Periodica Mathematica Hungarica Vol. 39 (1–3), (1999), pp. 139–144

ALMOST EQUIDISTANT POINTS ON S^{D-1}

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Abstract

Let S^{d-1} denote the $(d-1)$-dimensional unit sphere centered at the origin of the d-dimensional Euclidean space. Let $0 < \alpha < \pi$. A set $\mathcal{P}$ of points in S^{d-1} is called almost α-equidistant if among any three points of $\mathcal{P}$ there is at least one pair lying at spherical distance α. In this note we prove upper bounds on the cardinality of $\mathcal{P}$ depending only on d.

1. Introduction

Let S^{d-1} denote the $(d-1)$-dimensional unit sphere centered at the origin $\mathbf{o}$ of the d-dimensional Euclidean space $\mathbf{E^d}$ that is let $S^{d-1} = \{\mathbf{x} \in \mathbf{E^d} | \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$, where $\langle .,. \rangle$ stands for the standard inner product of $\mathbf{E^d}$. Let $0 < \alpha < \pi$. It is an elementary exercise to show that the number of points in S^{d-1} having pairwise distances equal to α is at most $d+1$. This motivates the following notion. A set $\mathcal{P}$ of points in S^{d-1} is called *almost α-equidistant* if among any three points of $\mathcal{P}$ there is at least one pair lying at spherical distance α. (As usual we measure the spherical distance between any two points of S^{d-1} by the length of the shortest geodesic arc connecting the two points.) It is proved in [1] in a very elegant way that the number of almost $\frac{\pi}{2}$-equidistant points in S^{d-1} is at most $2d$. In this paper we prove the following extensions of this result.

THEOREM 1. *For any $\frac{\pi}{2} \le \alpha < \pi$ the number of almost α-equidistant points in S^{d-1} is at most $2d+2$, where $d \ge 2$.*

REMARK 1. The spherical distance between any two vertices of a regular d-dimensional simplex inscribed S^{d-1} is equal to $\alpha_d = 2 \cdot \arcsin\sqrt{(d+1)/(2d)}$. (Notice

Mathematics subject classification numbers, 51F99, 51K99, 52C99, 05A99.

Key words and phrases. Almost equidistant points, spherical geometry.

[1] The author was partially supported by the Hung. Nat. Sci. Found. (OTKA), grant no. T029786 and by the Combinatorial Geometry Project of the Research Found. FKFP0151/1999.

0031-5303/99/\$5.00

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Kluwer Academic Publishers, Dordrecht

that $\frac{\pi}{2} < \alpha_d < \pi$ moreover, $\lim_{d\to+\infty} \alpha_d = \frac{\pi}{2}$.) Thus, if one takes the $2(d+1)$ vertices $\mathcal{V}$ of two regular d-dimensional simplices inscribed S^{d-1}, then among any three vertices of $\mathcal{V}$ there is a pair lying at spherical distance α_d. This shows that the upper bound $2(d+1)$ in Theorem 1 is sharp.

THEOREM 2. *For any positive integer $d \geq 2$ there exists a positive real number $\epsilon(d)$ such that the maximum number of almost α-equidistant points in S^{d-1} with $|\alpha - \frac{\pi}{2}| \leq \epsilon(d)$ is equal to $2d$.*

REMARK 2. Recall that for the angle α_d introduced in Remark 1 we have that $\lim_{d\to+\infty} \alpha_d = \frac{\pi}{2}$. As a result the construction of Remark 1 shows that $\lim_{d\to+\infty} \epsilon(d) = 0$. Finally, if one takes two congruent copies of a regular spherical $(d-1)$-dimensional simplex of spherical edge length $\alpha, 0 < \alpha \leq \alpha_{d-1}$, then the $2d$ vertices of the two spherical simplices form an almost α-equidistant pointset in S^{d-1}. This shows that the upper bound $2d$ in Theorem 2 cannot be improved.

