm{dist}_Q(a_i, a_j) \geq 1$ for $i = 5, 6, 7$, $j = 1, 2, 3, 4$, we have $\{a_5, a_6, a_7\} \subset [m_1, m, m_4] \setminus \{m\}$. This yields that $\mathrm{dist}_Q(i, j) < 1$ for some $\{i, j\} \subset \{5, 6, 7\}$.

Subcase 2.2, $a_3 \in [m_2, b] \cup [b, m_3]$ and $a_3 \neq b$. Let $a_3 \in [m_2, b]$. Let L_1 be the line passing through a_1 and a_4, and L_2 be the line through a_2 and a_3. Let $u = (a_1 + a_3)/2$ and $v = (a_3 + a_4)/2$. Since $\mathrm{dist}_Q(a_i, a_j) \geq 1$ for $i = 5, 6, 7$, $j = 1, 2, 3, 4$, we have

$\{a_5, a_6, a_7\} \subset [m, m_4, u] \cup [m_1, v]$. Since $\mathrm{dist}_Q(a_i, a_j) \geq 1$ for $\{i, j\} \subset \{5, 6, 7\}$, we may assume that $q_5 = m_4$, $q_6, q_7 \in [m_1, v]$, with $2\|a_7 - a_6\| \geq \|a_4 - a_1\|$, and that $||a_6 - m_1|| < ||a_7 - m_1||$.

Note that, from $\mathrm{dist}_C(a_4, a_7) \geq 1$, we have that L_1 and L_2 are supporting lines of C. Let L_3 and L_4 be two parallel supporting lines of C passing through a_1 and a_4, respectively. Then the parallelogram P, with sidelines L_1, L_2, L_3 and L_4, satisfies the conditions in (9.1.6.2).

Subcase 2.3, $a_3 = b$. Since $\mathrm{dist}_Q(a_i, a_j) \geq 1$ for $i = 5, 6, 7$, $j = 1, 2, 3, 4$, we have $\{a_5, a_6, a_7\} \subset [m_1, m3] \cup [m_2, m_4]$. Hence, we may assume that $a_5 = m_4$, $a_6 = m_2$ and $a_7 \in [m_1, m_3]$. Now a consideration similar to that in Subcase 2.2 yields the theorems.

Case 3. Let $V(Q) = \{a_1, a_2, a_3, a_4, a_5\}$. Then $a_6, a_7 \in \mathrm{int}\, Q$. We omit a simple argument that yields that the set $X = \{x \in \mathrm{int}\, Q : \mathrm{dist}_Q(a_i, x) \geq 1 \text{ for } i = 1, 2, 3, 4, 5\}$ does not contain two points at a Q-distance at least one.

Chapter 10

Conclusions

In the proofs presented in the thesis, I used the following methods to attack problems.

Due to the nature of the problems, my main tool was geometry. In particular, geometric transformations and their properties helped much in my research.

For optimization problems, Balázs Csikós taught me the following important principle: "Never compute and differentiate a quantity you want to maximize or minimize. Instead, find a way to compute the derivate directly." In other words, I learned the importance of *linearization*. Chapter 7 shows the difference between the efficiency of the two methods.

Besides linearization, I learned the importance of another area in mathematics: topology. Understanding the topology of a geometric configuration might help much in solving a geometric problem.

Most results have been published. Chapters 5, 6 and 8 appeared in [7]. Chapter 7 appeared in [17]. Chapter 9 appeared in [35]. Chapter 4 is submitted as [19].

Despite my efforts, there are still many open questions regarding the problems I dealt with. The first, naturally, is the following. *Is it possible to give a non-computer-based proof of the Erdős-Szekeres Conjecture for hexagons?*

A way to generalize the questions examined in Chapters 5 and 6 is to define the closed K-spindle of points a and b as the intersection of all the translates of a given convex body K containing a and b. This notion leads to the notion the Überkonvexität defined by Mayer [42]. Thus, one may develop a theory of "convexity" using K-spindles, similar to the one discussed in Chapters 5 and 6. For me, variants of Kirchberger's problem are especially intriguing: How to separate finite point sets with translates or positive homothets of a given convex body? With Márton Naszódi, we have already begun a research in this direction.

In Chapter 7, there is another way to define the area of an outer k_g-polygon: the area of the union of the interior of the underlying polygon and the ears. Similarly, we may define the area of an inner k_g-polygon as the area of the subset of the interior of the underlying polygon that does not belong to any of the ears. For outer k_g-polygons, our theorem still holds with this definition due to the fact that there are no overlapping ears in a regular outer k_g-polygon. Nevertheless, the isoperimetric problem with the alternative area definition is still open for inner k_g-polygons.

Maehara's problem is still open in 3-dimensions. *Does there exist a family $\mathfrak{F}$ of five unit 2-spheres in $\mathbb{E}^3$ such that any four members of $\mathfrak{F}$ have a nonempty intersection, but $\bigcap \mathfrak{F} = \emptyset$?*

The results in Chapter 9 assert that $F_7 = 1$. With this result, the values of F_k are determined for $k = 2, 3, \ldots, 9$. Clearly, one can determine the values of F_k for larger values of k. Nevertheless, the problem of finding f_k is still open even for the first few values of k, and thus, might be more interesting. For the conjectures regarding the values of f_3 and f_4, the interested reader is referred to [38].

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