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Relative Distance of Points of a Convex Body

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Introduction

Finding sets of points on the sphere or in the ball of a Euclidean n-space E^n such that the pairwise distances of the points are as large as possible is a long-standing question of geometry. In particular, problems of this kind are stated by Coxeter, Greening and Graham [5]. The best possible configurations of k points in the disc are found for $k \leq 12$ and for $k = 19$. The proofs are given in papers of Fodor [10] and [11], Graham [14], Kravitz [17] and Pirl [31], and also in the dissertation of Mellisen [28].

A generalization of this problem is presented by Lassak [24], and by Doyle, Lagarias and Randall [8]. The authors of [8] consider points on the boundary of the unit ball C of a Minkowski space, and the distance of the points is measured by the Minkowski distance. In [24] we see a more general approach. Here C is allowed to be an arbitary convex body. The question is finding configurations of points in C, or on the boundary of C, whose pairwise distances are large in the sense of the following notion of C-distance of points.

For arbitrary points $p, q \in E^n$ let $|pq|$ denote the Euclidean length of the segment pq. Let $p'q'$ be a chord of C parallel to pq such that there is no longer chord of C parallel to pq. The *C-distance $d_C(p, q)$ of points p and q* is defined by the ratio of $|pq|$ to $\frac{1}{2}|p'q'|$. We also use the term *C-length of the segment pq*. If there is no doubt about C, we may use the terms *relative distance of p and q*, or *relative length of pq*. Observe that for every $p, q \in E^n$ the C-distance of p and q is equal to their $[\frac{1}{2}(C - C)]$-distance. Thus the metric $d_C(p, q)$ is the metric of the Minkowski space whose unit ball is $\frac{1}{2}(C - C)$.

Let $d_k(C)$ denote the greatest possible number d such that C contains k points in pairwise C-distances at least d. Similarly, let $b_k(C)$ be the greatest possible number d such that the boundary of C contains k points in pairwise C-distances at least d. We are looking for the infima and the suprema of the numbers $d_k(C)$ and $b_k(C)$, where C runs over the family of n-dimensional convex bodies. Compactness arguments show that the above numbers are attained. We also provide an analogous investigation for C restricted to be a centrally symmetric convex body.

In Chapter 1 we find systems of boundary points of an arbitrary convex body in large pairwise relative distances. In Chapter 2 we find points in large pairwise relative distances which are allowed to be also in the interior of the body. In other words, in these two chapters we look for the infima of $b_k(C)$ and $d_k(C)$ for some values of k. The aim of Chapters 3 and 4 is to find upper bounds of the minimum of the pairwise relative distances of k points on the boundary of a convex body, and in the body, respectively. Hence in these two chapters we look for the suprema of $b_k(C)$ and $d_k(C)$. In the first four chapters only plane convex bodies are considered. Chapter 5 deals with the connection between the existence of points of a convex body in possibly large pairwise relative distances, and the existence of large homothetical copies of a convex body packed into, or touching the body.

In Chapter 6 we examine another question. It is a well-known fact that if P is a set of points on the sphere S^{n-1} of E^n such that the Euclidean distances of all pairs of points of P are equal, then the cardinality of P is at most $n+1$. From the paper [12] of Füredi, Lagarias and Morgan we see that the number of points on the boundary of an n-dimensional convex body C having equal pairwise C-distances is at most 2^n. A set P of points is called *almost d-equidistant*, if among every three points of P there exists a pair in the distance d. Rosenfeld [32] proved that if P is a set of almost $\sqrt{2}$-equidistant points on S^{n-1}, then the cardinality of P is at most $2n$. In this chapter we prove that if P is a set of almost d-equidistant points of S^{n-1}, where $\sqrt{2} < d < 2$, then the cardinality of P is at most $2n+2$. We also show estimates about the cardinality of almost d-equidistant sets of points of S^{n-1} for some other values of d.

In this treatise we use the standard notation $\mathcal{C}$ for the family of plane convex bodies, and $\mathcal{M}$ for the family of centrally symmetric plane convex bodies.

Chapter 1

Boundary Points of a Convex Body in Large Relative Distances

Let $k \geq 2$ be an integer. In this chapter we are looking for the greatest possible number d such that every plane convex body contains k boundary points in pairwise relative distances at least d. Remember that $b_k(C)$ denotes the greatest possible number d such that C contains k boundary points in pairwise relative distances at least d. In this chapter we are looking for the infimum of $b_k(C)$, where C runs over the family of plane convex bodies. We also consider an analogous question about centrally symmetric plane convex bodies.

Observe that two points of support in any two parallel opposite supporting lines of a convex body C are of C-distance 2. Apparently, no convex body contains two points of relative distance greater than 2. Thus we immediately see that for two points the infimum that we are looking for is 2.

Bezdek, Fodor and Talata [1] proved that in the boundary of every plane convex body there exist three points in pairwise relative distances at least $\frac{4}{3}$. A conjecture says that in the boundary of every plane convex body there exist three points in pairwise relative distances at least $\frac{1}{2}(\sqrt{5}+1)$ (see, for example, [23] or [24]). This value is attained for the regular pentagon (we examine this example in Chapter 5 in detail). Doliwka [6] proved that the boundary of every plane convex body contains five points in pairwise relative distances at least 1. Observe that no triangle contains four boundary points in pairwise relative distances greater than 1. Hence we see that the estimate 1 cannot be improved even for four points.

Let us denote by $b_k(\mathcal{C})$ the infimum of $b_k(C)$, where C runs over the family of plane convex bodies. Simple compactness arguments show that this infimum is attained. Using our notation we can reformulate the above results in a shorter form. We have $b_2(\mathcal{C}) = 2$, and $\frac{4}{3} \leq b_3(\mathcal{C}) \leq \frac{1}{2}(\sqrt{5}+1)$. Moreover, $b_4(\mathcal{C}) = b_5(\mathcal{C}) = 1$.

Now let us comment the case of centrally symmetric bodies. It is proved in [1] that in the boundary of every centrally symmetric plane convex body there exist three points in pairwise relative distances at least $\frac{3}{2}$. Lassak [26] improved it up to $1 + \frac{\sqrt{3}}{3}$. In [8] and in

[24] it is conjectured that the best possible estimate is $1+\frac{\sqrt{2}}{2}$. This value is attained for the regular octagon. Lassak [24] and Doyle, Lagarias and Randall [8] showed that every centrally symmetric plane convex body contains four boundary points in pairwise relative distances at least $\sqrt{2}$. As it follows from the example of the circle, this estimate cannot be improved. From [8] and from [24] we see that every centrally symmetric plane convex body contains six boundary points in pairwise relative distances at least 1. Obviously, the boundary of the square does not contain five points in pairwise relative distances greater than 1. Hence the estimate 1 is the best possible one for five points and for six points in the boundary of an arbitrary centrally symmetric plane convex body.

Let us denote by $b_k(\mathcal{M})$ the infimum of $b_k(C)$, where C runs over the family of centrally symmetric plane convex bodies. Again, compactness arguments show that the above infimum is attained. Now we have $b_2(\mathcal{M}) = b_2(\mathcal{C}) = 2$, and $1+\frac{\sqrt{3}}{3} \leq b_3(\mathcal{M}) \leq 1+\frac{\sqrt{2}}{2}$. Furthermore, $b_4(\mathcal{M}) = \sqrt{2}$, and $b_5(\mathcal{M}) = b_6(\mathcal{M}) = 1$.

In this chapter first we prove the following result about the relative distances of seven points in the boundary of a plane convex body.

Theorem 1. *The boundary of an arbitrary plane convex body contains seven points in pairwise relative distances at least $\frac{2}{3}$ such that the relative distances of all pairs of successive points are equal.*

The example of a triangle shows that the value $\frac{2}{3}$ in our theorem cannot be increased. As it is explained after Lemma 6 of [27], Lemma 3 of [27] implies that if x is a boundary point of a plane convex body C, and if y moves counterclockwise in the boundary of C from x, then $d_C(x, y)$ does not decrease until it reaches 2, and it accepts all values from the interval $[0, 2]$. Hence, for every positive integer r our theorem implies the existence of $7r$ points on the boundary of an arbitrary plane convex body in pairwise relative distances at least $\frac{2}{3} \cdot \frac{1}{r}$. As mentioned earlier, Theorem of [6] says that every plane convex body contains five boundary points in pairwise relative distances at least 1. Thus, by Lemma 3 of [27] this theorem implies that for every positive integer r on the boundary of every plane convex body there exist $5r$ points in pairwise relative distances at least $\frac{1}{r}$. The example of a triangle shows that this estimate is the best possible one not only for $r = 1$ as proved in [6], but also for $r = 2$.

In the second half of this chapter we improve the estimate $\frac{4}{3}$ from [1] about three far boundary points.

Proposition 1. *In the boundary of every plane convex body there exist three points in equal pairwise relative distances at least* $\frac{1}{5}(2+2\sqrt{6}) \approx 1.3798$.

The above results are presented in [20].

Now we prove Theorem 1. The proof is based on the following lemma.

Lemma 1. *Let* $F = f_1 f_2 \ldots f_7$ *be a convex heptagon. Then every convex heptagon* $D = d_1 d_2 \ldots d_7$ *inscribed in* F *such that* $d_i \in f_i f_{i+1}$ *for* $i = 1, 2, \ldots, 7$, *where* $f_8 = f_1$, *has a side of* F*-length at least* $\frac{2}{3}$.

Proof. Let α_i denote the angle $\angle f_{i-1} f_i f_{i+1}$ $(i = 1, \ldots, 7)$, where $f_0 = f_7$. Since every heptagon is the limit of a sequence of nondegenerate heptagons, it is sufficient to prove our lemma under the assumptions that $\alpha_1 < \pi$, …, $\alpha_7 < \pi$.

First, we wish to show that if the sum of two consecutive angles of F is at most π, then D has a side of F-length at least 1 (see Figure 1).

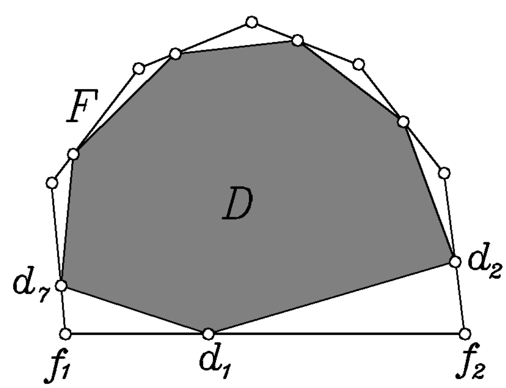

Figure 1

Assume, for example, that $\alpha_1 + \alpha_2 \le \pi$. Observe that in this case $d_F(f_1, f_2) = 2$. As mentioned earlier, Lemma 3 of [27] implies that if x is a boundary point of a plane convex body C, and if y moves counterclockwise in the boundary of C from x, then $d_C(x, y)$ does not decrease until it reaches 2, and it accepts all values from the interval $[0, 2]$. Thus we get that $d_F(d_7, d_1) + d_F(d_1, d_2) \ge d_F(f_1, d_1) + d_F(d_1, f_2) = d_F(f_1, f_2) = 2$, and therefore $d_F(d_7, d_1) \ge 1$ or $d_F(d_1, d_2) \ge 1$.

Next we show that if the sum of every pair of consecutive angles of D is greater than π and if D has three consecutive angles such that their sum is at most 2π, then D has a side of F-length at least $\frac{2}{3}$. Let us assume that $\alpha_1 + \alpha_2 + \alpha_3 \leq 2\pi$. In this case $d_F(f_1, f_3) = 2$. According to Lemma 3 of [27], we get that $d_F(d_7, d_1) + d_F(d_1, d_2) + d_F(d_2, d_3) \geq d_F(f_1, d_1) + d_F(d_1, d_2) + d_F(d_2, f_3) \geq d_(f_1, f_2) = 2$, where the last inequality is a consequence of the triangle inequality. Similarly to the previous consideration, we conclude that at least one of the numbers $d_F(d_7, d_1)$, $d_F(d_1, d_2)$, $d_F(d_2, d_3)$ is at least $\frac{2}{3}$.

Now consider the case when the sum of every three consecutive vertices of D is greater than 2π. Denote the intersection of the lines containing the segments f_2f_3 and f_4f_5 by a_3. Similarly, let a_5 be the intersection point of the lines containing the segments f_5f_6 and f_7f_1 (see Figure 2).

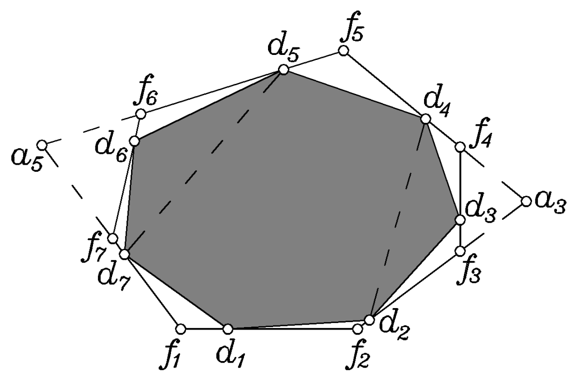

Figure 2

Consider the convex pentagon $D' = d_1d_2d_4d_5d_7$ inscribed in the convex pentagon $F' = f_1f_2a_3f_5a_5$. The angles of F' are $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3 + \alpha_4 - \pi$, $\beta_4 = \alpha_5$, $\beta_5 = \alpha_6 + \alpha_7 - \pi$. This implies that the sum of every two consecutive angles of F' is greater than π. For the sake of simplicity we use the following notation in the sequel: $a_1 = f_1$, $a_2 = f_2$, $a_4 = f_5$, $b_1 = d_1$, $b_2 = d_2$, $b_3 = d_4$, $b_4 = d_5$, $b_5 = d_7$.

We intend to show that the F'-length of b_2b_3 or b_4b_5 is at least $\frac{4}{3}$, or that the F'-length of another side of D' is at least $\frac{2}{3}$. We will show this indirectly. Hence let us assume that $d_{F'}(b_2, b_3) < \frac{4}{3}$, $d_{F'}(b_4, b_5) < \frac{4}{3}$, and that the remaining sides of D' are of F'-length less than $\frac{2}{3}$. Let c_1 and c_1' denote the trisection points of a_1a_2 such that c_1 is closer to a_1 (see Figure 3). Moreover, let c_2, c_3, c_4, c_5 be the trisection points of a_2a_3, a_3a_4, a_4a_5, a_5a_1 closer to the points a_2, a_4, a_4, a_1, respectively.

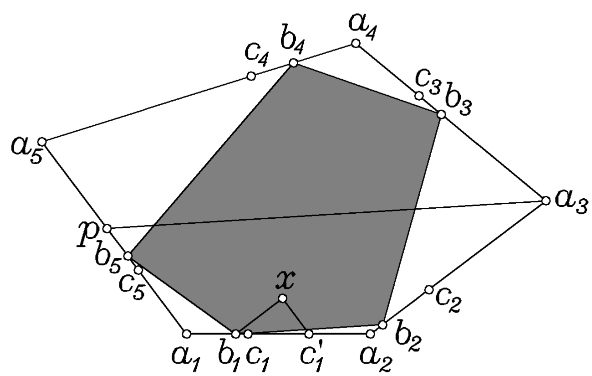

Figure 3

Observe that $d_{F'}(c_1', c_2) = d_{F'}(c_3, c_4) = d_{F'}(c_5, c_1) = \frac{2}{3}$, and that $d_{F'}(c_2, c_3) = d_{F'}(c_4, c_5) = \frac{4}{3}$. Thus, thanks to Lemma 3 of [27], if $b_1 \in a_1c_1'$ and $b_2 \in c_2a_3$, then $d_{F'}(b_1, b_2) \geq \frac{2}{3}$. Similarly, from $b_i \in a_ic_i$ and $b_{i+1} \in c_{i+1}a_{i+2}$, where $a_6 = a_1$, $a_7 = a_2$, $b_6 = b_1$ and $c_6 = c_1$, we get that $d_{F'}(b_i, b_{i+1}) \geq \frac{2}{3}$ if $i = 3$ or if $i = 5$, and that $d_{F'}(b_i, b_{i+1}) \geq \frac{4}{3}$ if $i = 2$ or if $i = 4$. Therefore, with respect to our assumption, b_1 cannot be an inner point of the segment c_1c_1'. Without loss of generality, we can assume that $b_1 \in a_1c_1$ (in the opposite case the proof is analogous). In this case $b_i \in a_ic_i$ for $i = 2, 3, 4, 5$.

Take the common point p of the straight line containing the segment a_5a_1 and of the straight line through a_3 parallel to b_1b_2. Notice that $d_{F'}(b_1, b_2) \geq 2|b_1b_2|/|a_3p|$. Let x be the intersection point of the line through b_1 parallel to a_2a_3 and of the line through c_1' parallel to a_5a_1. Let L_0 be the line containing a_1a_5. Consider the lines L_1, L_2, L_3 and L_4 parallel to a_1a_5, through b_1, c_1', b_2 and c_2, respectively. The relative distance $d_{F'}(c_2, c_1')$ is equal to the ratio of the width of the strip between the lines L_2 and L_4 to the half of the width of the strip between the lines L_0 and L_4. From $d_{F'}(b_1, b_2) \geq 2|b_1b_2|/|a_3p|$ we have that $d_{F'}(b_1, b_2)$ is at least the ratio of the width of the strip between the lines L_1 and L_3 to the half of the width of the strip between the lines L_0 and L_3. Hence $d_{F'}(b_1, b_2) < \frac{2}{3}$ implies that $|xb_1| < |b_2c_2|$. Now consider the triangle $b_1c_1'x$. We have $|b_1c_1'|/\sin(\beta_1 + \beta_2 - \pi) = |xb_1|/\sin(\pi - \beta_1)$. Thus,

$$\sin(\pi - \beta_1)|b_1c_1| < \sin(\pi - \beta_1)|b_1c_1'| < \sin(\beta_1 + \beta_2 - \pi)|b_2c_2|.$$

We omit an analogous calculation that $\sin(\pi - \beta_i)|b_ic_i| < \sin(\beta_i + \beta_{i+1} - \pi)|b_{i+1}c_{i+1}|$ for

$i = 2, 3, 4, 5$, where $\beta_6 = \beta_1$. Hence

$$\prod_{i=1}^{5} \sin \beta_i < \prod_{i=1}^{5} \sin(\beta_i + \beta_{i+1} - \pi).$$

This contradicts Lemma 2 of [18], which says that for every $\beta_1, \ldots, \beta_5 \in (0, \pi)$ such that $\sum_{i=1}^{5} \beta_i = 3\pi$ and $\beta_i + \beta_{i+1} > \pi$ for every $i \in \{1, \ldots, 5\}$, where $\beta_6 = \beta_1$, we have

$$\prod_{i=1}^{5} \sin \beta_i > \prod_{i=1}^{5} \sin(\beta_i + \beta_{i+1} - \pi).$$

We have shown that b_2b_3 or b_4b_5 has F'-length at least $\frac{4}{3}$, or that another side of D' is of F'-length at least $\frac{2}{3}$. As $F \subset F'$, we get that $d_F(s,t) \geq d_{F'}(s,t)$ for every $s, t \in E^2$. Thus, if at least one of the numbers $d_{F'}(b_1, b_2)$, $d_{F'}(b_3, b_4)$ or $d_{F'}(b_5, b_1)$ is at least $\frac{2}{3}$, then we are done. If $d_{F'}(b_2, b_3)$ or $d_{F'}(b_5, b_1)$ is at least $\frac{4}{3}$, then the statement of our lemma is a consequence of the triangle inequality.

∎

Proof of Theorem 1. Let C be an arbitrary plane convex body. Theorem 1 from [27] implies that for every $k \geq 3$ there exists a k-gon inscribed in C whose sides are of equal C-length. Thus, it is sufficient to show that every convex heptagon inscribed in C has a side of C-length at least $\frac{2}{3}$. Consider an arbitrary convex heptagon D inscribed in C. At every vertex of D take a supporting line of C. Let F denote the intersection of the closed halfplanes containing C and bounded by the above supporting lines. Obviously, F is a convex heptagon circumscribed about D such that $D \subset C \subset F$. Observe that the C-length of every side of D is at least its F-length. Therefore our lemma implies that D has a side of C-length at least $\frac{2}{3}$.

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Proof of Proposition 1. Let C be a plane convex body. For the simplicity of considerations, during the proof we denote the value $\frac{1}{5}(1 + \sqrt{6})$ by τ. First we wish to show the existence of three points of C in pairwise C-distances at least 2τ.

