ON SEVEN POINTS IN THE BOUNDARY OF A PLANE CONVEX BODY IN LARGE RELATIVE DISTANCES

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Abstract. By the relative distance of points p and q of a convex body C we mean the ratio of the Euclidean length of the segment pq to the half of the Euclidean length of a longest chord of C parallel to pq. We show that in the boundary of every plane convex body there exist seven points in pairwise relative distances at least $\frac{2}{3}$. We also give an estimate in case of three points.

Finding sets of points on the sphere or ball of Euclidean n-space E^n such that their pairwise distances are as large as possible is a long-standing question of geometry. A generalization was presented by Lassak [6], and by Doyle, Lagarias and Randall [3]. In [3], points are considered in the boundary of the unit ball C of a Minkowski space, and their distance is measured by the Minkowski distance. In [6] we see a more general approach. Here C is allowed to be an arbitary convex body. The question is in finding configurations of points in the boundary of C which are far in the sense of the following notion of C-distance of points.

For arbitrary points $p, q \in E^n$ denote the Euclidean length of the segment pq by $|pq|$. Let $p'q'$ be a chord of C parallel to pq such that there is no longer chord of C parallel to pq. The *C-distance* $d_C(p,q)$ of points p and q is defined by the ratio of $|pq|$ to $\frac{1}{2}|p'q'|$ (see [6]). We also use the term *C-length* of the segment pq. If there is no doubt about C, we may use the terms *relative distance* of p and q, or *relative length* of pq.

Both papers [3] and [6] show that every centrally symmetric plane convex body contains four boundary points in pairwise relative distances at least $\sqrt{2}$, and six boundary points whose pairwise relative distances are at least 1. Doliwka [2] proved that in the boundary of every plane convex body there exist five points in at least unit pairwise relative distances.

In this paper we show a similar result about seven points in the boundary of a plane convex body. We also improve the estimate in [1] about three far boundary points.

Theorem. *The boundary of an arbitrary plane convex body contains seven points in pairwise relative distances at least $\frac{2}{3}$ such that the relative distances of every two successive points are equal.*

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The proof of Theorem is based on the following lemma.

Lemma. *Let* $F = f_1f_2\dots f_7$ *be a convex heptagon. Then every convex heptagon* $D = d_1d_2\dots d_7$ *inscribed in* F *such that* $d_i \in f_if_{i+1}$ *for* $i = 1, 2, \dots, 7$, *where* $f_8 = f_1$, *has a side of* F*-length at least* $\frac{2}{3}$.

Proof. Let α_i denote the angle $\angle f_{i-1}f_if_{i+1}$ $(i = 1, \dots, 7)$, where $f_0 = f_7$. Since every heptagon is the limit of a sequence of nondegenerate heptagons, it is sufficient to prove our lemma under the assumptions that $\alpha_1 < \pi$, …, $\alpha_7 < \pi$.

First, we intend to show that if the sum of two consecutive angles of F is at most π, then D has a side of F-length at least 1 (see Figure 1).

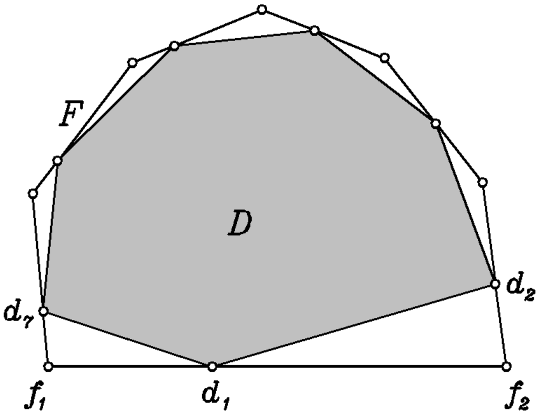

Figure 1

Assume, for example, that $\alpha_1 + \alpha_2 \le \pi$. Observe that in this case $d_F(f_1, f_2) = 2$. As it is explained after Lemma 6 of [7], Lemma 3 of [7] implies that if x is a boundary point of a plane convex body C, and if y moves counterclockwise in the boundary of C from x, then $d_C(x, y)$ does not decrease until it reaches 2, and it accepts all values from the interval $[0, 2]$. Thus we get that $d_F(d_7, d_1) + d_F(d_1, d_2) \ge d_F(f_1, d_1) + d_F(d_1, f_2) = d_F(f_1, f_2) = 2$, and therefore $d_F(d_7, d_1) \ge 1$ or $d_F(d_1, d_2) \ge 1$. We omit an analogous consideration which shows that if the sum of every pair of consecutive angles of D is greater than π and if D has three consecutive angles such that their sum is at most 2π, then D has a side of F-length at least $\frac{2}{3}$.