THEOREM 3. *For any $0 < \alpha < \frac{\pi}{2}$ the number of almost α-equidistant points in S^{d-1} is at most $d^2 + d - 2$, where $d \geq 2$.*

REMARK 3. Most likely the upper bound $d^2 + d - 2$ in Theorem 3 can be improved for all $d \geq 2$. Moreover, for any "small" $\alpha > 0$ we have the following construction. Take a regular spherical $(d-1)$-dimensional simplex of spherical edge length α with vertices say, $\mathbf{v}_1, \dots, \mathbf{v}_{d-1}, \mathbf{v}_d$ in S^{d-1}. Then reflect $\mathbf{v}_d$ about the $(d-2)$-dimensional great-sphere of S^{d-1} passing through the vertices $\mathbf{v}_1, \dots, \mathbf{v}_{d-1}$ and denote by $\mathbf{c}$ the point obtained. Finally, let $\mathbf{v}_i^*$ be the rotated copy of the point $\mathbf{v}_i$ about the point $\mathbf{c}$ through the same angle for all $1 \leq i \leq d$ in S^{d-1} such that the spherical distance between $\mathbf{v}_d$ and $\mathbf{v}_d^*$ is equal to α. It is easy to check that the points $\mathbf{c}, \mathbf{v}_1, \dots, \mathbf{v}_{d-1}, \mathbf{v}_d, \mathbf{v}_1^*, \dots, \mathbf{v}_{d-1}^*, \mathbf{v}_d^*$ form an almost α-equidistant pointset of cardinality $2d+1$ in S^{d-1}.

2. Proof of Theorem 1

The proof presented here follows the ideas of [1] with some proper modifications. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a maximal system of unit vectors in $\mathbf{E^d}$ with the property that among any three vectors $\{\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\} \in \mathcal{U}$ there are two say, $\mathbf{u}_i, \mathbf{u}_j$ with $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \cos\alpha$. We now consider the matrix $A = (\langle \mathbf{u}_i, \mathbf{u}_j \rangle - \cos\alpha)_{n\times n}$. Notice that $A = U - \cos\alpha E$, where $U = (\langle \mathbf{u}_i, \mathbf{u}_j \rangle)_{n\times n}$ is the Grammian matrix assigned to the vectors of $\mathcal{U}$ and E the matrix with entries being equal to 1 that is $E = (1)_{n\times n}$. Clearly, the matrices U and E are positive semidefinite matrices. As $\cos\alpha \leq 0$ the matrix A is positive semidefinite as well. Since the rank of U is at most d, it is easy to check that the rank of A is at most $d+1$ and so 0 is an eigenvalue of A with multiplicity at least $n - (d+1)$. As A is positive semidefinite all other eigenvalues of A are positive. Moreover, as the points (i.e. vectors) of $\mathcal{U} \subset S^{d-1}$ form an almost

α-equidistant pointset, for all triples $i \neq j \neq k \neq i$ we have that $a_{ij}a_{jk}a_{ki} = 0$, where $a_{ij} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle - \cos\alpha$. Finally, notice that the main diagonal entries of A are all $1 - \cos\alpha$. Thus, if I denotes the $n \times n$ identity matrix, then the matrix $B = A - (1 - \cos\alpha)I$ with the ij-entry $b_{ij}, 1 \leq i, j \leq n$ has the following properties:

(1) $b_{ii} = 0$ for all $i = 1, \ldots, n$;

(2) $-(1-\cos\alpha)$ is the smallest eigenvalue of B with multiplicity at least $n-(d+1)$;

(3) $b_{ij}b_{jk}b_{ki} = 0$ for all triples $1 \leq i, j, k \leq n$.

Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of B. (2) implies that without loss of generality we may assume that $\lambda_{d+2} = \cdots = \lambda_n = -(1 - \cos\alpha)$ and $\lambda_1 \geq -(1 - \cos\alpha), \ldots, \lambda_{d+1} \geq -(1 - \cos\alpha)$. (1) clearly implies that

$$\sum_{i=1}^{n} \lambda_i = \text{tr } B = 0,$$

where tr B denotes the trace of B. As an immediate result we get that

$$\sum_{i=1}^{d+1} \lambda_i = (n - d - 1)(1 - \cos\alpha). \tag{4}$$

Since, tr $B^3 = \sum_{1 \leq i,j,k \leq n} b_{ij}b_{jk}b_{ki}$, (3) yields that tr $B^3 = 0$. Notice that the eigenvalues of B^3 are $\lambda_1^3, \ldots, \lambda_n^3$ consequently, $\sum_{i=1}^{n} \lambda_i^3 = \text{tr } B^3 = 0$ that is

$$\sum_{i=1}^{d+1} \lambda_i^3 = (n - d - 1)(1 - \cos\alpha)^3. \tag{5}$$

In order to finish the proof of Theorem 1 we need the following lemma which is a somewhat stronger version of the lemma in [1].

LEMMA 1. *Let $\{x_1, \ldots, x_m\}$ be m real numbers with the property that there exists $y > 0$ such that $x_i \geq -y, i = 1, \ldots, m$ and $\sum_{i=1}^{m} x_i = (m + k)y, k \geq 0$. Then*

$$\sum_{i=1}^{m} x_i^3 \geq (m + 3k)y^3.$$

PROOF. If $x_i \geq 0$ for all $1 \leq i \leq m$, then the following is a well-known inequality:

$$(6) \qquad \sqrt[3]{\frac{\sum_{i=1}^{m} x_i^3}{m}} \geq \frac{\sum_{i=1}^{m} x_i}{m}.$$

From (6) it follows in a straightforward way that

$$\sum_{i=1}^{m} x_i^3 \geq \frac{(\sum_{i=1}^{m} x_i)^3}{m^2} = \frac{(m + k)^3 y^3}{m^2} \geq (m + 3k)y^3.$$

Now, we proceed by induction on the number t of indices i for which $x_i < 0$. If $t = 0$, then we are done. If $t > 0$, then without loss of generality we may assume that $x_1 = -ly < 0$ with some $0 < l \leq 1$. Then we replace x_1 by 0 to obtain m real numbers $\{0, x_2, \ldots, x_m\}$ the sum of which is equal to $(m+k+l)y$. The induction hypothesis implies that

$$\sum_{i=2}^{m} x_i^3 \geq [m + 3(k+l)]y^3.$$

Thus, we get that

$$\sum_{i=1}^{m} x_i^3 = (\sum_{i=2}^{m} x_i^3) - l^3y^3 \geq [m + 3(k+l) - l^3]y^3 \geq [m+3k]y^3.$$

This completes the proof of Lemma 1. □

Returning to the proof of Theorem 1 assume that $n > 2(d+1)$. Introducing the notations $k = n - 2(d+1) \geq 1$ and $y = 1 - \cos\alpha > 0$ we can rewrite (4) as follows:

$$\sum_{i=1}^{d+1} \lambda_i = (d+1+k)y.$$

Thus, Lemma 1 implies that

$$\sum_{i=1}^{d+1} \lambda_i^3 \geq (d+1+3k)y^3.$$

Finally, according to (5)

$$\sum_{i=1}^{d+1} \lambda_i^3 = (d+1+k)y^3,$$

a contradiction. This completes the proof of Theorem 1.

3. Proof of Theorem 2

As the proof presented in this section is a properly modified version of the proof of Theorem 1 given in §2 we describe the major steps only without going into details.

Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a maximal system of unit vectors in $\mathbf{E^d}$ with the property that among any three vectors $\{\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\} \in \mathcal{U}$ there are two say, $\mathbf{u}_i, \mathbf{u}_j$ with $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \cos\alpha$, where $|\alpha - \frac{\pi}{2}| \leq \epsilon(d)$ with some sufficiently small $\epsilon(d) > 0$ that will be chosen later. (Notice that as $\epsilon(d) > 0$ is small the angle α is close to $\frac{\pi}{2}$ and so $\cos\alpha$ is close to 0.)