According to Lemma 1 from [24] we circumscribe a parallelogram P about C such that the midpoints of two its parallel sides belong to C. As the C-distance of two points does not change under affine transformations, we can assume that P is a rectangle such that the length of the sides containing the mentioned midpoints is 2, and that the length of the other sides is 1. Consider the Cartesian coordinate system such that the above midpoints are $o = (0, 0)$ and $c = (0, 1)$. Since C is inscribed in P, it contains a point $a = (-1, \mu)$ and a point $b = (1, \nu)$, where $0 \le \mu \le 1$ and $0 \le \nu \le 1$.

Case 1, when $\mu + \nu \le \frac{\sqrt{6}}{3}$ or $\mu + \nu \ge 2 - \frac{\sqrt{6}}{3}$. We assume that $\mu + \nu \le \frac{\sqrt{6}}{3}$ (in the other case the proof is analogous). Observe that $\frac{\sqrt{6}}{3} = \frac{1-\tau}{2\tau-1}$. We intend to prove that the quadrangle $obca$ contains points r and s with y-coordinates at most $1 - \tau$ and with the difference of their x-coordinates at least 2τ. As $obca \subset C$, the points r, s and c are three points that we are looking for.

Subcase 1.1, when $\mu \ge 1 - \tau$ and $\nu \ge 1 - \tau$. Since the harmonic mean is not greater than the arithmetic mean, our assumptions imply that $\frac{1}{\mu} + \frac{1}{\nu} \ge \frac{4}{\mu+\nu} > \frac{2\tau}{1-\tau}$. Furthermore, a calculation shows that the intersection of the quadrangle $obca$ with the straight line $y = 1 - \tau$ is a segment of Euclidean length $(1 - \tau)(\frac{1}{\mu} + \frac{1}{\nu})$. Thus this length is at least 2τ. In the part of r and s we take the endpoints of this segment.

Subcase 1.2, when $\mu < 1-\tau$ or $\nu < 1-\tau$. Let $\mu < 1-\tau$ (if $\nu < 1-\tau$, our consideration is analogous). By the assumption of Case 1 we have $\nu \le \frac{1-\tau}{2\tau-1}$. Thus the quadrangle $obca$ contains the point $(2\tau - 1, 1 - \tau)$. We take it in the part of r. As s we take a.

Case 2, when $\frac{\sqrt{6}}{3} < \mu + \nu < 2 - \frac{\sqrt{6}}{3}$. We intend to show that C contains points w and z with the difference of their y-coordinates at least τ, and with their C-distances at least 2τ either from a or from b. Then w, z, and a or b are three promised points.

Let p and q denote the intersections of the straight line $x = -1 + \tau$ with the segments ao and ac, respectively.

Subcase 2.1, when $d_C(p, b) \ge 2\tau$ and $d_C(q, b) \ge 2\tau$. It is clear that $d_C(p, q) = 2\tau$. Thus we take p and q in the part of w and z.

Subcase 2.2, when $d_C(p, b) < 2\tau$ or $d_C(q, b) < 2\tau$. We can assume that $d_C(p, b) < 2\tau$ (in the other case our consideration is analogous). This assumption implies that there exists a point $t \in C$ whose translation u by $\vec{v} = \frac{1}{\tau}\overrightarrow{pb}$ is also a point of C. We intend to show that $g = (-(2\tau - 1), (2\tau - 1)\mu + 2 - 3\tau)$ or $h = (2\tau - 1, (2\tau - 1)\nu + \tau)$ belongs to C (see Figure 4). Suppose instead that $g \notin C$ and $h \notin C$.

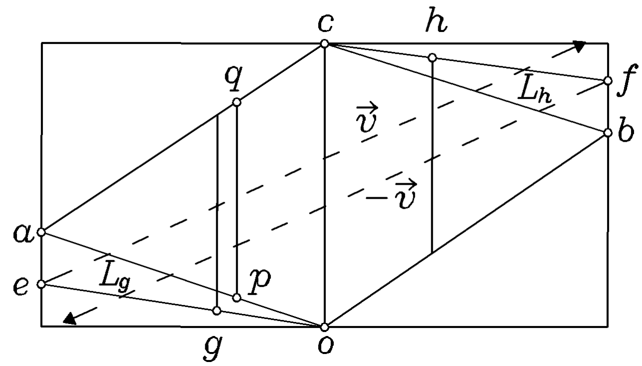

Figure 4

Let L_g be the line through o and g. Its equation is $y = -(\mu - \frac{3\tau-2}{2\tau-1})x$. Denote the right-hand side of this equation by $g(x)$. Let L_h be the line through c and h. Its equation is $y = (\nu - \frac{1-\tau}{2\tau-1})x + 1$. Denote its right-hand side by $h(x)$. Take the common point e of the lines L_g and $x = -1$. We have $e = (-1, \mu - \frac{3\tau-2}{2\tau-1})$. The common point of L_h and the line $x = 1$ is $f = (1, \nu + \frac{3\tau-2}{2\tau-1})$.

Let us denote the x-coordinate of a point m or a vector $\vec{m}$ by m^x, and its y-coordinate by m^y. Notice that $v^x > 1 + h^x = 1 + |g^x|$. This, and the assumption that $g \notin C$ and $h \notin C$ imply that the points t and u belong to the domain bounded by the sides of P and by the lines L_g and L_h. Hence we can take either e in the part of t, or f in the part of u. This depends on the directions of L_g and L_h. Then either of the following holds true.

(i) The translate of e by $\vec{v}$ is in the open half plane containing e bounded by the line L_h. In this case

$$0 > e^y + v^y - h(e^x + xv^x) = (\mu + \nu)(3 - \sqrt{6}) - (\sqrt{6} - 2).$$

Hence from $\mu + \nu > \frac{\sqrt{6}}{3}$ we obtain $0 > 0$, which is a contradiction.

(ii) The translate of f by $-\vec{v}$ is in the open half plane containing f bounded by the line L_g. We get

$$0 < f^y - v^y - g(f^x - v^x) = 7 - 3\sqrt{6} - (\sqrt{6} - 2)(\mu + \nu).$$

From $\mu + \nu > \frac{\sqrt{6}}{3}$ we conclude that the right-hand side of this inequality is negative, which is also impossible.

Thus $g \in C$ or $h \in C$. An easy calculation shows that the intersection of the pentagon $agobc$ and the line $x = -(2\tau - 1)$ is a segment of length τ. Therefore, if $g \in C$, then the intersection of the line $x = -(2\tau - 1)$ and C is a segment of length at least τ. We take the endpoints of this segment as w and z. Since the distance of the lines $x = -(2\tau - 1)$ and $x = 1$ is 2τ, and since $C \subset P$, we conclude that $d_C(b, w) \geq 2\tau$ and $d_C(b, z) \geq 2\tau$.

Similarly, the intersection of the pentagon $aobhc$ and the line $x = 2\tau - 1$ is a segment of length τ. Hence, if $h \in C$, then the intersection of C and the line $x = 2\tau - 1$ is a segment of length at least τ. Now we use the endpoints of this segment in the part of w and z.

We have shown that there exists a triangle in C whose all sides have relative lengths at least $\frac{1}{5}(2 + 2\sqrt{6})$. This permits to apply Theorem 2 of [27], which says that if an arbitrary convex body C contains a k-gon whose all sides are of relative lengths at least d, then there exists a k-gon inscribed in C whose sides are of equal relative length at least d.

$\blacksquare$

Chapter 2

Points of a Convex Body in Large Relative Distances

Let $k \geq 2$ be an integer. In this chapter we are looking for the greatest possible number d such that every plane convex body contains k points in pairwise relative distances at least d. Remember that $d_k(C)$ denotes the greatest possible d such that the convex body C contains k points in pairwise relative distances at least d. In this chapter we are looking for the infimum of $d_k(C)$, where C runs over the family of plane convex bodies. We also consider an analogous question about centrally symmetric plane convex bodies.

Apparently, if the boundary of a convex body contains k points in pairwise relative distances at least d, then also the body contains k points in pairwise relative distances at least d. Putting it in another form, for every $k \geq 2$ and for every convex body C we have $d_k(C) \geq b_k(C)$. Using this and the fact that no convex body contains two points in a relative distance greater than 2 we get that for every convex body C we have $d_2(C) = 2$.

In fact, apart from this trivial observation, the infimum of $d_k(C)$ over the family of plane convex bodies is known only for $k = 5$ and for $k = 8$. For other values of k we know only some estimates. In particular, the problem seems to be very difficult for three and for four points. The discrepancy between the simplicity of the formulation of those two questions and the trouble in finding a solution is especially striking.

To comment the case of three points we use Theorem 2 of [27], which says that if a convex body contains a k-gon whose all sides are of relative lengths at least d, then it permits to inscribe a k-gon whose sides are of equal relative length at least d. Thus $d_3(C) = b_3(C)$ for every plane convex body C. Therefore $d_3(C) \geq \frac{1}{5}(2 + 2\sqrt{6})$ for every $C \in \mathcal{C}$, and $d_3(P) = \frac{1}{2}(\sqrt{5} + 1)$, where P denotes the regular pentagon.

Now we examine the pairwise relative distances of four points in a plane convex body. We conjecture that every plane convex body contains four points in relative distances at least $\sqrt{5} - 1 \approx 1.236$. This value is attained for C being the pentagon $a_1'a_1a_2a_2'a_3$ such that the triangle $a_1a_2a_3$ is isosceles with $\angle a_2a_3a_1 = \frac{\pi}{2}$, and the quadrangle $a_1'a_1a_2a_2'$ is a rectangle with $|a_1a_2| = (\sqrt{5} + 1)|a_1a_1'|$ (see Figure 5).

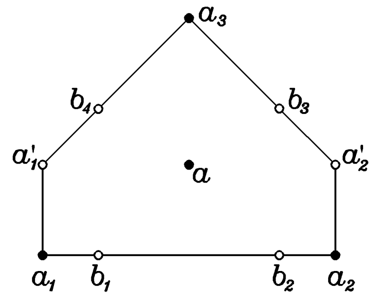

Figure 5

There are two configurations of four points in pairwise relative distances at least $\sqrt{5}-1$ here. The first configuration consists of three points a_1, a_2, a_3 on the boundary of the body and one point a inside of the triangle $a_1a_2a_3$. Here the point a is the midpoint of the segment $a'_1a'_2$. The second configuration consists of four points b_1, b_2, b_3, b_4 on the boundary of the pentagon.

Lassak proved [24] that every plane convex body contains five points in pairwise relative distances at least 1 (this fact also follows from [6]). From the example of the square we see that the estimate 1 is the best possible one. We conjecture that every plane convex body contains also six points in pairwise relative distances at least 1. This value is attained, for example, for triangles and for parallelograms. Moreover, we conjecture that every plane convex body contains seven points in pairwise relative distances at least $\frac{4}{5}$. This value is attained for triangles.

Let us denote by $d_k(\mathcal{C})$ the infimum of $d_k(C)$, where C runs over the family of plane convex bodies. According to the mentioned results and examples we have $d_2(\mathcal{C}) = 2$ and $\frac{1}{5}(2+2\sqrt{6}) \leq d_3(\mathcal{C}) \leq \frac{1}{2}(\sqrt{5}+1)$. Moreover, $d_4(\mathcal{C}) \leq \sqrt{5}-1$, $d_6(\mathcal{C}) \leq d_5(\mathcal{C}) = 1$ and $d_7(\mathcal{C}) \leq \frac{4}{5}$.

Now we summarize the known results about centrally symmetric plane convex bodies. We have mentioned in Chapter 1 that every centrally symmetric plane convex body contains three boundary points in pairwise relative distances at least $1+\frac{\sqrt{3}}{3}$ (see [26]). Thus $d_3(C) \geq 1+\frac{\sqrt{3}}{3}$ for every $C \in \mathcal{M}$. Moreover, $d_3(Q) = 1+\frac{\sqrt{2}}{2}$, where Q denotes the regular octagon.

In Chapter 1 we have written about the estimates $b_4(C) \geq \sqrt{2}$, $b_5(C) \geq b_6(C) \geq 1$ for every $C \in \mathcal{M}$ (see [8] and [24]). Thus we also have $d_4(C) \geq \sqrt{2}$, $d_5(C) \geq d_6(C) \geq 1$ for every $C \in \mathcal{M}$. Moreover, from the examples of the circle and the square we see that these estimates cannot be improved. It is observed in [24] and it also follows from [8] that $d_7(C) \geq 1$ for every $C \in \mathcal{M}$.

Denote by $d_k(\mathcal{M})$ the infimum of $d_k(C)$, where C runs over the family of centrally symmetric plane convex bodies. Using this notation we can put the above results in the following form. We have $d_2(\mathcal{M}) = d_2(\mathcal{C}) = 2$, and $1+\frac{\sqrt{3}}{3} \leq d_3(\mathcal{M}) \leq 1+\frac{\sqrt{2}}{2}$. Furthermore, $d_4(\mathcal{M}) = \sqrt{2}$ and $d_5(\mathcal{M}) = d_6(\mathcal{M}) = d_7(\mathcal{M}) = 1$.

In this chapter first we prove the following estimate about the relative distances of four points in an arbitrary plane convex body.

Theorem 2. *Every plane convex body C contains four points in pairwise C-distances at least $\frac{1}{3}(\sqrt{5}+1) \approx 1.079$.*

This result appeared in the joint paper [22] with M. Lassak.

In the next part we show the following general estimate about the relative distances of k points in an arbitrary plane convex body for certain values of k.

Proposition 2. *Let C be a plane convex body and let $t \geq 2$ be an integer. In C we can find at least $\frac{1}{8}(t^2+4t+q)$ points in pairwise relative distances at least $\frac{4}{t}$, where $q = 3$ for t odd, where $q = 4$ for every even t which is not a multiple of 4, and where $q = 8$ if t is a multiple of 4.*

From Proposition 2 we obtain a number of reasonable estimates for the relative distances of k points in a plane convex body when k is not very large, as follows. We have $d_3(\mathcal{C}) \geq \frac{4}{3}$, $d_4(\mathcal{C}) \geq d_5(\mathcal{C}) \geq 1$, $d_6(\mathcal{C}) \geq \frac{4}{5}$, $d_7(\mathcal{C}) \geq d_8(\mathcal{C}) \geq \frac{2}{3}$, $d_9(\mathcal{C}) \geq d_{10}(\mathcal{C}) \geq \frac{4}{7}$, and $d_{11}(\mathcal{C}) \geq d_{12}(\mathcal{C}) \geq d_{13}(\mathcal{C}) \geq \frac{1}{2}$. Pay attention that for $d_3(\mathcal{C})$ we get nothing else but the estimate from [1] and that for $d_5(\mathcal{C})$ we get again the estimate from [24]. Observe that the above estimate 1 for $d_4(\mathcal{C})$ is weaker than the estimate $\frac{1}{3}(\sqrt{5}+1) \approx 1.079$ from Theorem 2, which is still far from the conjectured value $\sqrt{5}-1 \approx 1.236$. The example of any triangle shows that $d_8(\mathcal{C}) = \frac{2}{3}$.

Let us show a system of eight points in pairwise relative distances at least $\frac{2}{3}$, which is different from that presented in Proposition 2. For this purpose we apply a result of Neumann [30], who proved that every plane convex body C contains a translate of $-\frac{1}{2}C$.

Thus, $\frac{3}{2}C$ contains a translate of $\frac{1}{2}(C-C)$. Hence, C contains a translate of $\frac{1}{3}(C-C)$. In other words, C contains a point in the relative distance at least $\frac{2}{3}$ from every boundary point of C. Theorem 1 in Chapter 1 shows the existence of seven boundary points of C in pairwise relative distances at least $\frac{2}{3}$. Thus C contains eight points in pairwise relative distances at least $\frac{2}{3}$ such that seven of them are boundary points of C. The construction presented in Proposition 2 guarantees that at least two of those points are in the interior of C. We see that (though the estimate $\frac{2}{3}$ cannot be improved) every plane convex body contains at least two different configurations of eight points in pairwise relative distances at least $\frac{2}{3}$.

Finally we examine the case of centrally symmetric plane convex bodies. We show the following estimate.

Claim. *Let C be a centrally symmetric plane convex body and let s be a positive integer. In C we can find at least $3s^2+3s+1$ points in pairwise relative distances at least $\frac{1}{s}$.*

Observe that the thesis of Claim does not depend on the area of C like the estimates in the last section of [8]. From Claim, in particular, we obtain that $d_7(\mathcal{M}) \geq 1$. It also gives the estimates $d_8(\mathcal{M}) \geq \ldots \geq d_{19}(\mathcal{M}) \geq \frac{1}{2}$.

Proposition 2 and Claim appear in the common paper [23] with M. Lassak.

First we prove Theorem 2. During the proof, by the *C-distance of two parallel lines* we mean the minimum C-distance of two points from those lines, respectively. It is easy to see that the C-distance of two parallel lines is nothing else but the ratio of the width of the strip between those lines to the half of the width of C in the perpendicular direction.

Proof of Theorem 2. In our proof we are looking for four points positioned analogously to those in the pentagon shown in Figure 5. If the shape of the body is somehow "similar" to a triangle, the first kind of configuration gives larger relative distances of points, and if not, then the second kind of configuration.

Consider a triangle $T = a_1a_2a_3$ of the largest possible area inscribed in C. Since the C-distance of points does not change under affine transformations, we may assume that T is a regular triangle of sides of length 2 (see Figure 6).

During the proof it is convenient to imagine the direction of the side a_2a_3 as horizontal.

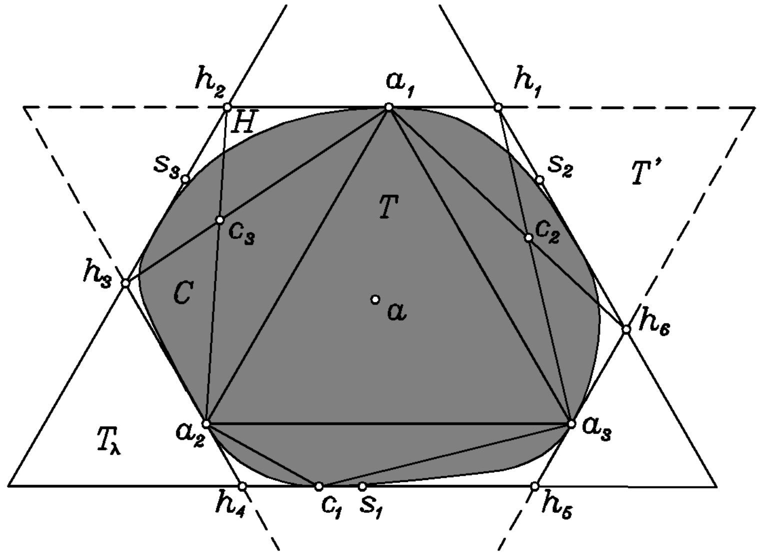

Figure 6

From the maximality of the area we conclude that the straight lines through the vertices parallel to the opposite sides of T are supporting lines of C. By T' we denote the triangle bounded by the above supporting lines. Consider the smallest positive homothetic image T_λ of T which contains C. Here λ denotes the ratio of the homothety which transforms T into T_λ. The intersection of the triangles T_λ and T' is a hexagon $H = h_1h_2h_3h_4h_5h_6$. The notation is chosen such that $a_1 \in h_1h_2$, $a_2 \in h_3h_4$, $a_3 \in h_5h_6$. We denote the common value of $|a_2h_4|$ and $|h_5a_3|$ by x_1, the common value of $|a_3h_6|$ and $|h_1a_1|$ by x_2, and the common value of $|a_1h_2|$ and $|h_3a_2|$ by x_3. Clearly, we have

$$\lambda = \frac{1}{2}(x_1 + x_2 + x_3) + 1.$$

Since $C \subset H$, in order to find four points of C in pairwise C-distances at least $\frac{1}{3}(\sqrt{5}+1)$ it is sufficient to find four points of C in pairwise H-distances at least $\frac{1}{3}(\sqrt{5}+1)$.

We intend to show that the pairwise H-distances of a_1, a_2, a_3 are over $\frac{1}{3}(\sqrt{5}+1)$. Consider the three triangles T_1, T_2, T_3 which are copies of T' under homotheties with centers at the vertices of T' and with ratio $\frac{1}{8}$. The sides of T' are of length 4, and since $\lambda \le \frac{5}{2}$ (see [25]), the sides of T_λ are of length at most 5. As H is contained in T_λ, we conclude that among T_1, T_2 and T_3 there exists at most one such a triangle that H has a point in its interior. Thus, the maximal chords of H parallel to the sides of T are of lengths at most $\frac{7}{2}$. Since the sides of T are of length 2, we see that the H-distances of the vertices of T are at least $\frac{8}{7}$, and thus they are greater than $\frac{1}{3}(\sqrt{5}+1)$.