Now consider the case when the sum of every three consecutive vertices of D is greater than 2π. Denote the intersection of the lines containing the segments f_2f_3 and f_4f_5 by a_3. Similarly, let a_5 be the intersection point of the lines containing the segments f_5f_6 and f_7f_1 (see Figure 2). Consider the convex pentagon $D' = d_1d_2d_4d_5d_7$ inscribed in the convex pentagon $F' = f_1f_2a_3f_5a_5$. The angles of F' are $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3 + \alpha_4 - \pi$, $\beta_4 = \alpha_5$, $\beta_5 = \alpha_6 + \alpha_7 - \pi$. This implies that the sum of every two consecutive angles of F' is greater than π. For the sake of simplicity we use the following notation in the sequel: $a_1 = f_1$, $a_2 = f_2$, $a_4 = f_5$, $b_1 = d_1$, $b_2 = d_2$, $b_3 = d_4$, $b_4 = d_5$, $b_5 = d_7$.

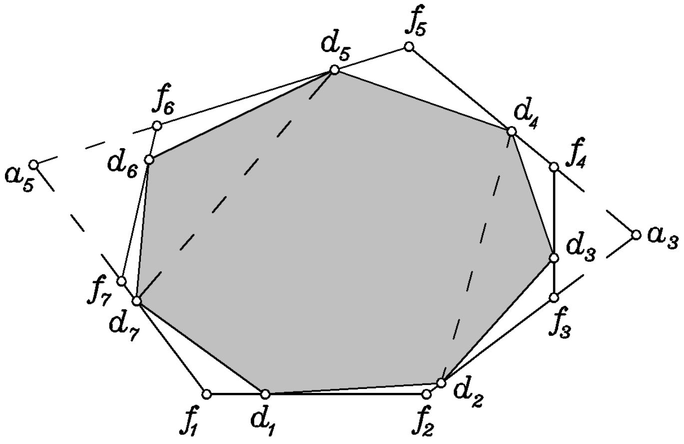

Figure 2

We intend to show that the F'-length of b_2b_3 or b_4b_5 is at least $\frac{4}{3}$, or that the F'-length of another side of D' is at least $\frac{2}{3}$. We will show this indirectly. Hence let us assume that $d_{F'}(b_2, b_3) < \frac{4}{3}$, $d_{F'}(b_4, b_5) < \frac{4}{3}$, and that the remaining sides of D' are of F'-length less than $\frac{2}{3}$. Let c_1 and c_1' denote the trisection points of a_1a_2 such that c_1 is closer to a_1 (see Figure 3). Moreover, let c_2, c_3, c_4, c_5 be the trisection points of a_2a_3, a_3a_4, a_4a_5, a_5a_1 closer to the points a_2, a_4, a_4, a_1, respectively.

