ON FOUR POINTS OF A CONVEX BODY IN LARGE RELATIVE DISTANCES

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Let C be a convex body in Euclidean plane E^2. By the *C-distance* $\mathrm{dist}_C(x,y)$ of points $x, y \in E^2$ we understand the ratio of the Euclidean distance $|xy|$ of x and y to the half of the maximum distance of points a and b in C such that the segments xy and ab are parallel (see [3]). When there is no doubt about the body C, we also use the term *relative distance*.

From [2] and [3] we know that every centrally-symmetric planar convex body contains four points in relative distances at least $\sqrt{2}$. Analogical question about four far points in an arbitrary planar convex body seems to be difficult. Here is our conjecture.

CONJECTURE. Every planar convex body contains four points in paiwise relative distances at least $\sqrt{5}-1$.

This value is attained for C being the pentagon $a_1'a_1a_2a_2'a_3$ such that the triangle $a_1'a_2'a_3$ is isosceles with $\angle a_2a_3a_1 = \frac{\pi}{2}$, and such that the quadrangle $a_1'a_1a_2a_2'$ is a rectangle with $|a_1a_2| = (\sqrt{5}+1)|a_1a_1'|$ (see Fig. 1).

There are two configurations of four points in pairwise relative distances at least $\sqrt{5}-1$ here. The first configuration consists of three points a_1, a_2, a_3 on the boundary of the body

and one point a inside of the triangle $a_1a_2a_3$. The second configuration consists of four points b_1, b_2, b_3, b_4 on the boundary of the pentagon.

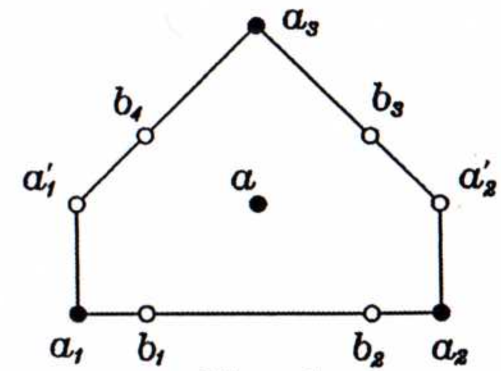

Figure 1

Analogical approach is developed in the proof of Theorem. If the shape of the body is somehow "similar" to a triangle, the first kind of configuration gives larger relative distances of points, and if not, then the second kind of configuration.

Let us add that the papers [1] and [3] consider analogical question about a few points in possibly large C-distance in a convex body $C \subset E^2$ while [2] concentrates on the situation when C is centrally-symmetric.

By the *C-distance of two parallel lines* we mean the minimum C-distance of two points from those lines, respectively. It is easy to see that the C-distance of two parallel lines is nothing else but the ratio of the width of the strip between those lines to the half of the width of C in the perpendicular direction.

Whenever we say about the *distance* of points, we mean the Euclidean distance.

THEOREM. *Every planar convex body C contains four points in pairwise C-distances at least $\frac{1}{3}(\sqrt{5}+1)$.*

Proof. Consider a triangle $T = a_1a_2a_3$ of the largest possible area inscribed in C. Since the C-distance of points does not change under affine transformations, we may assume that T is a regular triangle of sides of length 2 (see Fig. 2). During the proof it is convenient to imagine the direction of the side a_2a_3 as horizontal.

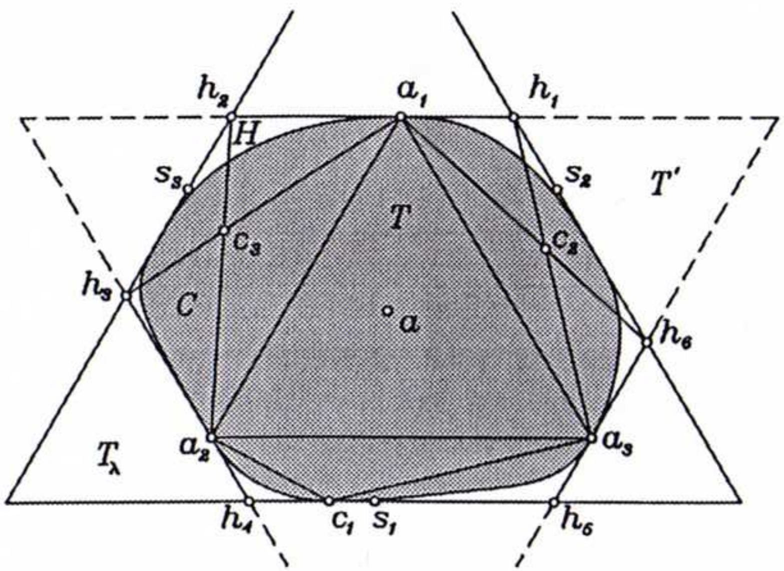

Figure 2

From the maximality of the area we conclude that the straight lines through the vertices parallel to the opposite sides of T are supporting lines of C. By T' we denote the triangle bounded by the above supporting lines. Consider the smallest positive homothetic image T_λ of T which contains C. Here λ denotes the ratio of the homothety which transforms T into T_λ. The intersection of the triangles T_λ and T' is a hexagon $H = h_1h_2h_3h_4h_5h_6$. The notation is chosen such that $a_1 \in h_1h_2$, $a_2 \in h_3h_4$, $a_3 \in h_5h_6$. We denote the common value of $|a_2h_4|$ and $|h_5a_3|$ by x_1, the common value of $|a_3h_6|$ and $|h_1a_1|$ by x_2, and the common value of $|a_1h_2|$ and $|h_3a_2|$ by x_3. Clearly, $\lambda = \frac{1}{2}(x_1 + x_2 + x_3) + 1$.