Case 1, when $\lambda \leq 3\sqrt{5}-5$. Let S be the triangle bounded by segments connecting the centers of sides of T_λ. As S is a homothetic image of T_λ of ratio $-\frac{1}{2}$, it is a homothetic image of T of ratio $-\frac{1}{2}\lambda$. Denote by a the center of the homothety that transforms T into S. Denote the images of the points a_1, a_2, a_3 by s_1, s_2, s_3, respectively. The segments a_1s_1, a_2s_2, a_3s_3 intersect at a. We show that $a \in C$. If $S \cap T = \emptyset$, then T_λ has a side which does not intersect T', which means that C does not intersect this side of T_λ, contrary to the definition of T_λ. So the intersection of T and S is not empty. This and the description of a give $a \in T \cap S$. Since $T \subset C$, we get $a \in C$.

Of course, $\frac{|as_i|}{|aa_i|} = \frac{\lambda}{2}$ for $i = 1, 2, 3$. This implies that

$$\frac{|aa_i|}{\frac{1}{2}|a_is_i|} = \frac{2|aa_i|}{|aa_i| + |as_i|} = \frac{4}{2 + \frac{2|as_i|}{|aa_i|}} = \frac{4}{2+\lambda}$$

for $i = 1, 2, 3$. Thus the H-distances of a from the points a_1, a_2, a_3 are at least $\frac{4}{2+\lambda}$. Hence, those H-distances, and thus also C-distances are at least $\frac{4}{2+3\sqrt{5}-5} = \frac{1}{3}(\sqrt{5}+1)$.

The pairs of points a_1, a_2, a_3 are also in C-distances at least $\frac{1}{3}(\sqrt{5}+1)$ as explained earlier. So the points a_1, a_2, a_3, a of C are in pairwise C-distances at least $\frac{1}{3}(\sqrt{5}+1)$.

Case 2, when $\lambda \geq 3\sqrt{5}-5$. As $\lambda = \frac{1}{2}(x_1+x_2+x_3)+1$, in this case we have $x_1+x_2+x_3 \geq 6\sqrt{5}-12$. We do not make our proof narrower assuming that $x_1 \leq x_2 \leq x_3$.

Denote by c_1 a point of the body C on the side h_4h_5 of the hexagon H, by c_2 the common point of segments a_3h_1 and a_1h_6, by c_3 the common point of segments a_1h_3 and a_2h_2. From the convexity of C we conclude that the hexagon $G = a_1c_3a_2c_1a_3c_2$ is a subset of C.

Before Case 1 we have explained that the H-distance of a_2 and a_3 is greater than $\frac{1}{3}(\sqrt{5}+1)$. Thus there is a horizontal segment S_1 whose endpoints are on the segments a_2c_1 and a_3c_1 in the H-distance $\frac{1}{3}(\sqrt{5}+1)$.

Denote by e the intersection point of a_1c_3 and the horizontal line containing c_2. We show that under the assumption of Case 2 there is also a horizontal segment S_2 whose endpoints are on the segments a_1c_2 and a_1c_3 in the H-distance $\frac{1}{3}(\sqrt{5}+1)$. For this it is sufficient to show that $d_H(c_2, e) \geq \frac{1}{3}(\sqrt{5}+1)$.

Denote by p_2 and by p_3 the points of intersection of the line through a_2 and a_3 with the lines through a_1, h_3, and through a_1, h_6, respectively. Observe that $\frac{x_2}{|a_2p_3|-x_2} = \frac{2-x_2}{x_2}$.

Thus

$$|a_3p_3| = \frac{x_2^2}{2-x_2} + x_2 = \frac{2x_2}{2-x_2} = \frac{4}{2-x_2} - 2.$$

We get analogously that $|a_2p_2| = \frac{4}{2-x_3} - 2$. Hence

$$|p_2p_3| = |p_2a_2| + |a_2a_3| + |a_3p_3| = \frac{4}{2-x_2} + \frac{4}{2-x_3} - 2.$$

Let f be the common point of a_1a_3 and c_2e. As the triangles $a_1a_3c_2$ and $h_1c_2h_6$ are similar, we have $\frac{|a_3c_2|}{|h_1c_2|} = \frac{|a_1a_3|}{|h_1h_6|} = \frac{2}{2-x_2}$. From this we get that $\frac{|a_1f|}{|a_1a_3|} = \frac{|h_1c_2|}{|h_1a_3|} = \frac{1}{1+|a_3c_2|/|h_1c_2|} = \frac{1}{1+2/(2-x_2)} = \frac{2-x_2}{4-x_2}$. Notice that $\frac{|a_1f|}{|a_1a_3|} = \frac{|c_2e|}{|p_2p_3|}$. Therefore

$$|c_2e| = \frac{2-x_2}{4-x_2}\left(\frac{4}{2-x_2} + \frac{4}{2-x_3} - 2\right).$$

Since the opposite sides of H are parallel, we conclude that the longest horizontal segment in H is of length $2+x_2$. Thanks to this a simple calculation gives

$$d_H(c_2,e) = \frac{4}{(2+x_2)(4-x_2)}\left(x_2 + \frac{2(2-x_2)}{2-x_3}\right).$$

As $x_2 \le x_3$, we have $d_H(c_2,e) \ge \frac{4}{4-x_2}$. Hence, if $x_2 > 7-3\sqrt{5}$, then $d_H(c_2,e) > \frac{4}{4-(7-\sqrt{3})} > \frac{1}{3}(\sqrt{5}+1)$. If $x_2 \le 7-3\sqrt{5} < 1$, then $(2+x_2)(4-x_2)$ is maximal for $x_2 = 7-3\sqrt{5}$. Thus $\frac{4}{(2+x_2)(4-x_2)} \ge \frac{4}{(9-3\sqrt{5})(3\sqrt{5}-1)} = \frac{\sqrt{5}+2}{9}$. Moreover, $x_1 \le x_2$ implies that $x_3 \ge 6\sqrt{5} - 12 - 2(7-3\sqrt{5}) = 12\sqrt{5} - 26$. Therefore $\frac{2-x_2}{2-x_3} \ge \frac{2-(7-3\sqrt{5})}{2-(12\sqrt{5}-26)} = \frac{5+3\sqrt{5}}{8}$. From these calculations we conclude that also in the case when $x_2 \le 7-3\sqrt{5}$, we have $d_H(c_2,e) \ge \frac{\sqrt{5}+2}{9} \cdot \frac{5+3\sqrt{5}}{4} > \frac{1}{3}(\sqrt{5}+1)$.

We have shown that under the assumption of Case 2 there is a horizontal segment S_2 whose endpoints are on the segments a_1c_2 and a_1c_3 in the H-distance $\frac{1}{3}(\sqrt{5}+1)$.

We see that the four endpoints of the segments S_1 and S_2 belong to G and thus to C. We intend to show that they are in pairwise H-distances at least $\frac{1}{3}(\sqrt{5}+1)$. Thus it remains to show that the H-distance l of the lines L_1 and L_2 containing the segments S_1 and S_2 is at least $\frac{1}{3}(\sqrt{5}+1)$.

We wish to check the behavior of the H-distance l in dependence on x_2 and x_3, but under the condition that x_1 and x_2+x_3 are fixed.

As x_2+x_3 is fixed, the value $2+x_2$ is maximal for $x_2 = x_3$. From this and from the fact that the H-distance of the endpoints of S_i is fixed for $i=1$ and $i=2$ we get that

$|S_1| = |S_2|$ is maximal for $x_2 = x_3$. So the distance d_1 of the line L_1 and the line through a_2 and a_3 is minimal for $x_2 = x_3$.

We have shown that $|p_2p_3| = \frac{4}{2-x_2} + \frac{4}{2-x_3} - 2$. When $x_2 + x_3$ is fixed, this expression is minimal for $x_2 = x_3$. Consider the distance d_2 of the line L_2 and the line through a_2 and a_3. As $|S_2|$ is maximal and $|p_2p_3|$ is minimal for $x_2 = x_3$, from the triangle $a_1p_2p_3$ we see that d_2 is minimal for $x_2 = x_3$. Remember that also d_1 is minimal for $x_2 = x_3$. Therefore the distance between the straight lines L_1 and L_2 is minimal for $x_2 = x_3$. Since the distance of the horizontal lines through a_1 and c_1 does not change, we conclude that l is minimal for $x_2 = x_3$.

It remains to consider the special case when $x_2 = x_3$. We intend to show that $l \geq \frac{1}{3}(\sqrt{5}+1)$. We have seen that the length of the maximal chord of H in the horizontal direction is $2 + x_2$. Thus $|S_1| = |S_2| = \frac{1}{3}(\sqrt{5}+1)\frac{2+x_2}{2}$. Since $|a_2a_3| = 2$, the ratio of the homothety wich maps a_2a_3 into S_1 is $\frac{1}{3}(\sqrt{5}+1)\frac{2+x_2}{4}$. Therefore

$$d_1 = \frac{\sqrt{3}}{2}x_1\left(1 - \frac{1}{3}(\sqrt{5}+1)\frac{2+x_2}{4}\right).$$

From $x_2 = x_3$ we have $|p_2p_3| = \frac{8}{2-x_2} - 2 = \frac{4+2x_2}{2-x_2}$. This implies that the ratio of the homothety which maps p_2p_3 into S_2 is $\frac{1}{3}(\sqrt{5}+1)\frac{2-x_2}{4}$. Hence

$$d_2 = \sqrt{3}\left(1 - \frac{1}{3}(\sqrt{5}+1)\frac{2-x_2}{4}\right).$$

From the above calculations we get that the distance of the lines L_1 and L_2 is $d_1 + d_2 = \frac{\sqrt{3}}{2}(2+x_1)\left(\frac{1}{6}(5-\sqrt{5}) + \frac{1}{6}(\sqrt{5}+1)\frac{x_2}{2}\cdot\frac{2-x_1}{2+x_1}\right)$. As the width of H in the direction parallel to the lines L_1 and L_2 is $\frac{\sqrt{3}}{2}(2+x_1)$, we have

$$l = \frac{1}{3}(5-\sqrt{5}) + \frac{1}{3}(\sqrt{5}+1)\frac{x_2}{2}\cdot\frac{2-x_1}{2+x_1}.$$

When x_2 decreases and x_1 increases, then l decreases. Hence l becomes minimal for $x_1 = x_2$.

We see that the worst case is when $x_1 = x_2 = x_3$. Thus now $\lambda = \frac{3}{2}x_1 + 1$. By the assumption of Case 2 we have $\lambda \geq 3\sqrt{5}-5$, and by [25] we have $\lambda \leq \frac{5}{2}$. So $2\sqrt{5}-4 \leq x_1 \leq 1$. From the preceding calculation we get that now

$$l = \frac{1}{3}(11+5\sqrt{5}) - \frac{1}{3}(\sqrt{5}+1)\cdot 2\sqrt{2}\cdot\left(\frac{x_1+2}{2\sqrt{2}} + \frac{2\sqrt{2}}{x_1+2}\right).$$

The form of the expression in the parenthesis shows that l is always at least the minimum of its values at the ends of the interval $[2\sqrt{5}-4,\ 1]$ in which x_1 changes. Thus it is always at least $\frac{1}{3}(\sqrt{5}+1)$. We conclude that the four endpoints of the segments S_1 and S_2 are in pairwise H-distances at least $\frac{1}{3}(\sqrt{5}+1)$. Since $C \subset H$, their C-distances are also at least this number.

∎

Proof of Proposition 2. By Lemma 1 from [24] there is a parallelogram P circumscribed about C such that the midpoints of two its parallel sides belong to C (see Figure 7). Denote them by a and c. Let b and d be points of C in the two remaining sides of P. Denote by D the quadrangle $abcd$.

Put $w = t/2$ for t even, and $w = (t-1)/2$ for t odd. We provide segments $S_0, \ldots, S_w$ with endpoints in the boundary of the quadrangle D which are parallel to the segment ac; the line containing S_i should be in the C- distance $4i/t$ from d, where $i = 0, \ldots, w$. So the C-distances of those lines are at least $4/t$.

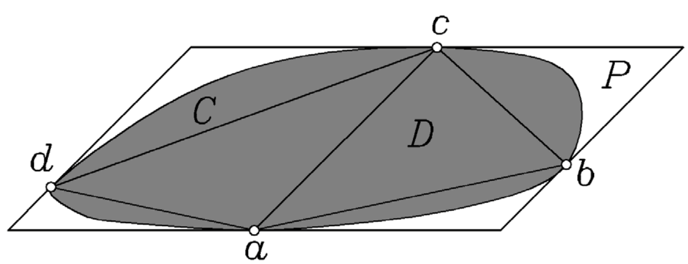

Figure 7

In Figures 8-10 we see the cases when $t = 5$, $t = 6$ and $t = 8$. They illustrate the three cases in Proposition 2. If $4i/t \leq 1$, then S_i contains $k = 2i + 1$ points in pairwise relative distances at least $4/t$. If $4i/t > 1$, then S_i contains $k = 2(w-i)+1$ points in pairwise relative distances at least $4/t$ when t is even (see Fig. 9 and 10), and S_i contains $k = 2(w-i)+2$ points in pairwise relative distances at least $4/t$ when t is odd (see Figure 8).

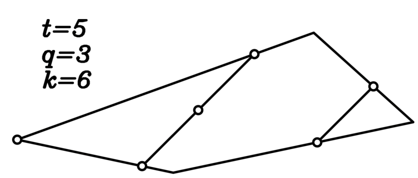

Figure 8

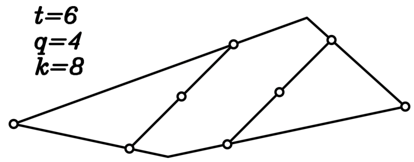

Figure 8

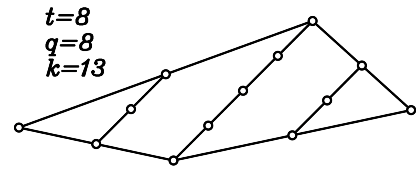

Figure 9

An easy calculation shows that the total number of those points in all the segments $S_0, \ldots, S_w$ is exactly like in the formulation of Proposition 2.

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Proof of Claim. It is well known that we can inscribe in C an affine regular hexagon H (under the assumption of the central symmetry this was proved in many papers; the earliest of them seems to be [13]).

The central symmetry and convexity of C implies that for every diagonal of H there is no longer parallel segment in C. Take a hexagonal configuration of points in H like in Figure 11.

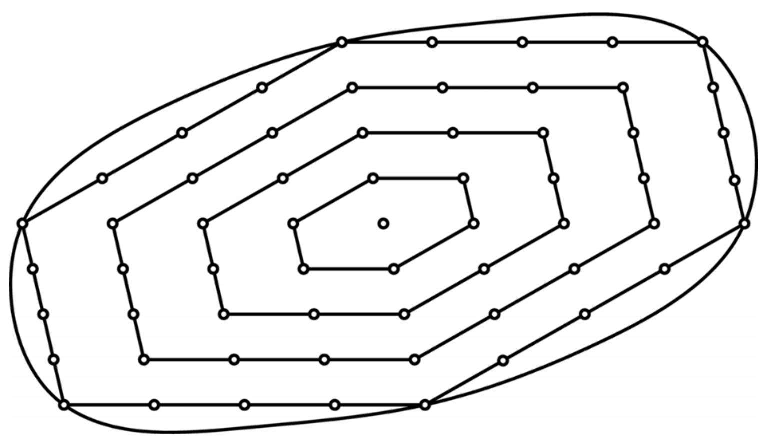

Figure 11

Considering s hexagons containing them on the boundaries we easily evaluate the number of those points: $1 + 6 + \ldots + 6s = 1 + 6 \cdot \frac{s(s+1)}{2} = 3s^2 + 3s + 1$.

∎

Chapter 3

Upper Bounds of the Minimum Relative Distance of Boundary Points of a Convex Body

Let $k \geq 2$ be an integer. In this chapter we are looking for the least upper bound of the minimum pairwise relative distance of k points in the boundary of an arbitrary plane convex body. Remember that $b_k(C)$ denotes the greatest possible number d such that the convex body C contains k boundary points in pairwise relative distances at least d. In this chapter we are looking for the supremum of $b_k(C)$, where C runs over the family of plane convex bodies. We also examine an analogous question about centrally symmetric plane convex bodies.

Since no convex body contains a pair of points in a relative distance greater than 2, we have $b_k(C) \leq 2$ for all values of k and for every plane convex body C. The boundary of the square contains four points in pairwise relative distances 2. Thus the estimate 2 cannot be improved for $k \leq 4$.

Doliwka and Lassak [7] proved that there exists no plane convex body whose boundary contains five points in pairwise relative distances greater than $\sqrt{5}-1$. This value is attained for the regular pentagon and decagon.

According to (266) on page 71 in [13], the circumference of every centrally symmetric plane convex body C measured in the metric $d_C(x, y)$ is at least 6 and at most 8. From Theorem 2 of [9] we see that for every $C \in \mathcal{C}$ the circumferences of C and $\frac{1}{2}(C - C)$ are equal in every Minkowski space. Since for every $C \in \mathcal{C}$ the metric $d_C(x, y)$ is the metric of the Minkowski space whose unit ball is $\frac{1}{2}(C - C)$, we conclude that for every $C \in \mathcal{C}$ the circumference of C in the metric $d_C(x, y)$ is at least 6 and at most 8. Hence for $r \geq 2$ there exists no plane convex body containing $4r$ boundary points in pairwise relative distances greater than $\frac{2}{r}$. Moreover, the boundary of the square contains $4r$ points in pairwise relative distances at least $\frac{2}{r}$.

Let us denote by $c_k(\mathcal{C})$ the supremum of $b_k(C)$, where C runs over the family of plane convex bodies. Using this notation we have $c_k(\mathcal{C}) = 2$ for $k = 2, 3, 4$. Furthermore, we have $c_5(\mathcal{C}) = \sqrt{5} - 1$, and $c_{4r}(\mathcal{C}) = \frac{2}{r}$ for every $r \geq 2$.

Now let us remember the results about centrally symmetric plane convex bodies. Since $\mathcal{M} \subset \mathcal{C}$, the least upper bound of $b_k(C)$ over the family of plane convex bodies is an upper bound of $b_k(C)$ over the family of centrally symmetric plane convex bodies. Moreover, from the examples of the square and of the regular decagon we see that all the mentioned estimates for plane convex bodies are also attained for centrally symmetric ones. Thus the least upper bound of the minimum pairwise relative distance of k boundary points of an arbitrary centrally symmetric plane convex body is 2 for $k = 2, 3, 4$, it is $\sqrt{5} - 1$ for $k = 5$, and it is $\frac{2}{r}$ for $k = 4r$, where $r \geq 2$.

It is shown in [8] that there exists no centrally symmetric plane convex body whose boundary contains six points in pairwise relative distances greater than 1. The value 1 is attained for every $C \in \mathcal{M}$. Since $b_6(C) \geq b_7(C) \geq b_8(C)$ for every convex body C, there is no centrally symmetric plane convex body whose boundary contains seven or eight points in pairwise relative distances greater than 1.

Let $c_k(\mathcal{M})$ denote the infimum of $b_k(C)$, where C runs over the family of centrally symmetric plane convex bodies. Now we can reformulate the mentioned results as follows. We have $c_k(\mathcal{M}) = 2$ for $k = 2, 3, 4$. Besides, $c_5(\mathcal{M}) = \sqrt{5} - 1$, $c_6(\mathcal{M}) = c_7(\mathcal{M}) = 1$, and $c_{4r}(\mathcal{M}) = \frac{2}{r}$ for every $r \geq 2$.

In this chapter we determine the least upper bounds of the minimum pairwise relative distance of six and seven points on the boundary of a plane convex body. These results appear in [19]. Moreover, we conjecture that there exists no plane convex body whose boundary contains nine points in pairwise relative distances greater than $4\sin(10°) \approx 0.6946$. This value is attained for the regular nine-gon and for the regular eighteen-gon. We also conjecture that there exists no plane convex body whose boundary contains ten points in pairwise relative distances greater than $\frac{2}{3}$. The value $\frac{2}{3}$ is attained for the square even for eleven points. In other words, we conjecture that $c_9(\mathcal{C}) = 4\sin(10°)$, and that $c_{10}(\mathcal{C}) = c_{11}(\mathcal{C}) = \frac{2}{3}$. Observe that these values are also attained for centrally symmetric plane convex bodies, namely for the regular eighteen-gon in case of nine points, and for the square in cases of ten and eleven points. Thus we conjecture that $c_9(\mathcal{M}) = 4\sin(10°)$, and that $c_{10}(\mathcal{M}) = c_{11}(\mathcal{M}) = \frac{2}{3}$.