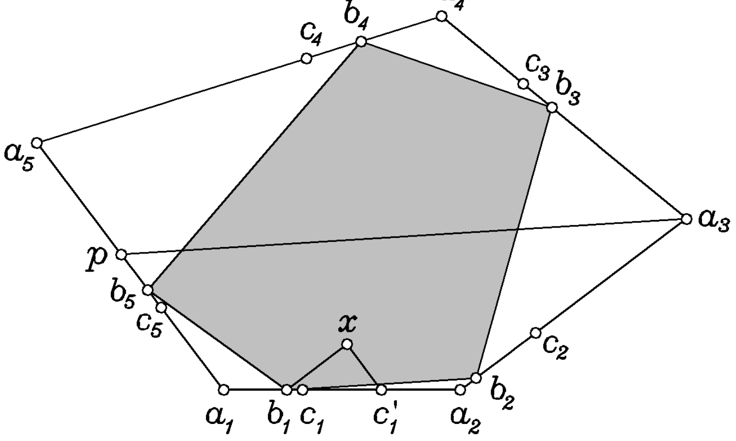

Figure 3

With respect to our assumption, b_1 cannot be an inner point of the segment c_1c_1'. Without loss of generality, we can assume that $b_1 \in a_1c_1$ (in the opposite case the proof is analogous). Observe that $b_1 \in a_1c_1$ implies that $b_i \in a_ic_i$ for $i = 2, 3, 4, 5$. Take the common point p of the straight line containing the segment a_5a_1 and of the straight line through a_3 parallel to b_1b_2. Notice that $d_{F'}(b_1, b_2) \geq 2|b_1b_2|/|a_3p|$. Let x be the intersection point of the line through b_1 parallel to a_2a_3 and of the line through c_1' parallel to a_5a_1. As $d_{F'}(b_1, b_2) < \frac{2}{3}$, we see that $|xb_1| < |b_2c_2|$. Now consider the triangle $b_1c_1'x$. We have $|b_1c_1'|/\sin(\beta_1 + \beta_2 - \pi) = |xb_1|/\sin(\pi - \beta_1)$. Thus, $\sin(\pi - \beta_1)|b_1c_1| < \sin(\pi - \beta_1)|b_1c_1'| < \sin(\beta_1 + \beta_2 - \pi)|b_2c_2|$. We omit an analogous calculation that $\sin(\pi - \beta_i)|b_ic_i| < \sin(\beta_i + \beta_{i+1} - \pi)|b_{i+1}c_{i+1}|$ for $i = 2, 3, 4, 5$, where $\beta_6 = \beta_1$, $b_6 = b_1$ and $c_6 = c_1$. Hence

$\prod_{i=1}^{5} \sin\beta_i < \prod_{i=1}^{5} \sin(\beta_i + \beta_{i+1} - \pi)$. This contradicts Lemma 2 of [4], which says that for every $\beta_1, \dots, \beta_5 \in (0, \pi)$ such that $\sum_{i=1}^{5} \beta_i = 3\pi$ and $\beta_i + \beta_{i+1} > \pi$ for every $i \in \{1, \dots, 5\}$, where $\beta_6 = \beta_1$, we have $\prod_{i=1}^{5} \sin\beta_i > \prod_{i=1}^{5} \sin(\beta_i + \beta_{i+1} - \pi)$.

We have shown that b_2b_3 or b_4b_5 has F'-length at least $\frac{4}{3}$, or that another side of D' is of F'-length at least $\frac{2}{3}$. As $F \subset F'$, we get that $d_F(s,t) \geq d_{F'}(s,t)$ for every $s, t \in E^2$. Thus, if at least one of the numbers $d_{F'}(b_1, b_2)$, $d_{F'}(b_3, b_4)$ or $d_{F'}(b_5, b_1)$ is at least $\frac{2}{3}$, then we are done. If $d_{F'}(b_2, b_3)$ or $d_{F'}(b_5, b_1)$ is at least $\frac{4}{3}$, then the statement of our Lemma is the consequence of the triangle inequality.

■

Proof of Theorem. Let C be an arbitrary plane convex body. Theorem 1 from [7] implies that for every $n \geq 3$ there exists an n-gon inscribed in C whose sides are of equal C-length. Thus, it is sufficient to show that every convex heptagon inscribed in C has a side of C-length at least $\frac{2}{3}$. Consider an arbitrary convex heptagon D inscribed in C. At every vertex of D take a supporting line of C. Let F denote the intersection of the closed halfplanes containing C bounded by the above supporting lines. Obviously, F is a convex heptagon circumscribed about D such that $D \subset C \subset F$. Observe that the C-length of every side of D is at least its F-length. Therefore our Lemma implies that D has a side of C-length at least $\frac{2}{3}$.

■

The example of a triangle shows that the estimate $\frac{2}{3}$ in our theorem cannot be improved. Notice that by Lemma 3 of [7] for every positive integer r our theorem implies the existence of $7r$ points in the boundary of every plane convex body in pairwise relative distances at least $\frac{2}{3} \cdot \frac{1}{r}$. Theorem of [2] says that every plane convex body contains five boundary points in pairwise relative distances at least 1. Thus, by Lemma 3 of [7] this theorem implies that for every positive integer r in the boundary of every plane convex body there exist $5r$ points in pairwise relative distances at least $\frac{1}{r}$. The example of a triangle shows that this estimate is the best possible one not only for $r = 1$ as proved in [2], but also for $r = 2$.