Since $C \subset H$, in order to find four points of C in pairwise C-distances at least $\frac{1}{3}(\sqrt{5}+1)$ it is sufficient to find four points

of C in pairwise H-distances at least $\frac{1}{3}(\sqrt{5}+1)$.

We intend to show that the pairwise H-distances of a_1, a_2, a_3 are over $\frac{1}{3}(\sqrt{5}+1)$. Consider the three triangles T_1, T_2, T_3 which are copies of T' under homotheties of centers at the vertices of T' and ratio $\frac{1}{8}$. The sides of T' are of length 4, and since $\lambda \leq \frac{5}{2}$ (see [4]), the sides of T_λ are of length at most 5. As H is contained in T_λ, we conclude that among T_1, T_2 and T_3 there exists at most one such that H has a point in its interior. Thus, the maximal chords of H parallel to the sides of T are of lengths at most $\frac{7}{2}$. Since the sides of T are of length 2, we see that the H-distances of the vertices of T are at least $\frac{8}{7}$, and thus over $\frac{1}{3}(\sqrt{5}+1)$.

Case 1, when $\lambda \leq 3\sqrt{5} - 5$.

Let S be the triangle bounded by segments connecting the centers of sides of T_λ. As S is a homothetic image of T_λ of ratio $-\frac{1}{2}$, it is a homothetic image of T of ratio $-\frac{1}{2}\lambda$. Denote by a the center of the homothety that transforms T into S. Denote the images of the points a_1, a_2, a_3 by s_1, s_2, s_3, respectively. The segments a_1s_1, a_2s_2, a_3s_3 intersect at a.

We show that $a \in C$. If $S \cap T = \emptyset$, then T_λ has a side which does not intersect T' which means that C does not intersect this side of T_λ, contrary to the definition of T_λ. So the intersection of T and S is not empty. This and the description of a give $a \in T \cap S$. Since $T \subset C$, we get $a \in C$.

Of course, $\frac{|as_i|}{|aa_i|} = \frac{1}{2}\lambda$ for i = 1, 2, 3. We omit a calculation which shows that $\frac{|aa_i|}{(1/2)|a_is_i|} = \frac{4}{2+\lambda}$ for $i = 1, 2, 3$. Thus the H-distances of a from the points a_1, a_2, a_3 are at least $\frac{4}{2+\lambda}$. Hence, those H-distances, and thus also C-distances are at least $\frac{4}{2+3\sqrt{5}-5} = \frac{1}{3}(\sqrt{5}+1)$.

The pairs of points a_1, a_2, a_3 are also in C-distances at least $\frac{1}{3}(\sqrt{5}+1)$ as explained earlier. So the points a_1, a_2, a_3, a of C are in pairwise C-distances at least $\frac{1}{3}(\sqrt{5}+1)$.

Case 2, when $\lambda \geq 3\sqrt{5} - 5$.

We do not make our proof narrower assuming that $x_1 \leq x_2 \leq x_3$.

Denote by c_1 a point of the body C on the side h_4h_5 of the hexagon H, by c_2 the common point of segments a_3h_1 and a_1h_6, by c_3 the common point of segments a_1h_3 and a_2h_2. From the convexity of C we conclude that the hexagon $G = a_1c_3a_2c_1a_3c_2$ is a subset of C.

Before Case 1 we have explained that the H-distance of a_2 and a_3 is over $\frac{1}{3}(\sqrt{5}+1)$. Thus there is a horizontal segment S_1 whose endpoints are on the segments a_2c_1 and a_3c_1 in the H-distance $\frac{1}{3}(\sqrt{5}+1)$. We omit a tedious calculation that under the assumption of Case 2 there is also a horizontal segment S_2 whose endpoints are on the segments a_1c_2 and a_1c_3 in the H-distance $\frac{1}{3}(\sqrt{5}+1)$.

We see that the four endpoints of the segments S_1 and S_2 belong to G and thus to C.

We intend to show that they are in pairwise H-distances at least $\frac{1}{3}(\sqrt{5}+1)$. Thus it remains to show that the H-distance l of the lines L_1 and L_2 containing the segments S_1 and S_2 is at least $\frac{1}{3}(\sqrt{5}+1)$.

We wish to check the behavior of the H-distance l in dependence on x_2 and x_3, but under the condition that x_1 and $x_2 + x_3$ are fixed.