In order to formulate our results about the relative distances of six and seven points in a plane convex body first let us present some elementary observations. Notice that for arbitrary points $p, q \in E^n$ and for arbitrary convex bodies $D \subset C$ we have $d_C(p,q) \leq d_D(p,q)$. Thus to find an upper bound of the minimum pairwise relative distance of k points in an arbitrary plane convex body it is enough to examine convex k-gons with the k points at the vertices of the k-gon. Moreover, according to Lemma 3 of [27], if x is a boundary point of a plane convex body C, and if y moves counterclockwise on the boundary of C from x, then $d_C(x,y)$ is a non-decreasing function until it accepts the value 2, and it accepts all the values from the interval $[0,2]$. Therefore to determine $c_k(\mathcal{C})$ it is enough to examine the relative lengths of the sides of convex k-gons.

Consider the hexagon H_0 which is the convex hull of a regular triangle and its homothetical copy with the homothety center in the center of the triangle with the homothety ratio $1-\sqrt{3}$ (we remark that H_0 is nothing else but the convex hull of the vertices and of the midpoints of the arcs of the Reuleaux triangle). The relative length of the sides of H_0 is $8-4\sqrt{3} \approx 1.071$. Doliwka and Lassak [7] conjectured that every convex hexagon has a side of relative length at most $8-4\sqrt{3}$. First we prove their conjecture and we show that the value $8-4\sqrt{3}$ is attained only for the affine images of H_0.

Theorem 3. *Every convex hexagon H has a side of relative length at most $8-4\sqrt{3}$. Moreover, if the relative length of every side of H is at least $8-4\sqrt{3}$, then H is an affine image of H_0.*

In the remaining part of this chapter we prove a similar statement about convex heptagons.

Theorem 4. *Every convex heptagon has a side of relative length at most 1.*

The example of the degenerated heptagon with four vertices at the vertices of a square and with three remaining vertices at the midpoints of the sides of the square shows that this result is the best possible one.

First we prove Theorem 3. Our proof is based on two lemmas.

Lemma 2. *Let G be a convex k-gon, where $k \geq 6$. Assume that a triangle of the largest possible area inscribed in G has a side which coincides with a side of G. Then G has a side of G-length at most 1.*

Proof. Let $T = abc$ be a triangle mentioned in the formulation of our lemma. Observe that we can assume that ab is a side of G, and that c is a vertex of G. At least one of the two pieces of the boundary of G between a and c contains at least two additional vertices e and f of G. For instance, let e be between c and f on this piece (see Figure 12).

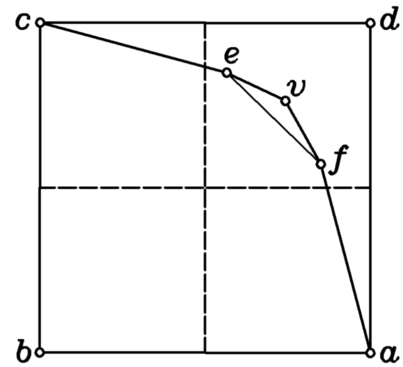

Figure 12

Since the ratio of the areas of two figures does not change under affine transformations, we may assume in our proof that abc is an isosceles triangle with right angle at b. Take the point d such that $S = abcd$ is a square. Since abc is a triangle of maximal area, we conclude that e and f belong to S.

Consider the convex pentagon $P = abcef$. First, we intend to show that at least one of the relative distances $d_P(c, e)$, $d_P(e, f)$, $d_P(f, a)$ is at most 1. We dissect S into four equal squares S_a, S_b, S_c, S_d containing a, b, c, d, respectively. Since G is convex, e and f are not in the interior of T. If $d_P(c, e) > 1$ and $d_P(f, a) > 1$, then $e \notin S_c$ and $f \notin S_a$, and thus $e \in S_d$ and $f \in S_d$. Hence $d_P(e, f) \leq 1$. We see that at least one of the numbers $d_P(c, e)$, $d_P(e, f)$, $d_P(f, a)$ is at most 1.

Finally, we intend to show that if one of the mentioned P-distances is at most 1, then G has a side of G-length at most 1. We assume that $d_P(e, f) \leq 1$ (analogous consideration can be applied for the remaining two cases). Examine the case when e and f are consecutive vertices of G. Since P is a subset of G, we have $d_G(p, q) \leq d_P(p, q)$ for arbitrary points p, q. Thus, in this case the thesis of our lemma holds true. Take into account the opposite case, when e and f are not consecutive vertices, and take a vertex v of G between them. Let V be a side of G with endpoint v. Consider the chords C_a and C_c of G with endpoints a and c, respectively, which are parallel to V. Observe that C_a or C_c is at least twice as long as V. Hence, the G-length of V is at most 1.

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Lemma 3. *Consider a convex hexagon $H = abcdef$ such that the triangle ace is regular. Let us take the lines through a, c, e parallel to the segments ce, ea, ac, respectively. The intersections of these lines are denoted by a_0, c_0, e_0 (they are opposite to a, c, e, respectively). Assume that b, d, f are in the triangle $a_0c_0e_0$ and that the angles $\angle cab, \angle acb, \angle aef, \angle eaf$ are equal to α. Denote the angle $\angle ecd$ by β, and denote the angle $\angle ced$ by γ. If $0 < \alpha < \frac{\pi}{6}$, $0 < \min(\beta, \gamma) < \frac{\pi}{6}$, $d_H(c, d) \geq 8 - 4\sqrt{3}$, and $d_H(d, e) \geq 8 - 4\sqrt{3}$, then $\min(\beta, \gamma) \geq \alpha$ with equality if and only if $\alpha = \beta = \gamma = \frac{\pi}{12}$.*

Proof. We choose a Cartesian coordinate system such that a, c, e are $(0, 0)$, $(1, \sqrt{3})$ and $(-1, \sqrt{3})$, respectively (see Figure 13). Since $d_H(d, e) \geq 8 - 4\sqrt{3}$, d is not in the interior of the homothetical copy C_1 of the quadrangle $cefa$ with the homothety ratio $-(4 - 2\sqrt{3})$ such that the image of c is e. Moreover, also d is not in the interior of the homothetical copy C_2 of the quadrangle $eabc$ with the homothety ratio $-(4 - 2\sqrt{3})$ such that the image of e is c. The boundaries of C_1 and C_2 inside of the triangle ca_0e intersect each other at one point. Denote it by d_0.

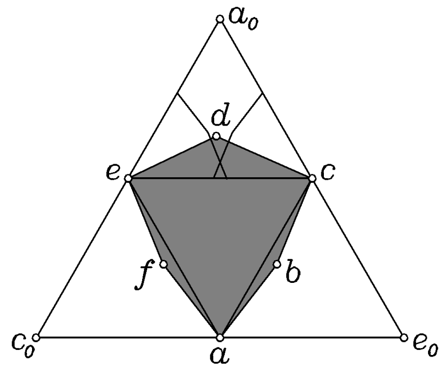

Figure 13

Case 1, when d_0 is on the images of the sides ef and bc. In this case the minimum of β and γ is attained for $d = d_0$. The y-coordinate of d_0 is

$$\tan(\alpha + \frac{\pi}{3})(7 - 4\sqrt{3}) + \sqrt{3}.$$

This implies the inequality

$$\tan(\min(\beta, \gamma)) \geq \tan(\alpha + \frac{\pi}{3})(7 - 4\sqrt{3}).$$

Using elementary trigonometric and algebraic identities we easily get that

$$\tan(\alpha + \frac{\pi}{3})(7 - 4\sqrt{3}) - \tan(\alpha) = \frac{\sqrt{3}(\tan(\alpha) + \sqrt{3} - 2)^2}{1 - \sqrt{3}\tan(\alpha)}.$$

Observe that $\tan(\frac{\pi}{12}) = 2 - \sqrt{3}$. Thus, if $0 < \alpha < \frac{\pi}{6}$, then $\tan(\alpha) \le \tan(\alpha + \frac{\pi}{3})(7 - 4\sqrt{3})$ with equality if and only if $\alpha = \frac{\pi}{12}$. Hence we have $\alpha \le \min(\beta, \gamma)$. Moreover, the equality can hold if and only if $\alpha = \min(\beta, \gamma) = \frac{\pi}{12}$. But when β or γ is equal to $\frac{\pi}{12}$, d_0 is the only point on the segment in the triangle eca_0 determined by the angle $\frac{\pi}{12}$ which is not in the interiors of C_1 and C_2. That is, we have $\beta = \gamma = \frac{\pi}{12}$.

Case 2: when d_0 is on the images of the sides fa and ab. In this case we get the minimum of β when d is d_0 or when d is the homothetic image of a in C_1. Hence $\beta \ge \frac{\pi}{6}$. A similar inequality holds for γ. Therefore $\min(\beta, \gamma) \ge \frac{\pi}{6}$, contrary to the hypothesis.

∎

Proof of Theorem 3. Consider a convex hexagon $H = abcdef$. If a triangle of the largest possible area inscribed in H has a side which coincides with a side of H, then we apply Lemma 2.

Let us look to the opposite possibility, when no triangle of the maximum area inscribed in H contains a side of H. Observe that then ace or bdf is a triangle of maximal area. Consider the first possibility (in the other one, further consideration is analogous). Since the relative distance is affine invariant, we can assume that ace is a regular triangle with vertices $a(0,0)$, $c(1, \sqrt{3})$, $e(-1, \sqrt{3})$ in a rectangular coordinate system. We provide straight lines L_a, L_c, L_e through a, c, e parallel to the segments ce, ea, ac, respectively. Denote the point of intersection of L_c and L_e by a_0. Similarly, let c_0 be the intersection of L_a and L_e. Moreover, let e_0 be the intersection of L_a and L_c. Since ace is a triangle of maximum area inscribed in H, the points b, d and f belong to the triangle $a_0c_0e_0$. Denote the angles $\angle cab, \angle acb, \angle ecd, \angle ced, \angle aef, \angle eaf$ by $\alpha_1, \alpha_2, \dots, \alpha_6$, respectively.

We intend to show that if the relative lengths of the sides of H are at least $8 - 4\sqrt{3}$, then $\alpha_i = \frac{\pi}{12}$ for $i = 1, \dots, 6$.

In further consideration we exclude the case when $\alpha_i = 0$ for a certain i because in this special situation the hexagon contains a closed segment containing three consecutive vertices which means that it has a side of relative length at most 1.

We do not make our consideration narrower assuming that $\alpha_4 = \min\{\alpha_1, \dots, \alpha_6\}$.

Case 1, when $\alpha_4 < \frac{\pi}{6}$. Consider first an auxiliary hexagon H' in which we have α_4 in the place of $\alpha_1, \alpha_2, \alpha_5, \alpha_6$. Then, from $H' \subset H$ we get that $d_H(c,d) \le d_{H'}(c,d)$ and that $d_H(d,e) \le d_{H'}(d,e)$.

Now we apply Lemma 3 for H' putting α_3 in the part of β, and α_4 in the part of α and γ. We get that $\alpha_4 \leq \alpha_4$ with equality if and only if $\alpha_3 = \alpha_4 = \frac{\pi}{12}$. Since α_4 is the minimal angle from among $\alpha_1, \dots, \alpha_6$, all those angles are at least $\frac{\pi}{12}$. Let us take the homothetical copies of the quadrangles $cefa$ and $eabc$ with the homothety ratio $-(4-2\sqrt{3})$ such that the images of c and e are e and c, respectively. Since d is in the interior of neither of the two copies, we get that $\min(\alpha_1, \alpha_2)$ and $\min(\alpha_5, \alpha_6)$ are at most $\frac{\pi}{12}$. Consequently they are equal to $\frac{\pi}{12}$. Now we take an auxiliary hexagon H'' in which α_5 and α_6 are replaced by $\alpha_4 = \frac{\pi}{12}$. We apply Lemma 3 for H'' and we get that $\frac{\pi}{12} \leq \frac{\pi}{12}$ with equality if and only if $\alpha_1 = \alpha_2 = \frac{\pi}{12}$. Thus, we can apply Lemma 3 for H, and as a result we get that $\alpha_i = \frac{\pi}{12}$ for every $i \in \{1, 2, \dots, 6\}$.

It can be easily verified that this hexagon is nothing else but the hexagon H_0 mentioned at the beginning of this chapter.

Case 2, when $\alpha_4 \geq \frac{\pi}{6}$. According to our previous assumption about α_4, all the angles are at least $\frac{\pi}{6}$. Notice that in this case the area of the triangle bdf is not less than the area of the triangle ace, with equality if and only if all the six angles are $\frac{\pi}{6}$. Hence this case concerns only the regular hexagon, whose sides are of relative length 1.

Finally, from the proof we see that if the relative length of every side of H is at least $8-4\sqrt{3}$, then H is an affine image of the hexagon H_0 constructed by Doliwka and Lassak.

$\blacksquare$

Proof of Theorem 4. Let $H = abcdefg$ be a convex heptagon, such that all the relative lengths of its sides are greater than 1. According to Lemma 2 we can assume that acf is a triangle of maximal area inscribed in H. As the relative distance is affine invariant, we can assume that the triangle acf is regular (see Figure 14).

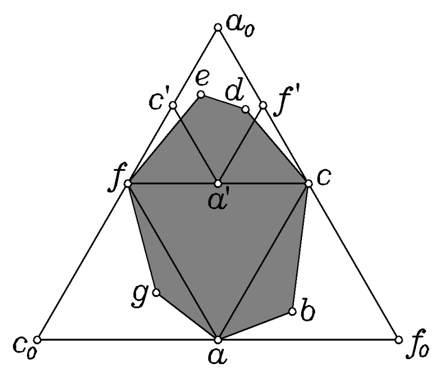

Figure 14

Let us take the Cartesian coordinate system such that a, c, and f are $(0,0)$, $(1,\sqrt{3})$, $(-1,\sqrt{3})$, respectively. We define the points a_0, c_0, f_0 similarly like in the proof of Theorem 3. Since acf is a triangle of maximal area, b, d, e, g are in the triangle $a_0c_0f_0$. Let a', c', f' be the midpoints of the segments cf, a_0f and a_0c, respectively. As $d_H(c,d)$ and $d_H(e,f)$ are greater than 1, the points d and e belong to the rhombus $a'f'a_0c'$. The convexity of H implies that the slope of the segment de is between $-\sqrt{3}$ and $\sqrt{3}$. Hence $d_H(d,e) \leq 1$. But this contradicts the assumption that the relative lengths of all the sides of H are greater than 1. ∎

Chapter 4

Upper Bounds of the Minimum Relative Distance of Points of a Convex Body

Let $k \geq 2$ be an integer. In this chapter we are looking for the least upper bound of the minimum pairwise relative distance of k points in an arbitrary plane convex body. Remember that $d_k(C)$ denotes the greatest possible number d such that C contains k points in pairwise relative distances at least d. In this chapter we are looking for the supremum of $d_k(C)$, where C runs over the family of plane convex bodies. We also consider an analogous problem for centrally symmetric plane convex bodies.

Since no convex body contains points in a relative distance greater than 2, we have $d_k(C) \leq 2$ for every value of k and for every convex body C. Moreover, in the square there exist four points in pairwise relative distance 2. Thus for $k \leq 4$ the least upper bound of the minimum relative distance of k points in an arbitrary plane convex body is 2.

A simple consideration applying the result of [7] leads to the conclusion that there exists no plane convex body which contains five points in pairwise relative distances greater than $\sqrt{5} - 1$. The value $\sqrt{5} - 1$ is attained for the regular pentagon and for the regular decagon.

Considering the area of homothetical copies we get that for any $r \geq 2$ no plane convex body can be packed by its r^2 homothetical copies of ratio greater than $\frac{1}{r}$. Applying Theorem 6 we get that no plane convex body contains r^2 points in pairwise relative distances greater than $\frac{2}{r-1}$. From the example of the square we see that the value $\frac{2}{r-1}$ is attained.

Let us denote by $e_k(\mathcal{C})$ the supremum of $d_k(C)$, where C runs over the family of plane convex bodies. Using this notation we have $e_k(\mathcal{C}) = 2$ for $k \leq 4$, and $e_5(\mathcal{C}) = \sqrt{5} - 1$. Moreover, $e_{r^2}(\mathcal{C}) = \frac{2}{r-1}$ for every $r \geq 2$.

Now we comment the case of centrally symmetric plane convex bodies. Apparently, the supremum of $d_k(C)$ over the family of plane convex bodies is an upper bound of $d_k(C)$ over the family of centrally symmetric plane convex bodies. From the examples of the square and the regular decagon we see that the mentioned least upper bounds for plane convex bodies are also attained for centrally symmetric ones. Thus, the least upper bound of the minimum pairwise relative distance of k points in a centrally symmetric plane convex body is 2 for $k \le 4$, it is $\sqrt{5}-1$ for $k=5$, and it is $\frac{2}{r-1}$ for $k=r^2$, where $r \ge 2$.

In Chapter 3 we have seen that there is no centrally symmetric plane convex body containing six boundary points in pairwise relative distances greater than 1. An elementary consideration shows that there is no centrally symmetric plane convex body containing six points in pairwise relative distances greater than 1, even if the points are not obligatorily on the boundary. Furthermore, the square contains also nine points in pairwise relative distances at least 1.

Denote by $e_k(\mathcal{M})$ the supremum of $d_k(C)$, where C runs over the family of centrally symmetric plane convex bodies. According to the above results and observations we have $e_k(\mathcal{C}) = 2$ for $k \le 4$, and $e_5(\mathcal{M}) = \sqrt{5}-1$. Moreover, $e_6(\mathcal{M}) = e_7(\mathcal{M}) = e_8(\mathcal{M}) = e_9(\mathcal{M}) = 1$, and also $e_{r^2}(\mathcal{M}) = \frac{2}{r-1}$ for every $r \ge 2$.

In this chapter we find the least upper bound of the minimum pairwise relative distance of six points in a plane convex body.

Theorem 5. *No plane convex body contains six points in pairwise relative distances greater than* $2-\frac{2\sqrt{5}}{5} \approx 1.106$. *Furthermore, if* $p_1,\dots,p_6$ *are points in a plane convex body* C *such that all their pairwise relative distances are at least* $2-\frac{2\sqrt{5}}{5}$, *then* C *is an affine regular pentagon, and the points are its vertices and its center.*

Besides, we conjecture that there exists no plane convex body containing seven points in pairwise relative distances greater than 1. Since the square contains even nine points in pairwise relative distances at least 1, we also conjecture an analogous statement about eight points. Theorem 5 and the above conjectures are presented in the joint paper [4] with K. Böröczky.

Now we prove Theorem 5. During the proof we denote points by small Latin letters. In a Cartesian coordinate system, the x-coordinate and the y-coordinate of a point $p \in E^2$ are denoted by p^x and by p^y, respectively. We denote the straight line through the points $p,q \in E^2$ by $L(p,q)$. The value $2-\frac{2\sqrt{5}}{5}$ is denoted by λ, the value $\frac{\lambda}{2} = 1-\frac{\sqrt{5}}{5} \approx 0.553$ by τ, and the value $\frac{\lambda}{2-\lambda} = \sqrt{5}-1$ by ν. By *the kernel of a convex pentagon* P we mean the convex pentagon which is bounded by the diagonals of P.

The proof is based on three lemmas.

Lemma 4. *Take a convex pentagon $P = a_1a_2a_3a_4a_5$ and take a point p in the kernel of P. Denote* $\min\{d_P(p, a_i)|i = 1, \ldots, 5\}$ *by $\lambda(P, p)$. Then $\lambda(P, p) \leq \lambda$, and the equality holds if and only if P is an affine regular pentagon and p is its center.*

Proof. Compactness arguments show that the maximal value of $\lambda(P, p)$ is attained on the family of convex pentagons P and of points p of the kernel of P. Moreover, if P is an affine regular pentagon and if p is its center, then $\lambda(P, p)$ is equal to λ. Hence it is enough to show that if P is not affine regular or if P is affine regular but p is not its center, then $\lambda(P, p)$ cannot be maximal. During the proof we denote the intersection point of the line $L(p, a_i)$ and of the segment $a_{i+2}a_{i+3}$ by b_i, for $i = 1, \ldots, 5$. Moreover, we denote the kernel of P by Q.

Observe that if p is on the boundary of Q, then $\lambda(P, p) \leq 1$, which is less than λ. Thus in this case $\lambda(P, p)$ cannot be maximal. Therefore in the sequel we assume that p is in the interior of Q.