Below we improve the estimate of Bezdek, Fodor and Talata from [1] for three points in the boundary of a plane convex body.

Proposition. *In the boundary of every plane convex body there exist three points in equal pairwise relative distances at least* $\frac{1}{5}(2 + 2\sqrt{6}) \approx 1.3798$.

Proof. Let C be a plane convex body. For the simplicity of considerations, during the proof we denote the value $\frac{1}{5}(1 + \sqrt{6})$ by k. First we are looking for three points in C in pairwise C-distances at least $2k$.

According to Lemma 1 from [5] we circumscribe a parallelogram P about C such that the midpoints of its two parallel sides belong to C. As the C-distance of two points does not change under affine transformations, we can assume that P is a rectangle such that the length of the sides containing the mentioned midpoints is 2, and that the length of the other sides is 1. Consider the Cartesian coordinate system such that the above midpoints are $o = (0, 0)$ and $c = (0, 1)$. Since C is inscribed in P, it contains a point $a = (-1, \alpha)$ and a point $b = (1, \beta)$, where $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$.

Case 1, when $\alpha + \beta \leq \frac{\sqrt{6}}{3}$ or $\alpha + \beta \geq 2 - \frac{\sqrt{6}}{3}$. We assume that $\alpha + \beta \leq \frac{\sqrt{6}}{3}$ (in the other case the proof is analogous). Observe that $\frac{\sqrt{6}}{3} = \frac{1-k}{2k-1}$. We intend to prove that the quadrangle $obca$ contains points r and s with y-coordinates at most $1 - k$ and with the difference of their x-coordinates at least $2k$. As $obca \subset C$, the points r, s and c are three points that we are looking for.

Subcase 1.1, when $\alpha \geq 1 - k$ and $\beta \geq 1 - k$. Since the harmonic mean is not greater than the arithmetic mean, our assumptions imply that $\frac{1}{\alpha} + \frac{1}{\beta} \geq \frac{4}{\alpha+\beta} > \frac{2k}{1-k}$. Furthermore, a calculation shows that the intersection of the quadrangle $obca$ with the straight line $y = 1 - k$ is a segment of Euclidean length $(1 - k)(\frac{1}{\alpha} + \frac{1}{\beta})$. Thus this length is at least $2k$. In the part of r and s we take the endpoints of this segment.

Subcase 1.2, when $\alpha < 1 - k$ or $\beta < 1 - k$. Let $\alpha < 1 - k$ (if $\beta < 1 - k$, considerations are similar). By the assumption of Case 1 we have $\beta \leq \frac{1-k}{2k-1}$. Thus the quadrangle $obca$ contains the point $(2k - 1, 1 - k)$. We take it in the part of r. As s we take a.

Case 2, when $\frac{\sqrt{6}}{3} < \alpha + \beta < 2 - \frac{\sqrt{6}}{3}$. We intend to show that C contains points w and z with the difference of their y-coordinates at least k, and with their C-distances at least $2k$ either from a or from b. Then w, z, and a or b are three promised points.

Let p and q denote the intersections of the straight line $x = -1 + k$ with the segments ao and ac, respectively.

Subcase 2.1, when $d_C(p, b) \geq 2k$ and $d_C(q, b) \geq 2k$. It is clear that $d_C(p, q) = 2k$. Thus we take p and q in the part of w and z.

Subcase 2.2, when $d_C(p, b) < 2k$ or $d_C(q, b) < 2k$. We can assume that $d_C(p, b) < 2k$ (in the other case our consideration is analogous). This assumption implies that there exists a point $t \in C$ whose translation u by $\vec{v} = \frac{1}{k}\overrightarrow{pb}$ is also a point of C. We intend to show that $g = \big(-(2k - 1), (2k - 1)\alpha + 2 - 3k\big)$ or $h = \big(2k - 1, (2k - 1)\beta + k\big)$ belongs to C (see Figure 4). Suppose instead that $g \notin C$ and $h \notin C$.

Let L_g be the line through o and g. Its equation is $y = -\big(\alpha - \frac{3k-2}{2k-1}\big)x$. Denote the right-hand side of this equation by $g(x)$. Let L_h be the line through c and h. Its equation is $y = \big(\beta - \frac{1-k}{2k-1}\big)x + 1$. Denote its right-hand side by $h(x)$. Take the common point e of the lines L_g and $x = -1$. We have $e = (-1, \alpha - \frac{3k-2}{2k-1})$. The common point of L_h and the line $x = 1$ is $f = (1, \beta + \frac{3k-2}{2k-1})$.