Assume that $n > 2d$. Then let $A = (\langle \mathbf{u}_i, \mathbf{u}_j \rangle)_{(2d+1)\times(2d+1)}$ be the Grammian matrix assigned to the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{2d+1}\}$. Clearly, the matrix A is positive semidefinite of rank at most d and so 0 is an eigenvalue of A with multiplicity at least $(2d+1) - d = d+1$ and all other eigenvalues of A are positive. Finally, if I is the $(2d+1)\times(2d+1)$ identity matrix, then the matrix $B = A - I$ with the ij-entry $b_{ij}, 1 \leq i, j \leq 2d+1$ has the following properties:

(7) $b_{ii} = 0$ for all $i = 1, \dots, 2d+1$;
(8) -1 is the smallest eigenvalue of B with multiplicity at least $d+1$;
(9) $b_{ij}b_{jk}b_{ki}$ is close to 0 for all $1 \le i, j, k \le 2d+1$ if $\epsilon(d) > 0$ is sufficiently small.

Let $\lambda_1, \dots, \lambda_{2d+1}$ denote the eigenvalues of B. (8) implies that without loss of generality we may assume that $\lambda_{d+1} = \dots = \lambda_{2d+1} = -1$ and $\lambda_1 \ge -1, \dots, \lambda_d \ge -1$. (7) clearly implies that

$$\sum_{i=1}^{d} \lambda_i = d+1. \tag{10}$$

Since, tr $B^3 = \sum_{1 \le i,j,k \le 2d+1} b_{ij}b_{jk}b_{ki}$, (9) yields that tr B^3 is close to 0 if $\epsilon(d)$ is sufficiently small that is

$$\sum_{i=1}^{d} \lambda_i^3 \text{ is close to } d+1. \tag{11}$$

Finally, based on (10) Lemma 1 implies that

$$\sum_{i=1}^{d} \lambda_i^3 \ge d+3. \tag{12}$$

As (12) clearly contradicts (11) for any sufficiently small $\epsilon(d)$ the proof of Theorem 2 is complete.

4. Proof of Theorem 3

Let $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an almost α-equidistant pointset on S^{d-1}, with some $0 < \alpha < \frac{\pi}{2}$. Let G be the graph defined on the points of $\mathcal{U}$ as vertices such that two points of $\mathcal{U}$ are connected by an edge if the spherical distance between them is equal to α. Finally, let $f(d-1)$ denote the maximum cardinality of almost α-equidistant pointsets of S^{d-1}.

If the spherical distance between any two points of $\mathcal{U}$ is equal to α, then it is easy to see that $n \le d$ and so we are done. Thus, we are left with the case that there are two points of $\mathcal{U}$ say, $\mathbf{u}_1$ and $\mathbf{u}_2$ lying at spherical distance different from α. This means that there is no edge of G between the vertices $\mathbf{u}_1$ and $\mathbf{u}_2$. Now, let $\mathcal{U}_1$ (resp., $\mathcal{U}_2$) denote the set of the vertices of G that are not connected by an edge to the vertex $\mathbf{u}_1$ (resp., $\mathbf{u}_2$). Moreover, let $\mathcal{U}_3 = \mathcal{U} \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)$. As $\mathcal{U}$ is an almost α-equidistant pointset the graph G restricted to $\mathcal{U}_1$ (resp., $\mathcal{U}_2$) is a complete graph. Thus,

$$\operatorname{card}(\mathcal{U}_1) \le d \quad \text{and} \quad \operatorname{card}(\mathcal{U}_2) \le d. \tag{13}$$

Finally, notice that the vertices of $\mathcal{U}_3$ are connected by an edge to $\mathbf{u}_1$ as well as to $\mathbf{u}_2$. As a result $\mathcal{U}_3$ lies on a $(d-2)$-dimensional great-sphere of S^{d-1}. Hence,

$$\operatorname{card}(\mathcal{U}_3) \le f(d-2). \tag{14}$$

Thus, (13) and (14) imply that

$$n = \text{card}(\mathcal{U}) \leq 2d + f(d-2). \tag{15}$$

(15) immediate yields that

$$f(d-1) \leq 2d + f(d-2). \tag{16}$$

Finally, (16) with $f(1) = 4$ completes the proof of Theorem 3.

ACKNOWLEDGEMENTS. We are indebted to B. Csikós and Gy. Kiss for helpful comments.

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