Case 1, when P has a side of P-length 2. For instance, let a_1a_2 be such a side. Instead of the condition that p is in the kernel of P, during the proof in this case we use only the facts that $p \in a_1a_3a_5$ and $p \in a_2a_3a_5$. For $i = 1, \ldots, 5$ let us denote a maximal chord of P parallel to a_ip by u_iv_i. As $|u_iv_i| \geq |a_ib_i|$, we have $d_P(a_i, p) = \frac{|a_ip|}{\frac{1}{2}|u_iv_i|} \leq \frac{2|a_ip|}{|a_ib_i|}$.

If $|a_5p| > |pb_5|$ and if $|a_3p| > |pb_3|$, then $L(a_1, a_2)$ separates p and the intersection point of $L(a_1, a_5)$ and $L(a_2, a_3)$. Thus $d_P(a_1, a_2) < 2$, which is a contradiction. If $|a_3p| \leq |pb_3|$, then $d_P(a_3, p) \leq \frac{2|a_3p|}{|a_3b_3|} = \frac{2}{1+\frac{|pb_3|}{|a_3p|}} \leq 1$. Hence $\lambda(P, p) \leq 1$. Similarly, if $|a_5p| \leq |pb_5|$, then $\lambda(P, p) \leq d_P(a_5, p) \leq 1$.

Case 2, when P has no side of P-length 2. In this case $d_P(p, a_i) = \frac{2|a_ip|}{|a_ib_i|}$ for $i = 1, \ldots, 5$.

Subcase 2.1, when P has two consecutive vertices in P-distance from p greater than $\lambda(P, p)$. Assume, for example, that $d_P(a_4, p) > \lambda(P, p)$ and that $d_P(a_5, p) > \lambda(P, p)$ (see Figure 15). For $i = 1, \ldots, 5$ let us denote by H_i the open halfplane bounded by the line through p parallel to $a_{i+2}a_{i+3}$ such that $a_i \notin H_i$. Observe that if p' is in H_i, then $d_P(a_i, p) < d_P(a_i, p')$. Let $H = H_1 \cap H_2 \cap H_3$. Notice that $H' = H \cap \mathrm{intQ}$ is a nonempty open set, and that p is a boundary point of H'. If p' is a point of H', then $d_P(a_i, p') > d_P(a_i, p) \geq \lambda(P, p)$ for $i = 1, 2, 3$. Moreover, if p' is close enough to p, then $d_P(a_j, p') > \lambda(P, p)$ for $j = 4, 5$. Thus, we can choose a point $p' \in \mathrm{int}Q$ such that $\lambda(P, p) < \lambda(P, p')$. Hence $\lambda(P, p)$ cannot be maximal.

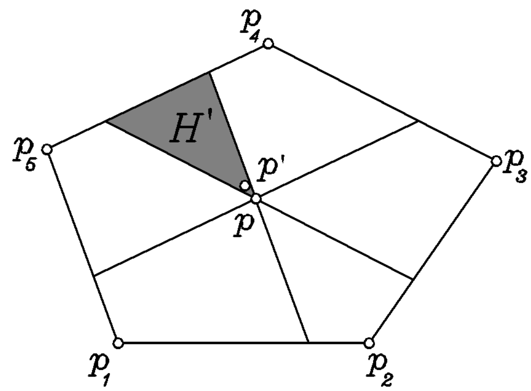

Figure 15

Subcase 2.2, when P has exactly two vertices in P-distance from p greater than $\lambda(P,p)$, and these vertices are nonconsecutive. Without loss of generality, let $d_P(a_1,p) = d_P(a_2,p) = d_P(a_4,p) = \lambda(P,p)$, $d_P(a_3,p) > \lambda(P,p)$ and $d_P(a_5,p) > \lambda(P,p)$. Take the convex pentagon $P' = a_1a_2a_3a_4a_5'$, where a_5' is an interior point of the segment a_5a_1. We have $d_{P'}(a_1,p) = d_P(a_1,p) = \lambda(P,p)$, $d_{P'}(a_2,p) > d_P(a_2,p) = \lambda(P,p)$, $d_{P'}(a_3,p) = d_P(a_3,p) > \lambda(P,p)$, $d_{P'}(a_4,p) = d_P(a_4,p) = \lambda(P,p)$, $d_{P'}(a_5',p) < d_P(a_5,p)$. Moreover, if a_5' is close enough to a_5, then $d_{P'}(a_5',p) > \lambda(P,p)$, and p is in the kernel of P'. Hence, according to Subcase 2.1, there exists a point p' in the kernel of P' such that $\lambda(P,p) = \lambda(P',p) < \lambda(P',p')$.

Subcase 2.3, when P has exactly one vertex in P-distance from p greater than $\lambda(P,p)$. Let this vertex be a_5. Take the convex pentagon $P^* = a_1a_2a_3a_4a_5^*$, where a_5^* is an interior point of the segment a_5a_1. We have that $d_{P^*}(a_1,p) = d_P(a_1,p) = \lambda(P,p)$, $d_{P^*}(a_2,p) > d_P(a_2,p) = \lambda(P,p)$, $d_{P^*}(a_3,p) = d_P(a_3,p) = \lambda(P,p)$, $d_{P^*}(a_4,p) = d_P(a_4,p) = \lambda(P,p)$, $d_{P^*}(a_5^*,p) < d_P(a_5,p)$. Moreover, if a_5^* is close enough to a_5, then $d_{P^*}(a_5^*,p) > \lambda(P,p)$. Hence, thanks to Subcase 2.2, there exist a convex pentagon P' and a point p' in the kernel of P' such that $\lambda(P,p) = \lambda(P^*,p) < \lambda(P',p')$.

Subcase 2.4, when $d_P(a_i,p) = \lambda(P,p)$ for $i = 1,\ldots,5$. As we are looking for the maximal value of $\lambda(P,p)$, we assume that $\lambda(P,p) > 1$. For the sake of simplicity, we use the notation $\nu(P,p) = \frac{\lambda(P,p)}{2-\lambda(P,p)}$. Thus $\frac{|a_ip|}{|pb_i|} = \nu(P,p)$ for $i = 1,\ldots,5$. Observe that $\nu(P,p)$ is a srictly increasing function of $\lambda(P,p)$. Additionally, $\lambda(P,p) > 1$ implies that $\nu(P,p) > 1$. Let h_p be the homothety with homothety center p and with homothety ratio $-\frac{1}{\nu(P,p)}$. Then $h_p(a_i) = b_i$ for $i = 1,\ldots,5$.

Consider the intersection point a of the lines $L(a_1,a_5)$ and $L(a_2,a_3)$. Let us take a Cartesian coordinate system. As the relative distance of two points does not change under affine transformations, we can assume that the points a, a_1, a_2 are $(0,0)$, $(1,-1)$, $(-1,-1)$, respectively (see Figure 16). We intend to show that if $p^x \neq 0$, then $\lambda(P,p)$ is not maximal. Assume that $p^x > 0$ (in the other case the proof is analogous).

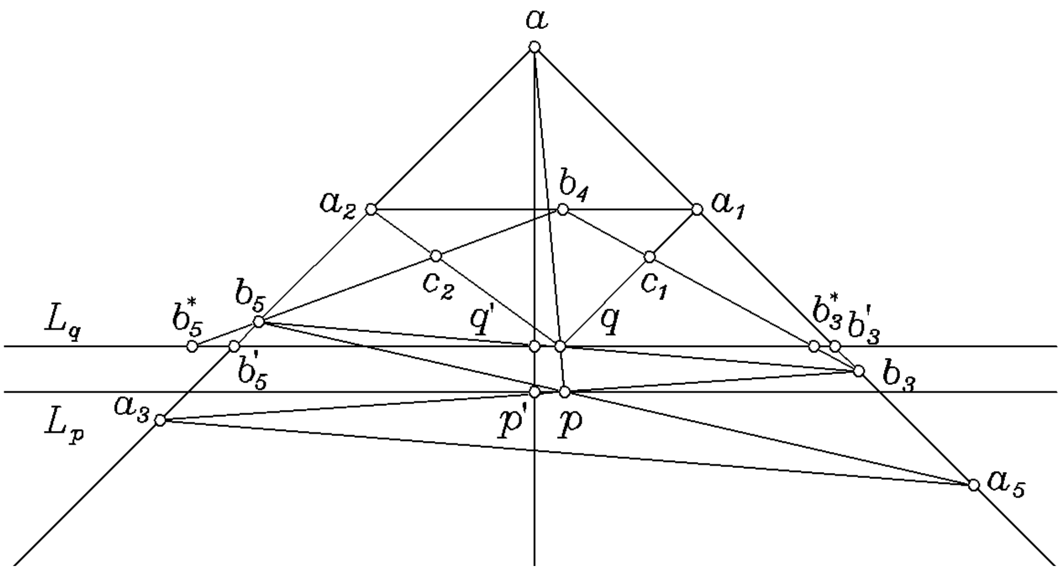

Figure 16

Take the intersection point q of the segments ap and b_3b_5. Denote the straight lines $y = p^y$ and $y = q^y$ by L_p and L_q, respectively. Let p' and q' be the points $(0, p^y)$ and $(0, q^y)$, respectively, and let $h_{p'}$ be the homothety with center p', and with homothety ratio $-\frac{1}{\nu(P,p)}$. Let us denote by b_3' and by b_5' the intersections of L_q and of the straight lines $L(a, a_1)$ and $L(a, a_2)$, respectively. Let a_3' be the intersection of $L(a, a_2)$ and $L(p', b_3')$. Similarly, let a_5' be the intersection of $L(a, a_1)$ and $L(p', b_5')$.

We show that $b_3' = h_{p'}(a_3')$ and that $b_5' = h_{p'}(a_5')$. Observe that $pb_3b_5 = h_p(pa_3a_5)$. Thus a_3a_5 and b_3b_5 are parallel, and b_3b_5 is the homothetic image of a_3a_5 of ratio $\frac{1}{\nu(P,p)}$, where the center of homothety is a. Since $a_3'a_5'$ and $b_3'b_5'$ are also parallel, $b_3'b_5'$ is the homothetic image of $a_3'a_5'$ of ratio $\frac{1}{\nu(P,p)}$, where the center of homothety is a. Hence $\frac{|b_3'b_5'|}{|a_3'a_5'|} = \frac{1}{\nu(P,p)}$. From this we get that $b_3'b_5'p' = h_{p'}(a_3'a_5'p')$. That is, $b_3' = h_{p'}(a_3')$ and $b_5' = h_{p'}(a_5')$.

Denote a_1 by a_1', a_2 by a_2', $h_{p'}(a_1)$ by b_1', and $h_{p'}(a_2)$ by b_2'. Let a_4' be the common point of the straight lines $L(a_3', b_1')$ and $L(a_5', b_2')$, and let b_4' denote $h_{p'}(a_4')$. Using these notations we have $\frac{2|a_i'p'|}{|a_i'b_i'|} = \lambda(P,p)$ for $i = 1, \ldots, 5$. We omit a consideration which shows that $P' = a_1'a_2'a_3'a_4'a_5'$ is a convex pentagon, that p' is in the kernel of P' and that P' has no side of P'-length 2. From the above properties of P' and p' we get that $d_{P'}(p', a'_i) = \lambda(P, p)$ for $i = 1, 2, 3, 5$. We show that $d_{P'}(p', a'_4) > \lambda(P, p)$.

Take the points $c_1 = h_p(b_1)$, $c_2 = h_p(b_2)$, $c'_1 = h_{p'}(b'_1)$ and $c'_2 = h_{p'}(b'_2)$. As the homothety ratios of h_p and $h_{p'}$ are equal, we have $c_1^y = c_2^y = {c'_1}^y = {c'_2}^y$, and $|c_1c_2| = |c'_1c'_2|$. Since b_3b_5 and a_3a_5 are parallel, the quadrangle $a_5b_3b_5a_3$ is a trapezoid. Thus $|b_3q| = |qb_5|$. Consider the triangles $b_3b'_3q$ and $b_5b'_5q$. We get that $|b_3b'_3| = |b_5b'_5|$. Let b_3^* be the intersection point of the segment b_3b_4 and the straight line L_q. Similarly, let b_5^* be the intersection point of the segment b_4b_5 and the straight line L_q. Notice that $b_4^x > 0$, $b_3 \in b'_3a_5$, and $b_5 \in b'_5a_2$. These observations and the equality of $|b_3b'_3|$ and $|b_5b'_5|$ imply that $|b_3^*b_5^*| > |b'_3b'_5|$. Consider that b_4 is the intersection of $L(b_3^*, c_1)$ and $L(b_5^*, c_2)$, and that b'_4 is the intersection of $L(b'_3, c'_1)$ and $L(b'_5, c'_2)$. As $|b_3^*b_5^*| > |b'_3b'_5|$ and $|c_1c_2| = |c'_1c'_2|$, we get that $b_4^y < {b'_4}^y$. Take the intersection point b_4^* of a_1a_2 and $p'b'_4$. Since $\frac{|p'a'_4|}{|p'b_4^*|} > \nu(P, p)$, we have $d_{P'}(p', a'_4) > \lambda(P, p)$. Obviously, $\lambda(P, p) = \lambda(P', p')$. Thus, according to Subcase 2.3, the value $\lambda(P, p)$ cannot be maximal.

Notice that our choice of the side a_1a_2 was arbitrary. This implies that $\lambda(P, p)$ can be maximal only if P is affine symmetric to every line containing the midpoint of a side of P and the opposite vertex of P, and if p is on every one of the above lines. But this holds only if P is an affine regular pentagon and if p is its center.

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Lemma 5. *Let $P = a_1a_2a_3a_4a_5$ be a convex pentagon and let p be a point of P which is not in the kernel of P. Then among $p, a_1, \ldots, a_5$ there exists a pair of points in P-distance less than λ.*

Proof. If P is a degenerate pentagon, then it has a chord containing at least 3 vertices of P. Thus in this case P has a side of P-length at most 1, which is less than λ.

In the sequel we deal with the case when P is nondegenerate. Take a Cartesian coordinate system. As the P-distance of two points is affine invariant, we assume that the points a_1, a_2 and a_5 are $(0, 0)$, $(1, 0)$ and $(0, 1)$, respectively. Let b be the point $(1, 1)$. Denote the square $a_1a_2ba_5$ by S. Furthermore, for every $i, j \in \{1, \ldots, 5\}$, where $i \neq j$, we denote the slope of the line $L(a_i, a_j)$ by m_{ij}, provided it exists.

Case 1, when P has more than one side of P-length 2. Consider the case when P has two nonconsecutive sides of P-length 2. We assume that $d_P(a_1, a_2) = 2$ and that

$d_P(a_3, a_4) = 2$ (the proof of the other cases is analogous). Let the angle of P at the vertex a_i be denoted by α_i, for $i = 1, \ldots, 5$. From $d_P(a_1, a_2) = 2$ we get that $\alpha_1 + \alpha_2 \leq \pi$. Similarly, from $d_P(a_1, a_2) = 2$ we have $\alpha_3 + \alpha_4 \leq \pi$. The convexity of P implies that $\alpha_5 \leq \pi$. Obviously, $\sum_{i=1}^{5} \alpha_i = 3\pi$. Thus, $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = \alpha_5 = \pi$. Therefore P is a degenerate pentagon.

Let us assume that P has two consecutive sides of P-length 2. Without loss of generality, let these sides be a_5a_1 and a_1a_2. Hence $P \subset S$. Denote the triangle $a_1a_2a_5$ by S_1. Take the homothetical copies S_2, S_3, S_4 of a_2ba_5, S, a_2ba_5 with ratio $\frac{1}{2}$ and with homothety centers a_2, b, a_5, respectively. As P is convex, S_1 contains neither of the points a_3 and a_4 in its interior. If S_2 contains a_3 or if S_4 contains a_4, then $d_P(a_2, a_3) \leq 1$ or $d_P(a_4, a_5) \leq 1$, respectively. Finally, if S_3 contains both a_3 and a_4, then $m_{34} < 0$ implies that $d_P(a_3, a_4) \leq 1$. Thus we get that P has a side of P-length at most 1.

Case 2, when P has exactly one side of P-length 2. We choose the indices of the points such that $d_P(a_1, a_2) = 2$, and that $a_3^y \geq 1$. The condition of this case and the convexity of P imply that $0 < a_4^x < a_3^x \leq 1$, and that $1 \leq a_3^y < a_4^y$. Moreover, either $d_P(a_1, a_5) < \lambda$ or $a_4^y \leq \frac{1}{\tau}$. In the following we assume that $a_4^y \leq \frac{1}{\tau} < 2$.

Observe that for arbitrary $w \in E^2$, the set of points whose P-distance from w is less than λ is the interior of the translate of $\frac{\tau}{2}(P - P)$ where the center of the body is w. From the previous considerations concerning the properties of P we get that the sides of the centrally symmetric convex decagon $\frac{\tau}{2}(P - P)$ are parallel to a_1a_5, a_4a_5, a_1a_2, a_3a_4, a_2a_3.

First we show that if every side of P has P-length greater than 1, then $m_{45} > m_{13}$. We show the statement indirectly. Denote the intersection point of $L(a_2, a_3)$ and $L(a_4, a_5)$ by s, and denote the intersection point of a_1a_3 and a_2a_5 by q. Let a_3' and a_4' be the homothetic images of a_3 and a_4, respectively, where the center of homothety is s and its ratio is 2. Denote the midpoint of the segment a_5s by s_5. Let t be the point of a_2a_3 such that a_1t and a_4a_5 are parallel. Similarly, let s' be the point of $L(a_2, a_3)$ such that a_5s' and a_1a_3 are parallel. Observe that a_1t is a maximal chord of P parallel to a_4a_5. Thus $d_P(a_4, a_5) > 1$ implies that $\frac{1}{2}|a_1t| < |a_4a_5|$. Moreover, we have $|a_5s_5| = \frac{1}{2}|a_5s| \leq \frac{1}{2}|a_1t|$. Therefore $|a_5s_5| < |a_4a_5|$, from which $a_4 \in s_5s$. Since $m_{13} \geq 1$, we get that $1 \leq \frac{|a_2q|}{|qa_5|} = \frac{|a_2a_3|}{|a_3s'|}$. Hence $\frac{|a_2a_3|}{|a_3s|} \geq \frac{|a_2a_3|}{|a_3s'|} \geq 1$. This and $a_4 \in s_5s$ imply that $a_3'a_4'$ is a chord of C. Thus $d_P(a_3, a_4) \leq 1$, which is a contradiction.

Now we prove the statement under the condition that $m_{45} > m_{13}$. If p is in both the triangles $a_1a_3a_5$ and $a_2a_3a_5$, then, according to the proof of Case 1 in Lemma 4 we have $d_P(a_3, p) \leq 1$ or $d_P(a_5, p) \leq 1$. We intend to examine the cases when $p \in a_3a_4a_5$, or $p \in a_1a_2a_5$, or $p \in a_2a_3q$.

Subcase 2.1, when p is in the triangle $a_3a_4a_5$. Denote the midpoints of the segments a_4a_5, a_3a_5 and a_3a_4 by c_3, c_4 and c_5, respectively. Notice that the triangles $a_3c_4c_5$, $a_5c_3c_4$ and the parallelogram $a_4c_5c_4c_3$ are contained in the homothetical copies of P with homothety ratio $\frac{1}{2}$ where the homothety centers are a_3, a_5 and a_4, respectively. Thus in this case at least one of the values $d_P(a_3, p)$, $d_P(a_4, p)$, $d_P(a_5, p)$ is at most 1.

Subcase 2.2, when p is in the triangle $a_1a_2a_5$. We show the statement indirectly, therefore we assume that among p and the vertices of P there is no pair in P-distance less than λ. Denote by Q_1 and by Q_5 the translates of $\frac{\tau}{2}(P-P)$ where the centers of the bodies are a_1 and a_5, respectively. Consider the points $b_1 = (1-\tau, 0)$ and $b_5 = (1-\tau, 1-\tau)$. As $d_P(a_2, p) \geq \lambda$, we have $p \in a_1b_1b_5a_5$. We show that $a_1b_1b_5a_5$ is covered by the interiors of Q_1 and Q_5. For this it is enough to show that b_1b_5 is in the interior of $Q_1 \cup Q_5$. Denote by d_1 the intersection of b_1b_5 and of the boundary of Q_1 such that $d_1 \neq b_1$. Similarly, let d_5 be the intersection of b_1b_5 and of the boundary of Q_5. We omit an easy calculation that if d_1 is not on the side of Q_1 parallel to a_2a_3, or if d_5 is not on the side of Q_5 parallel to a_4a_5, then b_1b_5 is in the interior of $Q_1 \cup Q_5$. In the opposite case we get that $d_1^y = m_{23}(1-2\tau)$, and that $d_5^y = m_{45}(1-2\tau) + 1 - \tau$. Thus $d_1^y - d_5^y = (2\tau-1)(m_{45} - m_{23}) + \tau - 1$.