Since the opposite sides of H are parallel, we conclude that the longest horizontal segment in H is of length $2 + x_2$. As $x_2 + x_3$ is fixed, the value $2 + x_2$ is maximal for $x_2 = x_3$. From this and from the fact that the H-distance of the endpoints of S_i is fixed for $i = 1$ and $i = 2$ we get that $|S_1| = |S_2|$ is maximal for $x_2 = x_3$. So the distance d_1 of the line L_1 and the line through a_2 and a_3 is minimal for $x_2 = x_3$.

Denote by p_2 and p_3 the points of the intersection of the line through a_2 and a_3 with the lines through a_1, h_3, and through a_1, h_6. An elementary calculation gives $|p_2p_3| = \frac{4}{2-x_2} + \frac{4}{2-x_3} - 2$. This expression is minimal for $x_2 = x_3$.

Consider the distance d_2 of the line L_2 and the line through a_2 and a_3. As $|S_2|$ is maximal and $|p_2p_3|$ is minimal for $x_2 = x_3$, from the triangle $a_1p_2p_3$ we see that d_2 is minimal for $x_2 = x_3$. Remember that also d_1 is minimal for $x_2 = x_3$. Therefore the distance between the straight lines L_1 and L_2 is minimal for $x_2 = x_3$. Since the distance of the horizontal lines through a_1 and c_1 does not change, we conclude that l is minimal for $x_2 = x_3$.

It remains to consider the case when $x_2 = x_3$. We intend to show that $l \geq \frac{1}{3}(\sqrt{5}+1)$. We omit a time consuming calculation which shows that the distance of L_1 and L_2 is $\frac{\sqrt{3}}{2}(2+x_1)(\frac{1}{6}(5-\sqrt{5})+\frac{1}{6}(\sqrt{5}+1)\frac{x_2}{2}\cdot\frac{2-x_1}{2+x_1})$. As the width of H in the direction parallel to the lines L_1 and L_2 is $\frac{\sqrt{3}}{2}(2+x_1)$, we get $l = \frac{1}{3}(5-\sqrt{5})+\frac{1}{3}(\sqrt{5}+1)\frac{x_2}{2}\cdot\frac{2-x_1}{2+x_1}$. When x_2 decreases and x_1 increases, then l decreases. Hence it is minimal for $x_1 = x_2$.

We see that the worst case is when $x_1 = x_2 = x_3$. Thus now $\lambda = \frac{3}{2}x_1 + 1$. By the assumption of Case 2 we have $\lambda \geq 3\sqrt{5} - 5$, and by [4] we have $\lambda \leq \frac{5}{2}$. So $2\sqrt{5} - 4 \leq x_1 \leq 1$. From the preceding calculation we get that now $l = \frac{1}{3}(5 - \sqrt{5}) - \frac{1}{3}(\sqrt{5}+1)\frac{x_1}{2}\cdot\frac{2-x_1}{2+x_1} = \frac{1}{3}(11+5\sqrt{5}) - \frac{1}{3}(\sqrt{5}+1)\cdot 2\sqrt{2}\cdot(\frac{x_1+2}{2\sqrt{2}} + \frac{2\sqrt{2}}{x_1+2})$. The form of the expression in the parenthesis shows that l is always at least the minimum of its values at the ends of the interval $[2\sqrt{5}-4,\ 1]$ in which x_1 changes. Thus it is always at least $\frac{1}{3}(\sqrt{5}+1)$. We conclude that the four endpoints of the segments S_1 and S_2 are in pairwise H-distances at least $\frac{1}{3}(\sqrt{5}+1)$. Since $C \subset H$, their C-distances are also at least this number. ∎

The endpoints of the segments S_1 and S_2 in Case 2 of the above proof are usually not in the boundary of C. But if we prolong the segments S_1, S_2 up to the intersection with the boundary of C, we obtain four boundary points b_1, b_2, b_3, b_4 in the pairwise C-distances at least $\frac{1}{3}(\sqrt{5}+1)$.

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CHROMATIC NUMBER OF THE PLANE & ITS RELATIVES
Part II: Polychromatic Number & 6-Coloring

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In Memory of Paul Erdős on Occasion of His 90th Birthday

This is the second installment of an essay [Soi7] that started in the previous issue. We will look here at polychromatic number of the plane, notion of the type of coloring, and 6-colorings of the plane.

3. POLYCHROMATIC NUMBER OF THE PLANE

When a great problem withstands all assaults, mathematicians create many related problems. It gives them something to solve :-). Sometimes there is a real gain in this process, when an insight into a related problem brings new ways to conquer the original one. Numerous problems were posed around the chromatic number of the plane. I would like to share with you my favorite among them.

It is convenient to say that a colored set S *realizes distance* d if S contains a monochromatic segment of length d.

Our knowledge about this problem starts with the celebrated 1959 book by Hugo Hadwiger ([HD2], and consequently its translations [HD3] and [HDK]). Hadwiger reported in the book that he had received a 9/9/1958 letter from the Hungarian mathematician A.