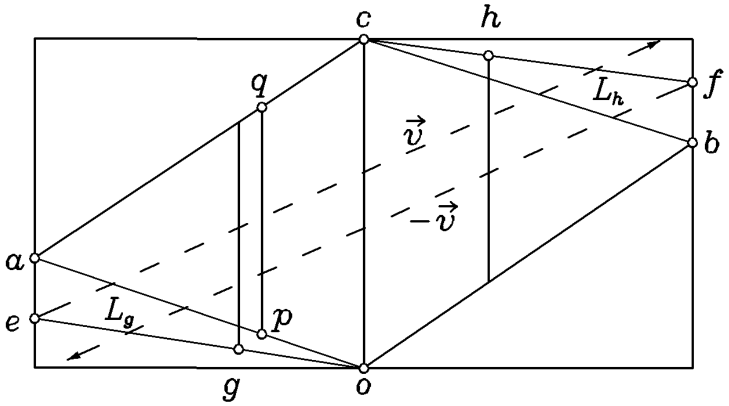

Figure 4

Let us denote the x-coordinate of a point m or vector $\vec{m}$ by x_m, and its y-coordinate by y_m. Notice that $x_v > 1 + x_h = 1 + |x_g|$. This, and the assumption that $g \notin C$ and $h \notin C$ imply that the points t and u belong to the domain bounded by the sides of P and by the lines L_g and L_h. Hence we can take either e in the part of t, or f in the part of u. This depends on the directions of L_g and L_h. Then either of the following holds true.

(i) The translate of e by $\vec{v}$ is in the open half plane containing e bounded by the line L_h. In this case $0 > y_e + y_v - h(x_e + x_v) = (\alpha + \beta)(3 - \sqrt{6}) + 2 - \sqrt{6}$. Hence from $\alpha + \beta > \frac{\sqrt{6}}{3}$ we obtain $0 > 0$, which is a contradiction.

(ii) The translate of f by $-\vec{v}$ is in the open half plane containing f bounded by the line L_g. We get $0 < y_f - y_v - g(x_f - x_v) = 7 - 3\sqrt{6} - (\sqrt{6} - 2)(\alpha + \beta)$. From $\alpha + \beta > \frac{\sqrt{6}}{3}$ we conclude that the right-hand side of this inequality is negative, which is also impossible.

Thus $g \in C$ or $h \in C$. We omit a calculation showing that if $g \in C$, then the intersection of the line $x = -(2k-1)$ and C is a segment of length at least k. We take the endpoints of this segment as w and z. Since the distance of the lines $x = -(2k-1)$ and $x = 1$ is $2k$, and since $C \subset P$, we conclude that $d_C(b, w) \geq 2k$ and $d_C(b, z) \geq 2k$.

Similarly, it can be shown that if $h \in C$, then the intersection of C and the line $x = 2k - 1$ is a segment of length at least k. Now we use the endpoints of this segment in the part of w and z.

We have shown that there exists a triangle in C whose all sides have relative lengths at least $\frac{1}{5}(2 + 2\sqrt{6})$. But Theorem 2 of [7] says that if an arbitrary convex body C contains an n-gon whose all sides are of relative lengths at least d, then C permits to inscribe an n-gon whose sides are of equal relative length at least d.

$\blacksquare$

In Theorem 3 of [7] it is shown that if $C \subset E^2$ is a convex body and if P is a polygon inscribed in C, then all sides of P have the same relative length d if and only if the consecutive homothetical copies of C with homothety centers at the vertices of P and with ratio $\frac{d}{2+d}$ touch each other (this also follows from Lemma 2 of [5]). Thus from Theorem and Proposition we get the following corollary.

Corollary. *Every plane convex body C can be packed by its seven homothetical copies with homothety ratio at least $\frac{1}{4}$ touching the boundary of C from inside such that every two succesive of those copies touch each other. Here the estimate $\frac{1}{4}$ cannot be improved. Moreover, every plane convex body C can be packed by its three homothetical copies with homothety ratio at least $\frac{1}{\sqrt{6}}$ touching its boundary from inside such that every two of those copies touch each other.*

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