Let us assume that $m_{34} \leq -1$. Take the point u on the line $L(a_2, a_3)$ such that a_3 is the midpoint of the segment a_2u. As $d_P(a_3, a_4) \geq \lambda > 1$, we have $\angle ua_5b < \angle a_4a_5b$. Thus, $0 < \frac{2a_3^y-1}{2a_3^x-1} < m_{45}$. This implies that $d_1^y - d_5^y \geq (2\tau-1)(\frac{2a_3^y-1}{2a_3^x-1} + \frac{a_3^y}{1-a_3^x}) + \tau - 1 \geq (2\tau-1)(\frac{1}{2a_3^x-1} + \frac{1}{1-a_3^x}) + \tau - 1$. But the last expression is always positive.

Now we discuss the case when $m_{34} > -1$. In this case from $d_P(a_3, a_4) \geq \lambda$ we conclude that $a_3^x - a_4^x \geq \tau$. Consider the point $m = (0, \frac{1}{1-\tau})$. Take the line L_m through m with slope -1. Since $m_{34} > -1$ and since $d_P(a_3, a_4) \geq \lambda$, a_3 and a_4 are in the open halfplane not containing a_1 bounded by L_m. As $a_4^y \leq \frac{1}{\tau}$, we have $a_4^x > \frac{1}{1-\tau} - \frac{1}{\tau}$, and thus $a_3^x > \frac{1}{1-\tau} - \frac{1}{\tau} + \tau = \frac{11\sqrt{5}-5}{20} \approx 0.980$. But this contradicts that d_1 is on the side of Q_1 parallel to a_2a_3, that is, that $a_3^x \leq \frac{1-\tau}{\tau} = \frac{\sqrt{5}+1}{4} \approx 0.809$. Hence b_1b_5 is in the interior of $Q_1 \cup Q_5$. Therefore every point of $a_1b_1b_5a_5$ is in P-distance from a_1 or from a_5 less than λ.

Subcase 2.3, when p is in the triangle a_2a_3q. Let Q_2 and Q_3 be the translates of $\frac{\tau}{2}(P-P)$ where the centers of the bodies are a_2 and a_3, respectively. If $p^y > 1$, then p is in the interior of Q_3. In the following we deal with the case when $p^y \leq 1$. From $d_P(a_5, p) \geq \lambda$ we have $p^x \geq \tau$. Let us show that the points of a_2a_3q with x-coordinates at least τ are in the interior of $Q_2 \cup Q_3$. Denote by e_2 the common point of the line $x = \tau$ and of the boundary of Q_2 with greater y-coordinate. Denote by e_3 the common point of the line $x = \tau$ and of the boundary of Q_3 with less y-coordinate. We show that $e_2^y - e_3^y$ is positive. We have $e_3^y \leq (1-\tau)a_3^y$. Moreover, $e_2^y = \tau a_4^y$ or $e_2^y = m_{45}(2\tau - 1) + \tau$. If $e_2^y = \tau a_4^y$, then $e_2^y - e_3^y \geq \tau(a_3^y + a_4^y) - a_3^y > 0$. In the sequel we assume that $e_2^y = m_{45}(2\tau-1)+\tau$. Observe that $m_{45} \geq m_{13} \geq 1$. Hence, if $a_3^y < \frac{3\tau-1}{1-\tau}$, then $e_2^y - e_3^y \geq 3\tau - 1 - (1-\tau)a_3^y > 0$. Let us assume the opposite case, when $a_3^y \geq \frac{3\tau-1}{1-\tau}$. In this case $a_4^y \leq \frac{1}{\tau} < a_3^y + \tau$. Thus $d_P(a_3, a_4) \geq \lambda$ implies that $a_4^x \leq a_3^x - \tau = 1 - \tau$. Take the points $m(0, \frac{1}{1-\tau})$ and $g(\frac{1}{\tau}, 1-\tau)$. Denote by h the intersection point of $L(m, g)$ and $x = 1$. We omit an elementary calculation which shows that $h^y = \frac{1}{\tau} - \frac{2\tau-1}{(1-\tau)^2}$. Since $d_P(a_3, a_4) \geq \lambda$, we get that a_3 is in the closed halfplane containing a_1 bounded by $L(a_4, m)$. Therefore $a_4^y - a_3^y \geq \frac{1}{\tau} - h^y$. Thus, $a_3^y \leq a_4^y - \frac{1}{\tau} + h^y \leq h^y \approx 1.281$. But this contradicts our assumption that $a_3^y \geq \frac{3\tau-1}{1-\tau} \approx 1.472$.

We have shown that $e_2^y - e_3^y$ is positive. But this implies that every point of the triangle a_2a_3q with x-coordinate at least τ is in the interior of $Q_2 \cup Q_3$.

Case 3, when P has no side of P-length 2. We assume that p is in the triangle $a_1a_2a_5$ and that m_{34} is at least -1 (the proof of the other cases is analogous). Since P has no side of P-length 2, we have $a_3^x > 1$ and $a_4^y > 1$. Observe that the points of $a_1a_2a_5$ with x-coordinates greater than $1-\tau$ are in P-distance from a_2 less than λ. Similarly, the points of $a_1a_2a_5$ with y-coordinates greater than $1-\tau$ are in P-distance from a_5 less than λ. Thus it is enough to deal with the case when both coordinates of p are at most $1-\tau$. Take the point $f = (\frac{1}{\tau} - 1, \frac{1}{\tau} - 1)$. We intend to show that if P has no side of P-length less than λ, then f is in the interior of P.

Consider the case when the maximal chord parallel to a_4a_5 has an endpoint at a_1. In this case the other endpoint of the above chord is on the segment a_3a_4. This and $d_P(a_4, a_5) \geq \lambda$ imply that the y-coordinate of the common point of $L(a_3, a_4)$ and of the line $y = 0$ is at least $\frac{1}{1-\tau} = \sqrt{5}$. Therefore, as $-1 \leq m_{34}$, we get that f is in the open halfplane containing a_1 bounded by $L(a_3, a_4)$. Thus f is in the interior of P.

Consider the case when the maximal chord of P parallel to a_4a_5 has an endpoint at a_3. In this case $d_P(a_4, a_5) \geq \lambda$ implies that $a_4^x \geq \tau a_3^x$. Since $d_P(a_3, a_4) \geq \lambda$, we have $a_3^x - a_4^x \geq \tau$. Therefore $a_4^x \geq \frac{\tau^2}{1-\tau}$. From $d_P(a_2, a_3) \geq \lambda$ we get that a_3 is not in the interior of the homothetical copy of $a_1a_2a_5$ with homothety ratio τ where the image of a_1 is a_2. Take the points $a_4' = (\frac{\tau^2}{1-\tau}, 1)$ and $a_3' = (1, 1-\tau)$. We omit an elementary calculation which shows that f is in the open halfplane containing a_1 bounded by $L(a_3', a_4')$. Thus f is in the open halfplane containing a_1 bounded by $L(a_3, a_4)$. Therefore f is in the interior of P.

We have shown that if P has no side of P-length less than λ, then f is in the interior of P. But the definition of f and our inequalities for the coordinates of p imply that in this case $d_P(p, a_1) < \lambda$.

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Lemma 6. *Let $a_1, \ldots, a_6$ be points such that their convex hull Q is a quadrangle or a triangle. Then among those points there exists a pair in Q-distance at most* 1.

Proof. We show the statement of our lemma indirectly. We assume that among the points $a_1, \ldots, a_6$ there is no pair in Q-distance at most 1. Let us take a Cartesian coordinate system. As the Q-distance of two points does not change under affine transformation, we assume that the points a_1, a_2 and a_3 are $(0,1)$, $(0,0)$ and $(1,0)$, respectively. Take the point $b(1,1)$ and the square $S = a_1a_2a_3b$. We choose the indices of our points such that $Q \subset S$. Let us denote the homothetical copies of S with homothety ratio $\frac{1}{2}$ and with centers a_1, a_2, a_3, b by S_1, S_2, S_3, S_4, respectively. Consider the center c of S, the center b_1 of the segment a_1a_2 and the center b_2 of the segment a_2a_3. Observe that every point of the triangle a_1b_1c is in Q-distance at most 1 from a_1. Similarly, every point of the triangles $a_2b_2b_1$ and b_2a_3c is in Q-distance at most 1 from a_2 and from a_3, respectively. Notice that there are no two points in the triangle b_1b_2c in Q-distance from each other greater than 1. Thus b_1b_2c contains at most one of the points a_4, a_5, a_6. Hence Q is a quadrangle. Let a_4 be the fourth vertex of Q. As $d_Q(a_1, a_4) > 1$ and $d_Q(a_3, a_4) > 1$, we have $a_4 \in S_4$. Hence every point of S_1 and S_3 is in Q-distance at most 1 from a_1 and a_3, respectively. Furthermore, $S_4 \cap Q$ is covered by the homothetical copy of Q with homothety center a_4

and with ratio $\frac{1}{2}$. Thus, every point of $S_4 \cap Q$ is in Q-distance at most 1 from a_4. Moreover, b_2cb_1 contains at most one of the points a_5 and a_6, which is a contradiction.

∎

Proof of Theorem 5. First, observe that if C is an affine regular pentagon, and if the six points are its vertices and its center, then the minimal pairwise C-distance of the points is λ. Take an arbitrary plane convex body C. Let $p_1, \ldots, p_6$ be points of C. Let us denote the convex hull of $p_1, \ldots, p_6$ by C'. As $C' \subset C$, the C'-distance of arbitrary two points is greater than or equal to their C-distance. If C' is a hexagon, from Theorem 3 we get that among $p_1, \ldots, p_6$ there is a pair in C'-distance at most $8 - 4\sqrt{3}$, which is less than λ. With respect to Lemma 4 and Lemma 5, if C' is a pentagon, then the minimal pairwise C'-distance of the points is at most λ, with equality if and only if C' is an affine regular pentagon and the points are its vertices and its center. According to Lemma 6, if C' is a quadrangle or a triangle, then there exists a pair of points in C'-distance at most 1, which is less than λ. We have proved the first statement of our theorem.

To prove the second statement, it remains to show that if C' is an affine regular pentagon and if the points are its vertices and its center, and if there is no pair of them in a C-distance less than λ, then $C = C'$. Let us choose the indices of the points such that C' is the pentagon $p_1p_2p_3p_4p_5$ and that p_6 is the center of C'. Assume that $C \neq C'$. In this case there exists a point $q \in C$, which is not a point of C' and the convex hull D of q and C' is a convex hexagon. It is enough to deal with the case when $D = p_1p_2p_3p_4p_5q$ (the proof of the other cases is analogous). But then $d_C(p_6, p_3) \leq d_D(p_6, p_3) < d_{C'}(p_6, p_3) = \lambda$.

∎

Chapter 5

Relative Distance and a Convex Body Packed or Touched by its Homothetical Copies

Let A and B be convex bodies in a Euclidean n-space E^n. If A is a subset of B, and if A contains a boundary point of B, we say that A *touches the boundary of* B *from inside*. If the intersection of A and B is not empty but their interiors are disjoint, we say that A *and* B *touch each other*. If the interiors of A and B have a common point, we call them *overlapping*. If $A_1, \ldots, A_k$ are mutually nonoverlapping convex bodies, and if B is a convex body such that $A_i \subset B$ for $i = 1, \ldots, k$, then we say that B *is packed by* $A_1, \ldots, A_k$, or that $A_1, \ldots, A_k$ *are packed into* B.

In this chapter we investigate the connection between the existence of points in a convex body in large relative distances, and the existence of large homothetical copies of a convex body packed into, or touching the body. The idea of such a connection first appeared in [8] and in [24]. Both papers deal with the case when some small number of homothetical copies of a plane convex body C with equal positive homothety ratio are packed into C. The case when k mutually nonoverlapping homothetical copies of C with equal homothety ratio touch the boundary of C from inside is discussed in [27]. In [27] the case when mutually nonoverlapping equal negative homothetical copies of C touch C is also examined.

In this chapter first we prove the following connection between the relative distances of k points of C and the ratio of k equal positive homothetical copies of C packed into C.

Theorem 6. *Let C be a convex body in E^n and let $k \geq 2$ be an integer. If C contains k points in relative distances at least d, then we can pack C by its k homothetical copies of ratio $\frac{d}{2+d}$. Vice-versa; if we can pack C by its k homothetical copies of a positive ratio $r < 1$, then we can find k points in C in relative distances at least $\frac{2r}{1-r}$.*

The below Figure 17 illustrates our theorem.

Let $r_k(C)$ denote the greatest possible positive ratio of k homothetical copies of C that can be packed into C. Simple compactness arguments show that for every convex body C in E^n and for every integer $k \geq 2$, the number $r_k(C)$ exist.

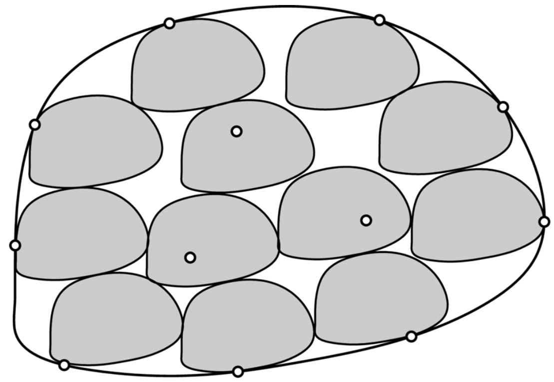

Figure 17

Using the notion of the values $d_k(C)$ and $r_k(C)$ we can rewrite Theorem 6 in the following form.

For every convex body $C \subset E^n$ and for every integer $k \geq 2$ we have

$$r_k(C) = \frac{d_k(C)}{2 + d_k(C)} \quad \text{and} \quad d_k(C) = \frac{2r_k(C)}{1 - r_k(C)}.$$

Recall that the infimum and the supremum of $d_k(C)$, where C runs over the family of plane convex bodies, are denoted by $d_k(\mathcal{C})$ and by $e_k(\mathcal{C})$, respectively. Also remember the notations $d_k(\mathcal{M})$ and $e_k(\mathcal{M})$ for the infimum and the supremum of $d_k(C)$, respectively, where C runs over the family of centrally symmetric plane convex bodies. In Chapters 2 and 4 we have collected the results and conjectures regarding the values of $d_k(\mathcal{C})$, $e_k(\mathcal{C})$, $d_k(\mathcal{M})$ and $e_k(\mathcal{M})$.

Let us denote by $r_k(\mathcal{C})$ and by $s_k(\mathcal{C})$ the infimum and the supremum of $r_k(C)$, respectively, where C runs over the family of plane convex bodies. Similarly, let $r_k(\mathcal{M})$ and $s_k(\mathcal{M})$ be the infimum and the supremum of $r_k(C)$, respectively, where C runs over the family of centrally symmetric plane convex bodies. Compactness arguments show that these infima and suprema are attained. Applying Theorem 6 we get a number of estimates for the values of $r_k(\mathcal{C})$, $s_k(\mathcal{C})$, $r_k(\mathcal{M})$ and $s_k(\mathcal{M})$.

From $d_2(\mathcal{C}) = 2$ we have $r_2(\mathcal{C}) = \frac{1}{2}$. Moreover, from Proposition 1 we obtain $d_3(\mathcal{C}) \geq \frac{1}{5}(2 + 2\sqrt{6})$. Using Theorem 6 we get that $r_3(\mathcal{C}) \geq \frac{\sqrt{6}}{6} \approx 0.4082$. We have mentioned a conjecture which says that $d_3(\mathcal{C}) = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ (see [23] or [24]). The example of the regular pentagon P in the part of C shows that this value cannot be replaced by a larger one.

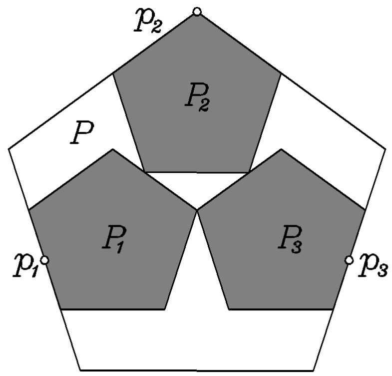

Figure 18

In Figure 18 we see three homothetical copies P_1, P_2, P_3 of P and corresponding centers p_1, p_2, p_3 of homotheties. Observe that they can be moved step by step around P so that the relative distance between pairs of them is always $\frac{1}{2}(1+\sqrt{5})$. Having in mind Theorem 6, instead of this we can say that P_1, P_2, P_3 may be moved around such that they touch themselves and the boundary of P all the time. In Figure 18 we first move p_1 up to the lower end of the corresponding side of P. This means that P_1 moves and makes some space which permits to move P_2. Simultaneously, p_2 moves on the boundary of P. Then we can move P_3 and so on. We conclude that for the regular pentagon the relative distance $\frac{1+\sqrt{5}}{2}$ cannot be increased. According to Theorem 6 we can also say that for the regular pentagon the homothety ratio $\frac{\sqrt{5}}{5}$ cannot be increased.

From Theorem 2 and from the convex pentagon shown in Figure 5 we have seen that $1.0787 \approx \frac{\sqrt{5}+1}{3} \leq d_4(\mathcal{C}) \leq \sqrt{5}-1 \approx 1.2361$. Thus we also have $0.3504 \approx \frac{1+3\sqrt{5}}{22} \leq r_4(\mathcal{C}) \leq \frac{3-\sqrt{5}}{2} \approx 0.3820$. From [8] and [24], and from the example of the square we see that $r_5(\mathcal{C}) = \frac{1}{3}$. Moreover, Proposition 2 and the example of any triangle imply that $r_8(\mathcal{C}) = \frac{1}{4}$.

For $k \leq 4$, from $e_k(\mathcal{C}) = 2$ we conclude that $s_k(\mathcal{C}) = \frac{1}{2}$. Thanks to [7] and the example of the regular pentagon we have $s_5(\mathcal{C}) = \frac{3-\sqrt{5}}{2} \approx 0.3820$. Theorem 5 and the example of the regular pentagon imply that $s_6(\mathcal{C}) = \frac{9-\sqrt{5}}{19} \approx 0.3360$.

From $d_2(\mathcal{M}) = 2$ we get that $r_2(\mathcal{M}) = \frac{1}{2}$. From [26] and from the example of the regular octagon we see that $0.4409 \approx \frac{4+\sqrt{3}}{13} \leq r_3(\mathcal{M}) \leq \frac{5+2\sqrt{2}}{17} \approx 0.4605$. Thanks to [8] and [24], and thanks to the example of the circle, we have $r_4(\mathcal{M}) = \sqrt{2}-1 \approx 0.4142$. From [8], from [24], and from the example of the square we have $r_5(\mathcal{M}) = r_6(\mathcal{M}) = r_7(\mathcal{M}) = \frac{1}{3}$.

In Chapter 4 we have seen that $e_k(\mathcal{M}) = 2$ for $k \leq 4$. Thus we also have $s_k(\mathcal{M}) = \frac{1}{3}$ for $k \leq 4$. From [7] and from the example of the regular decagon we get that $s_5(\mathcal{M}) =$

$\frac{3-\sqrt{5}}{2} \approx 0.3820$. Moreover, from [8] and from the example of the square we conclude that $s_k(\mathcal{M}) = \frac{1}{3}$ for $k = 6, \ldots, 9$.

Theorem 6 appears in the joint paper [23] with M. Lassak.

To formulate our next theorem we introduce the concept of *translative kissing number $H(C)$ of a convex body $C \subset E^n$*. By $H(C)$ we mean the maximal number of mutually nonoverlapping translates of C touching C. Hadwiger [16] showed that for every plane convex body its translative kissing number is always at least 6 and at most 8. Grünbaum [15] proved that if C is a parallelogram, then $H(C) = 8$, and the translative kissing number of every other plane convex body is 6.

We examine the following question. Let k be a fixed integer and let $C \subset E^n$ be a convex body. We are looking for the greatest possible number t such that there exist k mutually nonoverlapping homothetical copies of C with homothety ratio t touching C. In the second part of this chapter we prove the following theorem.

Theorem 7. *For every convex body $C \subset E^n$ and for every $t \in (0, \infty)$ the following two conditions are equivalent:*

(i) there exist k mutually nonoverlapping homothetical copies of C with homothety ratio t touching C,

(ii) there exist k points in the boundary of $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$ in pairwise C-distances at least $\frac{2t}{1+t}$.

In our proof we also conclude that our theorem remains true if we take disjoint homothetical copies in (i) and C-distances greater than $\frac{2t}{1+t}$ in (ii).

We consider the consequences of our theorem only for the planar case. For every $t \in (0, \infty)$ we denote by $\mathcal{C}_t$ the family of plane convex bodies that can be presented in the form $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$, where all $C \in \mathcal{C}$ are taken.

Let t_k, where $k \geq 3$, denote the greatest possible number such that for every plane convex body C there exist its k mutually nonoverlapping homothetical copies with ratio t_k touching C. Analogously, let u_k, where $k \geq 5$, denote the greatest possible number such that there exists a plane convex body C for which there are k mutually nonoverlapping homothetical copies of C with ratio u_k touching C. Here, compactness arguments show that the above maxima exist. Obviously, both $\{t_k\}$ and $\{u_k\}$ are nonincreasing sequences. Using Theorem 7, we get a number of estimates for some values of t_k and u_k. These estimates are collected in the following Corollary.

Corollary. *We have $t_5 = t_6 = 1$ and $\frac{1}{2} \leq t_7 \leq \frac{3}{4}$. Moreover, $u_5 = \frac{1}{2}(\sqrt{5}+1) \approx 1.618$, $u_6 = u_7 = u_8 = 1$, and for every integer $s \geq 2$ we have $u_{4s} = \frac{1}{s-1}$.*

Other values of t_k and u_k are not determined. We have conjectured in Chapter 3 that $e_9(\mathcal{C}) = e_9(\mathcal{M}) = 4\sin(10°) \approx 0.6946$, and that $e_{10}(\mathcal{C}) = e_{10}(\mathcal{M}) = e_{11}(\mathcal{C}) = e_{11}(\mathcal{M}) = \frac{2}{3}$. From the proof of our corollary we will see that verification of these conjectures imply $u_9 = \frac{2\sin(10°)}{1-2\sin(10°)}$ and $u_{10} = u_{11} = \frac{1}{2}$.

In the remaining part of this chapter we are looking for large negative homothetical copies of a convex body C with equal homothety ratio packed into C. Like in Theorem 7, we prove a connection between the ratio of the above homothetical copies and the relative distances of points in a convex body.

Proposition 3. *Let C be an arbitary convex body in E^n, and let $t \in (0,1]$. Denote by C_t the set of points of C whose C-distance from every boundary point of C is at least $\frac{2t}{1+t}$. Then the following two conditions are equivalent:*

(i) there exist k mutually nonoverlapping homothetical copies of C with homothety ratio $-t$ packed into C,

(ii) there exist k points in C_t in pairwise C-distances at least $\frac{2t}{1+t}$.

Analogously to the proof of Proposition 3, we can show the following.

If there exist k negative homothetical copies of C with ratio $-t$ touching the boundary of C from inside, then there are k points in the boundary of C_t in pairwise C-distances at least $\frac{2t}{1+t}$, and vica versa, if there exist k points in the boundary of C_t in pairwise C-distances at least $\frac{2t}{1+t}$, then there are k negative homothetical copies of C with ratio $-t$ touching the boundary of C from inside.

Theorem 7, Corollary and Proposition 3 are presented in the paper [21].

First we prove Theorem 6. In order to prove it we show Lemma 7, which implies Lemma 8. Our theorem is an immediate consequence of Lemma 8.

Lemma 7. *Let xy and ab be two parallel segments in E^n. Put $d = 2(|xy|/|ab|)$. The two segments being homothetical copies of the segment ab with homothety centers x and y, and with the homothety ratio $\frac{d}{2+d}$, have exactly one common point.*

Proof. Denote by w the point of intersection of the straight lines containing segments xb and ya (see Figure 19). We tacitly assume that the notation for a and b is taken such that the segments intersect. Through w we provide the straight line parallel to the segment xy. The intersections of this line with the segment xa is denoted by g, and with the segment yb is denoted by h. Thus $\frac{|gw|}{|ab|} = \frac{|wx|}{|bx|} = \frac{|wx|}{|bw|+|wx|} = (\frac{|bw|}{|wx|} + 1)^{-1} = (\frac{|ab|}{|xy|} + 1)^{-1}$.

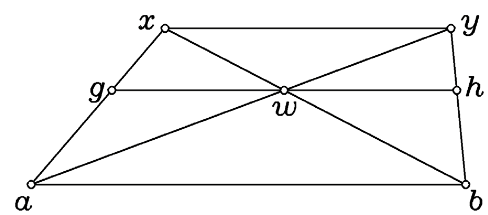

Figure 19

Analogousy, $\frac{|hw|}{|ab|} = (\frac{|ab|}{|xy|} + 1)^{-1}$. Consequently, for the homothety ratio $(\frac{|ab|}{|xy|} + 1)^{-1} = (\frac{2}{d} + 1)^{-1} = \frac{d}{2+d}$, the common part of the images of the segments ab under homotheties with centers x and y is just the point w. ∎

Lemma 8. *Let $C \subset E^n$ be a convex body and let x, y be boundary points of C. For every positive constant $d \leq 2$ the following conditions are equivalent.*

(i) $d_C(x, y) = d$,

(ii) the homothetical copies of C with homothety centers x and y, and with ratio $\frac{d}{2+d}$ touch each other.

Lemma 8 follows from Lemma 7 when in the part of ab we take a longest segment contained in C which is parallel to xy (see Figure 20). Observe that this way of proving permits to avoid using arguments of separation of the two copies of C by a hyperplane.

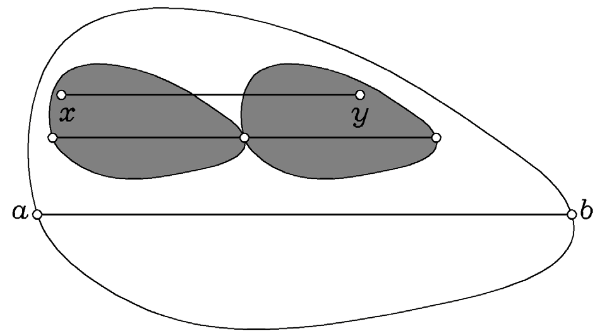

Figure 20

From the proof of Lemma 7 we see that only for the ratio $\frac{d}{2+d}$ we get exactly one point of the intersection of the two segments which are homothetical copies of ab. If the ratio is smaller, then the intersection is empty. If it is greater, then the intersection contains more than one point. So analogous equivalence like in Lemma 8 holds true if we have the inequality $d_C(x, y) < d$ in (i) and the condition about nonempty intersection of the interiors of the copies in (ii).

Proof of Theorem 7. First we show that (i) implies (ii).

Case 1, when $t \in (0, 1)$. Let us assume that $C_1, \ldots, C_k$ are mutually nonoverlapping homothetical copies of C with homothety ratio t touching C. Denote by c_i the center of the homothety h_i which maps C into C_i, for $i = 1, \ldots, k$ (see Figure 21). Let q_i be a common point of C and C_i.

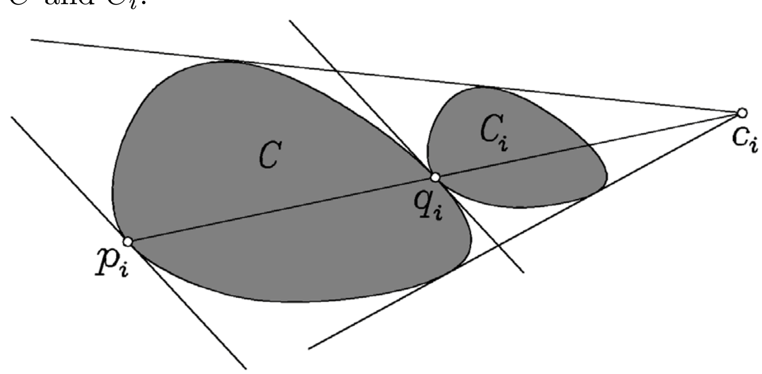

Figure 21

As C and C_i are not overlapping, they have a common supporting hyperplane H_i containing q_i. Take the point p_i of C for which $h_i(p_i) = q_i$. Obviously, $d_C(c_i, q_i) = td_C(c_i, p_i)$. Since there exist parallel supporting hyperplanes of C containing p_i and q_i (for instance, $h_i^{-1}(H_i)$ and H_i), we get $d_C(p_i, q_i) = 2$. That is, $t(d_C(c_i, q_i) + 2) = d_C(c_i, q_i)$. Thus, $d_C(c_i, q_i) = \frac{2t}{1-t}$. It is easy to see that for every point $z \in C$ we have $d_C(c_i, z) \geq$

$\frac{2t}{1-t}$. Observe that the set of points whose C-distance from a point $w \in E^n$ equals to d is the boundary of the convex body $w + \frac{d}{2}(C - C)$. Hence c_i is on the boundary of $C + \frac{t}{1-t}(C - C) = \frac{1}{1-t}C + \frac{t}{1-t}(-C)$.

Now we intend to show that $d_C(c_i, c_j) \geq \frac{2t}{1-t}$ for $i, j \in \{1, \ldots, k\}$, where $i \neq j$. Let us take a point r of C. Denote $h_i(r)$ by r_i, for $i = 1, \ldots, k$ (see Figure 22). Apparently, $|rr_i| = (1-t)|rc_i|$. As the triangles rr_ir_j and rc_ic_j are similar, we conclude that $|r_ir_j| = (1-t)|c_ic_j|$. It was noted by Minkowski in [29] that for an arbitrary convex body C, if $x+C$ and $y+C$ are overlapping, touching or disjoint, then $x+\frac{1}{2}(C-C)$ and $y+\frac{1}{2}(C-C)$ are overlapping, touching or disjoint, respectively (we will apply this property a few times). Thus, as C_i and C_j are not overlapping, we obtain that $d_C(r_i, r_j) \geq 2t$. Hence $d_C(c_i, c_j) \geq \frac{2t}{1-t}$.

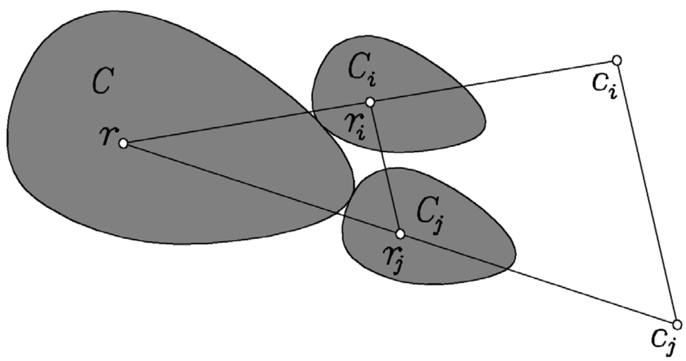

Figure 22

Finally, let us take the homothety h with the homothety ratio $\frac{1-t}{1+t}$ and with the center at the origin. Then $h(c_1), \ldots, h(c_k)$ are k points in pairwise C-distances at least $\frac{2t}{1+t}$ on the boundary of $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$.

Case 2, when $t = 1$. Let $p_1 + C, \ldots, p_k + C$ be mutually nonoverlapping translates of C touching C. Thanks to the mentioned result of [29], we see that $p_1, \ldots, p_k$ are points in C-distance 2 from the origin. Hence they are on the boundary of $C - C$. This result in [29] also implies that the pairwise C-distances of $p_1, \ldots, p_k$ are at least 2. Let h denote the homothety with the center at the origin and with the homothety ratio $\frac{1}{2}$. Then $h(p_1), \ldots, h(p_k)$ are k points on the boundary of $\frac{1}{2}C + \frac{1}{2}(-C)$ in pairwise C-distances at least 1.

Case 3, when $t \in (1, \infty)$. Let $C_1, \ldots, C_k$ be mutually nonoverlapping homothetical copies of C with homothety ratio t touching C. Denote by c_i the center of the homothety h_i that maps C into C_i, for $i = 1, \ldots, k$. We omit a consideration analogous to that in

Case 1 which shows that c_i is on the boundary of $C + \frac{1}{t-1}(C - C) = \frac{t}{t-1}C + \frac{1}{t-1}(-C)$. We also omit a consideration that $d_C(c_i, c_j) \geq \frac{2t}{t-1}$ for every $i, j \in \{1, \dots, k\}$, where $i \neq j$). Let h denote the homothety with the center at the origin and with the negative homothety ratio $\frac{1-t}{1+t}$. Then $h(c_1), \dots, h(c_k)$ are k points on the boundary of $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$ in pairwise C-distances at least $\frac{2t}{1+t}$.

Observe that the considerations in all the three cases are revertible. Thus (ii) implies (i). Also notice that analogous proof can be given if in (i) we write about disjoint homothetical copies of C, and in (ii) about points in pairwise C-distances greater than $\frac{2t}{1+t}$.

$\blacksquare$

Proof of Corollary. Notice that for every $r \in [-1, 1]$ and for every convex body C the C-distance of arbitrary two points is equal to their $[rC + (1-r)(-C)]$-distance. Hence for every $t \in (0, \infty)$, the C-distance of arbitrary two points is equal to their $[\frac{t}{1+t}C + \frac{1}{1+t}(-C)]$-distance. Thus, according to Theorem 7, t_k is the maximal number such that the boundary of every $C \in \mathcal{C}_{t_k}$ contains k points in pairwise C-distances at least $\frac{2t_k}{1+t_k}$. Similarly, u_k is the maximal number such that there exists $C \in \mathcal{C}_{u_k}$ whose boundary contains k points in pairwise C-distances at least $\frac{2u_k}{1+u_k}$. We use the notation $d = \frac{2t}{1+t}$. Thus $t = \frac{d}{2-d}$. Observe that $\mathcal{C}_1 = \mathcal{M}$. Furthermore, for every $t \in (0, \infty)$ we have $\mathcal{M} \subset \mathcal{C}_t \subset \mathcal{C}$. Hence, if the boundary of every plane convex body contains k points in pairwise relative distances at least d, and if there exists a centrally symmetric plane convex body whose boundary does not contain k points in pairwise relative distances greater than d, then $t_k = \frac{d}{2-d}$. Analogously, if there exists a centrally symmetric plane convex body whose boundary contains k points in pairwise relative distances at least d, and if there is no plane convex body whose boundary contains k points in pairwise relative distances greater than d, then $u_k = \frac{d}{2-d}$. We apply these two statements a few times in the remaining part of the proof.

In [6] it is proved that the boundary of every plane convex body contains five points in pairwise relative distances at least 1. It is easy to check that the boundary of the parallelogram does not contain five points in pairwise relative distances greater than 1. Therefore $t_5 = 1$.

In [8] and in [24] it is observed that the boundary of every centrally symmetric plane convex body contains six points in pairwise relative distances at least 1. As $\mathcal{C}_1 = \mathcal{M}$, we get $t_6 \geq 1$. It is also observed in [8] that there is no centrally symmetric plane convex body whose boundary contains six points in pairwise relative distances greater than 1. Consequently, Theorem implies that there is no plane convex body that can be touched by its six mutually disjoint translates. This means that there is no convex body that can be touched by its six mutually nonoverlapping homothetical copies with homothety ratio greater than 1. Hence $u_6 \leq 1$. Obviously $t_6 \leq u_6$. Thus $t_6 = u_6 = 1$.

We have proved in Theorem 1 that the boundary of every plane convex body contains seven points in pairwise relative distances at least $\frac{2}{3}$. Hence $t_7 \geq \frac{1}{2}$. We omit an elementary consideration which shows that the boundary of the regular hexagon does not contain seven points in pairwise relative distances greater than $\frac{6}{7}$. This gives the estimate $t_7 \leq \frac{3}{4}$.

In [7] it is proved that there exists no plane convex body whose boundary contains five points in pairwise relative distances greater than $\sqrt{5}-1$. The value $\sqrt{5}-1$ is attained for the regular pentagon and decagon. Therefore $u_5 = \frac{1}{2}(\sqrt{5}+1)$.

We have mentioned that the circumference of every plane convex body measured in the metric $d_C(x,y)$ is at most 8 (see page 22). The example of the parallelogram shows that for every integer $s \geq 2$, we have $u_{4s} = \frac{1}{s-1}$. Hence $u_8 = 1$.

We see that $u_6 = u_8 = 1$. As the sequence $\{u_k\}$ is nonincreasing, we get $u_7 = 1$.

∎

Now we prove Proposition 3. Since its proof is analogous to the proof of Theorem 7, we only sketch it.

Proof of Proposition 3. Consider a homothetical copy K of C with homothety ratio $-t$ packed into C. Denote by h the homothety which maps C into K, and let c be the center of homothety. For the sake of simplicity let us assume that c is the origin. Then $K = -tC$. Observe that for arbitrary sets A and B, and for arbitary $r \in [0,1]$, the set $rA+(1-r)B$ is contained in the convex hull of $A \cup B$. Therefore C contains $\frac{t}{1+t}C + \frac{1}{1+t}(-tC) = \frac{t}{1+t}(C-C)$. That is, the C-distance of c and every boundary point of C is at least $\frac{2t}{1+t}$. So c is in C_t.

We omit a consideration analogous to that in Theorem 7 that if $-tC$ is not contained in C, then $c \notin C_t$.

Finally, take two arbitrary homothetical copies K_1 and K_2 of C with homothety ratio $-t$. Let c_1 and c_2 be the centers of the homotheties which map C into K_1 and K_2, respectively. Similarly like in Theorem 7, we observe that K_1 and K_2 do not overlap if and only if $d_C(c_1, c_2) \geq \frac{2t}{1+t}$.

∎

Chapter 6

Almost Equidistant Points on the Sphere

Let S^{n-1} denote the $(n-1)$-dimensional unit sphere centered at the origin o of the n-dimensional Euclidean space E^n, and let $d \in (0, 2)$. It is an elementary exercise to show that the number of points on S^{n-1} having pairwise distances equal to d is at most $n+1$. Moreover, it follows from the paper [12] of Füredi, Lagarias and Morgan that the number of points on the boundary of an n-dimensional convex body C having equal pairwise C-distances is at most 2^n.

Here we present a generalization of the above problem. A set P of points on S^{n-1} is called *almost d-equidistant* if among any three points of P there is at least one pair lying in the Euclidean distance d. Rosenfeld [32] proved in a very elegant way that the maximal number of almost $\sqrt{2}$-equidistant points on S^{n-1} is $2n$. In this chapter we prove a similar result for almost d-equidistant points, where $\sqrt{2} < d < 2$. Moreover, we prove that the estimate $2n$ of Rosenfeld holds true also in a neighbourhood of $\sqrt{2}$. Finally we show an analogous estimate for every $d \in (0, \sqrt{2})$. These theorems appeared in the joint paper [2] with K. Bezdek. For the maximum cardinality of almost equidistant pointsets in various Minkowski spaces see also the paper [3] of Bezdek, Naszódi and Visy.

Between the spherical distance and the Euclidean distance of two points on S^{n-1} there is a one-to-one correspondence (as usual, we measure the spherical distance between any two points of S^{n-1} by the length of the shortest geodesic arc connecting the two points). Thus we can also measure the distance of points of an almost equidistant set on S^{n-1} by their spherical distance. Equivalently, we may also consider angles between unit vectors of E^n.

First we prove the following theorem.

Theorem 8. *For every $d \in (\sqrt{2}, 2)$ and for every integer $n \geq 2$ the number of almost d-equidistant points on S^{n-1} is at most $2n+2$.*

The Euclidean distance between any two vertices of a regular n-dimensional simplex inscribed in S^{n-1} is equal to $d_n = \sqrt{\frac{2n+2}{n}}$. Notice that $\sqrt{2} < d_n < 2$. Thus, if one takes the set V of $2(n+1)$ vertices of two regular n-dimensional simplices inscribed in S^{n-1}, then among any three vertices of V there is a pair lying in the Euclidean distance d_n. This shows that the upper bound $2(n+1)$ in Theorem 8 is sharp.

In the second part of this chapter we prove the following statement.

Theorem 9. *For every integer $n \geq 2$ there exists a positive real number $\epsilon(n)$ such that the maximum number of almost d-equidistant points on S^{n-1}, where $|d - \sqrt{2}| \leq \epsilon(n)$, is equal to $2n$.*

Observe that for the distance $d_n = \sqrt{\frac{2n+2}{n}}$ introduced after the formulation of Theorem 8 we have $\lim_{n\to+\infty} d_n = \sqrt{2}$. As a result, the construction after the formulation of Theorem 8 shows that $\lim_{n\to+\infty} \epsilon(n) = 0$. Finally, if one takes two congruent copies of a regular spherical $(n-1)$-dimensional simplex of edge length d, where $0 < s \leq d_{n-1}$, then the $2n$ vertices of the two spherical simplices form an almost d-equidistant pointset on S^{n-1}. This shows that the upper bound $2n$ in Theorem 9 cannot be improved.

In the last part of this chapter we give the following estimate about the maximum cardinality of almost d-equidistant pointsets of S^{n-1} for $d \in (0, \sqrt{2})$.

Theorem 10. *For every $d \in (0, \sqrt{2})$ and for every $n \geq 2$ the number of almost d-equidistant points on S^{n-1} is at most $n^2 + n - 2$.*

It is likely that the upper bound $n^2 + n - 2$ of Theorem 10 can be improved for every $n \geq 2$. Moreover, for any "small" $d > 0$ we provide the following construction. Take a regular spherical $(n-1)$-dimensional simplex of edge length d with vertices $v_1, \ldots, v_{n-1}, v_n$ on S^{n-1}. Then reflect v_n about the $(n-2)$-dimensional great-sphere of S^{n-1} passing through the vertices $v_1, \ldots, v_{n-1}$, and denote by c the point obtained. Finally, let v_i^* be the rotated copy of the point v_i about the point c through the same angle for all $i \in \{1, \ldots, n\}$ on S^{n-1} such that the distance between v_n and v_n^* is equal to d. It is easy to check that the points $c, v_1, \ldots, v_{n-1}, v_n, v_1^*, \ldots, v_{n-1}^*, v_n^*$ form an almost d-equidistant pointset of cardinality $2n + 1$ on S^{n-1}.

Now we prove Theorem 8. In order to prove our statement we need the following lemma which is a somewhat stronger version of the lemma from [32].

Lemma 9. *Let $\{x_1, \ldots, x_m\}$ be a set of m real numbers with the property such that there exists $y > 0$ for which $x_1 \geq -y, \ldots x_m \geq -y$ and $\sum_{i=1}^m x_i = (m+s)y$, where $s \geq 0$. Then*

$$\sum_{i=1}^m x_i^3 \geq (m+3s)y^3.$$

Proof. If $x_i \geq 0$ for all $i = 1, \ldots, m$, then the following well-known inequality holds true:

$$\sqrt[3]{\frac{\sum_{i=1}^m x_i^3}{m}} \geq \frac{\sum_{i=1}^m x_i}{m}. \tag{1}$$

From (1) it follows in a straightforward way that

$$\sum_{i=1}^m x_i^3 \geq \frac{(\sum_{i=1}^m x_i)^3}{m^2} = \frac{(m+s)^3 y^3}{m^2} \geq (m+3s)y^3.$$

Now, we proceed by induction on the number t of indices i for which $x_i < 0$. If $t = 0$, then we are done. If $t > 0$, then without loss of generality we may assume that $x_1 = -ly$ for some $l \in (0, 1]$. We replace x_1 by 0 in order to obtain m real numbers $0, x_2, \ldots, x_m$ whose sum is equal to $(m+s+l)y$. The induction hypothesis implies that

$$\sum_{i=2}^m x_i^3 \geq [m + 3(s+l)]y^3.$$

Thus, we get that

$$\sum_{i=1}^m x_i^3 = \left(\sum_{i=2}^m x_i^3\right) - l^3y^3 \geq [m + 3(s+l) - l^3]y^3 \geq (m+3s)y^3.$$

This completes the proof of Lemma 9.

∎

Proof of Theorem 8. The proof presented here follows the ideas of [32] with some necessary modifications. First observe that any two points of S^{n-1} lying in a Euclidean distance greater than $\sqrt{2}$ are in a spherical distance greater than $\frac{\pi}{2}$.

Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ be a maximal system of unit vectors in E^n with the property that among any three vectors $\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k \in U$ there are two, for instance $\mathbf{u}_i, \mathbf{u}_j$ with

$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \cos\alpha$. We now consider the matrix $A = (\langle \mathbf{u}_i, \mathbf{u}_j \rangle - \cos\alpha)_{r\times r}$. Notice that $A = G - \cos\alpha E$, where $G = (\langle \mathbf{u}_i, \mathbf{u}_j \rangle)_{r\times r}$ is the Grammian matrix assigned to the vectors of U, and E is the matrix with entries being equal to 1, that is $E = (1)_{r\times r}$. Clearly, U and E are positive semidefinite matrices. As $\cos\alpha < 0$, the matrix A is also positive semidefinite. Since the rank of U is at most n, it is easy to check that the rank of A is at most $n+1$ and so 0 is an eigenvalue of A with the multiplicity at least $r-(n+1)$. As A is positive semidefinite, all other eigenvalues of A are positive. Moreover, as the points (i.e. vectors) of $U \subset S^{n-1}$ form an almost α-equidistant pointset, for all pairwisely different $i, j, k \in \{1, \ldots, r\}$ we have $a_{ij}a_{jk}a_{ki} = 0$, where $a_{ij} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle - \cos\alpha$. Finally, notice that the main diagonal entries of A are all $1-\cos\alpha$. Thus, if I denotes the $r\times r$ identity matrix, then the matrix $B = A - (1-\cos\alpha)I$ with the ij-entry b_{ij}, where $1 \le i, j \le r$, has the following properties:

(2) $b_{ii} = 0$ for all $i = 1, \ldots, r$;

(3) $-(1-\cos\alpha)$ is the smallest eigenvalue of B with the multiplicity at least $r-(n+1)$;

(4) $b_{ij}b_{jk}b_{ki} = 0$ for all triples $1 \le i, j, k \le r$.

Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of B. According to (3), we may assume that $\lambda_{n+2} = \ldots = \lambda_r = -(1-\cos\alpha)$ and that $\lambda_1 \ge -(1-\cos\alpha), \ldots, \lambda_{n+1} \ge -(1-\cos\alpha)$. Apparently, (2) implies that

$$\sum_{i=1}^{r} \lambda_i = \operatorname{tr} B = 0,$$

where $\operatorname{tr} B$ denotes the trace of B. As an immediate result we get that

$$\sum_{i=1}^{n+1} \lambda_i = (r-n-1)(1-\cos\alpha). \tag{5}$$

Since $\operatorname{tr} B^3 = \sum_{1\le i,j,k\le r} b_{ij}b_{jk}b_{ki}$, (4) yields that $\operatorname{tr} B^3 = 0$. Notice that the eigenvalues of B^3 are $\lambda_1^3, \ldots, \lambda_r^3$. Consequently, $\sum_{i=1}^{r} \lambda_i^3 = \operatorname{tr} B^3 = 0$. In other words, we have

$$\sum_{i=1}^{n+1} \lambda_i^3 = (r-n-1)(1-\cos\alpha)^3. \tag{6}$$

Now we intend to use Lemma 9. Assume that $r > 2(n+1)$. Introducing the notations $s = r - 2(n+1) \ge 1$ and $y = 1 - \cos\alpha > 0$ we can rewrite (6) as follows:

$$\sum_{i=1}^{n+1} \lambda_i = (n+1+s)y.$$

Thus, Lemma 9 implies that

$$\sum_{i=1}^{n+1} \lambda_i^3 \geq (n+1+3s)y^3.$$

Finally, according to (6) we have

$$\sum_{i=1}^{n+1} \lambda_i^3 = (n+1+s)y^3,$$

a contradiction. This completes the proof of Theorem 8.

∎

Proof of Theorem 9. As the proof presented here is a properly modified version of the proof of Theorem 8, we describe the major steps only without going into details.

Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ be a maximal system of unit vectors in E^n with the property that among any three vectors $\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k \in U$ there are two, for example $\mathbf{u}_i, \mathbf{u}_j$ with $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \cos\alpha$, where $|\alpha - \frac{\pi}{2}| \leq \epsilon(n)$ for a sufficiently small $\epsilon(n) > 0$ that will be chosen later. (Notice that as $\epsilon(n) > 0$ is small, the angle α is close to $\frac{\pi}{2}$ and so $\cos\alpha$ is close to 0.)

Assume that $r > 2n$. Then let $A = (\langle \mathbf{u}_i, \mathbf{u}_j \rangle)_{(2n+1)\times(2n+1)}$ be the Grammian matrix assigned to the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{2n+1}$. Clearly, A is positive semidefinite of rank at most n and so 0 is an eigenvalue of A with the multiplicity at least $(2n+1) - n = n+1$, and all other eigenvalues of A are positive. Finally, if I is the $(2n+1)\times(2n+1)$ identity matrix, then the matrix $B = A - I$ with the ij-entry b_{ij}, where $1 \leq i, j \leq 2n+1$, has the following properties:

(7) $b_{ii} = 0$ for all $i = 1, \ldots, 2n+1$;

(8) -1 is the smallest eigenvalue of B with multiplicity at least $n+1$;

(9) $b_{ij}b_{jk}b_{ki}$ is close to 0 for all $1 \leq i, j, k \leq 2n+1$ if $\epsilon(n) > 0$ is sufficiently small.

Let $\lambda_1, \ldots, \lambda_{2n+1}$ denote the eigenvalues of B. Thanks to (8), we may assume that $\lambda_{n+1} = \ldots = \lambda_{2n+1} = -1$ and that $\lambda_1 \geq -1, \ldots, \lambda_n \geq -1$. From (7) we obtain that

$$\sum_{i=1}^{n} \lambda_i = n+1. \tag{10}$$

Since $\operatorname{tr} B^3 = \sum_{1\le i,j,k\le 2n+1} b_{ij}b_{jk}b_{ki}$, (9) yields that $\operatorname{tr} B^3$ is close to 0 if $\epsilon(n)$ is sufficiently small. That is,

$$\sum_{i=1}^{d} \lambda_i^3 \text{ is close to } n+1 \text{ if } \epsilon(n) \text{ is sufficiently small.} \tag{11}$$

Finally, applying Lemma 9 with the choice $y = 1$, we get

$$\sum_{i=1}^{n} \lambda_i^3 \ge n+3. \tag{12}$$

As (12) clearly contradicts (11) for any sufficiently small $\epsilon(n)$, the proof of Theorem 9 is complete.

∎

Proof of Theorem 10. Let $U = \{u_1, \ldots, u_r\}$ be an almost d-equidistant pointset on S^{n-1}, for a value $d \in (0, \sqrt{2})$. Let G be the graph defined on the points of U as vertices such that two points of U are connected by an edge if and only if the distance between them is equal to d. Finally, let $f(n-1)$ denote the maximum cardinality of almost d-equidistant pointsets of S^{n-1}.

If the distance between any two points of U is equal to d, then it is easy to see that $r \le n$ and so we are done. Thus, we are left with the case when there are two points of U, for instance u_1 and u_2 lying in a distance different from d. This means that there is no edge of G between the vertices u_1 and u_2. Now, let U_1 and U_2 denote the sets of the vertices of G that are not connected by an edge to the vertex u_1 and u_2, respectively. Moreover, let $U_3 = U \setminus (U_1 \cup U_2)$. As U is an almost d-equidistant pointset, the graphs G restricted to U_1 and U_2 are complete graphs. Thus,

$$\operatorname{card}(U_1) \le n \text{ and } \operatorname{card}(U_2) \le n. \tag{13}$$

Finally, notice that the vertices of U_3 are connected by an edge to u_1 as well as to u_2. As a result, U_3 lies on an $(n-2)$-dimensional great-sphere of S^{n-1}. Hence,

$$\operatorname{card}(U_3) \le f(n-2). \tag{14}$$

Thus, (13) and (14) imply that

$$r = \operatorname{card}(U) \le 2n + f(n-2). \tag{15}$$

From (15) we immediately get that

$$f(n-1) \leq 2n + f(n-2). \tag{16}$$

Finally, (16) with $f(1) = 4$ completes the proof of Theorem 10.

$\blacksquare$

Notations

$\langle .,. \rangle$	the standard inner product of the Euclidean n-space E^n
$b_k(C)$	the greatest possible number d such that the convex body C contains k boundary points in pairwise C-distances at least d
$b_k(\mathcal{C})$	the infimum of $b_k(C)$, where C runs over $\mathcal{C}$
$b_k(\mathcal{M})$	the infimum of $b_k(C)$, where C runs over $\mathcal{M}$
$\mathcal{C}$	the family of plane convex bodies
$\mathrm{card}(S)$	the cardinality of the set S
$c_k(\mathcal{C})$	the supremum of $b_k(C)$, where C runs over $\mathcal{C}$
$c_k(\mathcal{M})$	the supremum of $b_k(C)$, where C runs over $\mathcal{M}$
C_t	the set of points of the convex body C whose C-distance from every boundary point of C is at least $\frac{2t}{1+t}$
$\mathcal{C}_t$	the family of the plane convex bodies that can be presented in the form $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$, where all $C \in \mathcal{C}$ are taken
$d_C(p,q)$	the C-distance of points p and q
$d_k(C)$	the greatest possible number d such that the convex body C contains k points in pairwise C-distances at least d
$d_k(\mathcal{C})$	the infimum of $d_k(C)$, where C runs over $\mathcal{C}$
$d_k(\mathcal{M})$	the infimum of $d_k(C)$, where C runs over $\mathcal{M}$
d_n	the edge length of the n-dimensional regular simplex inscribed in the unit sphere S^{n-1} of E^n
$e_k(\mathcal{C})$	the supremum of $d_k(C)$, where C runs over $\mathcal{C}$
$e_k(\mathcal{M})$	the supremum of $d_k(C)$, where C runs over $\mathcal{M}$
E^n	the n-dimensional Euclidean space
$H(C)$	the translative kissing number of the convex body C
$L(p,q)$	the line containing the points p and q
$\mathcal{M}$	the family of centrally symmetric plane convex bodies

m^x	the x-coordinate of the point m or the vector $\vec{m}$
m^y	the y-coordinate of the point m or the vector $\vec{m}$
pq	the segment with endpoints p and q
$\lvert pq \rvert$	the Euclidean length of the segment pq
$r_k(C)$	the greatest possible number r such that the convex body C can be packed by its k homothetical copies of ratio r
$r_k(\mathcal{C})$	the infimum of $r_k(C)$, where C runs over $\mathcal{C}$
$r_k(\mathcal{M})$	the infimum of $r_k(C)$, where C runs over $\mathcal{M}$
$s_k(\mathcal{C})$	the supremum of $r_k(C)$, where C runs over $\mathcal{C}$
$s_k(\mathcal{M})$	the supremum of $r_k(C)$, where C runs over $\mathcal{M}$
S^{n-1}	the $(n-1)$-dimensional unit sphere of the Euclidean n-space E^n
t_k	the greatest possible number t such that every plane convex body C can be touched by k mutually nonoverlapping translates of tC
$\mathrm{tr}M$	the trace of the matrix M
u_k	the greatest possible number t such that there exists a plane convex body C that can be touched by k mutually nonoverlapping translates of tC

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Summary

Let $C \subset E^n$ be an arbitrary convex body, and let $p, q \in E^n$ be arbitrary points. Take a chord $p'q'$ of C parallel to pq such that there is no longer chord of C parallel to pq. The *C-distance $d_C(p,q)$ of points p and q* is defined by the ratio of $|pq|$ to $\frac{1}{2}|p'q'|$. If there is no doubt about C, we may use the term *relative distance of p and q*.

In our dissertation we examine the pairwise C-distances of points of C. In the main part we investigate the following problem. Let $k \geq 2$. By $d_k(C)$ (resp., by $b_k(C)$) we denote the greatest possible number d such that the convex body C (resp., the boundary of C) contains k points in pairwise C-distances at least d. Compactness arguments show that for every $k \geq 2$ and for every convex body C the above numbers exist. Let us denote the infimum and the supremum of $d_k(C)$, where C runs over the family of plane convex bodies by $d_k(\mathcal{C})$ and by $e_k(\mathcal{C})$, respectively. Moreover, let $d_k(\mathcal{M})$ and $e_k(\mathcal{M})$ denote the infimum and the supremum of $d_k(C)$, where C runs over the family of centrally symmetric plane convex bodies. We define the quantities $b_k(\mathcal{C})$, $c_k(\mathcal{C})$, $b_k(\mathcal{M})$, $c_k(\mathcal{M})$ analogously. Using compactness arguments one can easily show that for every $k \geq 2$ all the numbers $b_k(\mathcal{C})$, $c_k(\mathcal{C})$, $d_k(\mathcal{C})$, $e_k(\mathcal{C})$, $b_k(\mathcal{M})$, $c_k(\mathcal{M})$, $d_k(\mathcal{M})$, $e_k(\mathcal{M})$ exist. In the dissertation we determine the above defined eight numbers for some values of k. Moreover, we show general estimates about $d_k(\mathcal{C})$ and $d_k(\mathcal{M})$. We also examine the connection between the existence of k points of a convex body in large pairwise relative distances and the existence of large mutually nonoverlapping homothetical copies of a convex body packed into, or touching the body.

In the last chapter of the dissertation we deal with another problem. It is a well-known fact that the maximal number of points on the $(n-1)$-dimensional unit sphere S^{n-1} of E^n lying at equal pairwise distances is at most $n+1$. It is also proved that the cardinality of a pointset on the boundary of a convex body $C \subset E^n$ such that the pairwise relative distances of the points are equal is at most 2^n. A pointset P is called *almost d-equidistant*, if among every three points of P there exists a pair in the distance d. In our dissertation we find estimates about the maximal cardinality of almost d-equidistant pointsets on S^{n-1}.

Összefoglalás

Legyen $C \subset E^n$ egy tetszőleges konvex test, és legyenek $p, q \in E^n$ tetszőleges pontok. Vegyük C egy pq-val párhuzamos $p'q'$ húrját, amelynél nincs hosszabb pq-val párhuzamos C-beli húr. A *p és q pontok C-távolságát* $|pq|$-nak $\frac{1}{2}|p'q'|$-vel vett hányadosaként definiáljuk, és $d_C(p, q)$-val jelöljük. Ha nyilvánvaló, hogy melyik C konvex testről beszélünk, a *p és q pontok relatív távolsága* elnevezést is használjuk.

A disszertációban C-beli pontok páronkénti C-távolságait vizsgáljuk. Főként a következő probémával foglalkozunk. Legyen $k \geq 2$. A $d_k(C)$ (ill. a $b_k(C)$) jelölést használjuk a lehetséges legnagyobb d számra, melyre igaz, hogy a C konvex test (ill. C határa) tartalmaz k pontot, melyek páronkénti C-távolsága legalább d. Kompaktsági érvek mutatják, hogy a fenti számok léteznek minden C konvex testre k minden lehetséges értéke esetén. Jelöljük $d_k(C)$ infimumát és supremumát rendre $d_k(\mathcal{C})$-vel és $e_k(\mathcal{C})$-vel, ahol C végigfut a konvex síkidomok családján. Emellett jelöljük rendre $d_k(\mathcal{M})$-mel és $e_k(\mathcal{M})$-mel $d_k(C)$ infimumát és supremumát, ahol C végigfut a középpontosan szimmetrikus konvex síkidomok családján. Hasonlóan definiáljuk a $b_k(\mathcal{C})$, a $c_k(\mathcal{C})$, a $b_k(\mathcal{M})$ és a $c_k(\mathcal{M})$ mennyiségeket. Kompaktsági érvek alapján könnyen megmutatható, hogy $b_k(\mathcal{C})$, $c_k(\mathcal{C})$, $d_k(\mathcal{C})$, $e_k(\mathcal{C})$, $b_k(\mathcal{M})$, $c_k(\mathcal{M})$, $d_k(\mathcal{M})$ és $e_k(\mathcal{M})$ létezik $k \geq 2$ minden értéke esetén. A disszertációban az előbb definiált számokat határozzuk meg k különböző értékei esetén. A $d_k(\mathcal{C})$ és $d_k(\mathcal{M})$ mennyiségekre általános becslést is adunk. Emellett megvizsgáljuk egy konvex testbeli, nagy páronkénti relatív távolsággal rendelkező k pont létezésének kapcsolatát azzal, hogy található-e egy konvex testnek a test köré vagy a testbe írt, páronként diszjunkt belsővel rendelkező nagy homotetikus arányú homotetikus képe.

A disszertáció utolsó fejezetében egy másik problémával foglalkozunk. Jól ismert tény, hogy az n-dimenziós euklideszi tér egységgömbjén található, páronként egyenlő távolságra levő pontok maximális száma $n + 1$. Ugyancsak bizonyított az az állítás, hogy egy n-dimenziós konvex test határán található, páronként egyenlő relatív távolságra levő pontok száma legfeljebb 2^n. A P ponthalmazt majdnem egyenlő d távolságú pontok halmazának hívjuk, ha P bármely három eleme közül létezik kettő, melyek távolsága d. A disszertáció utolsó fejezetében becsléseket adunk az n-dimenziós euklideszi tér egységgömbjén található, majdnem egyenlő d távolságú ponthalmazok maximális számosságára.