Stochastic processes exam

24th Jan 2023

Theoretical part

1. (a) (3 points) How can we compute the generating function of a sum with a random amount of summands from an i.i.d. sequence? State it mathematically and prove it.
   (b) (2 points) Define branching processes.
   (c) (2 points) Express the probability that there is no individual in the nth generation of a branching process in terms of the generating function of the offspring distribution. *Hint:* Use the statement of the first part of this exercise.
   (d) (2 points) How can the extinction probability be computed in a branching process?

2. (2+6 points) Let a and b be two absorbing states in a finite-state Markov chain in continuous time. Consider the probability that the Markov chain hits a before b depending on the starting position. Write down and prove a system of equations for these probabilities which are sufficient to determine the values of these probabilities.

   Hint: Use the continuous version of the one-step argument and condition on the first jump of the Markov chain.

3. (a) (3 points) Define Brownian motion.
   (b) (5 points) Describe Lévy's construction for Brownian motion on $[0, 1]$ without proofs. *Hint:* Specify the values of the Brownian motion in dyadic points, write down how it is extended to all other points and explain that it defines a process with the required properties.

Exercise part

4. (6 points) Folk wisdom holds that in Ithaca in the summer it rains 1/3 of the time, but a rainy day is followed by a second one with probability 1/2. Suppose that Ithaca weather is a Markov chain. What is its transition probability?

5. (6 points) Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. Let X_t be the number of frogs in the sun at time t. Find the stationary distribution for X_t explicitly.

   Hint: Either solve the system of equations for stationarity or note that the three frogs are independent two-state Markov chains.

6. (7 points) Let $\Omega = \{1, 2, 3\}$, $\mathcal{F} = 2^\Omega$ and $\mathbf{P}(\{1\}) = 1/2$, $\mathbf{P}(\{2\}) = \mathbf{P}(\{3\}) = 1/4$. Consider also the sub σ-algebras

   $$\mathcal{G} = \{\emptyset, \{1, 2\}, \{3\}, \Omega\}, \quad \mathcal{H} = \{\emptyset, \{1, 3\}, \{2\}, \Omega\}.$$

   Let $X : \Omega \to \mathbb{R}$ be the random variable $X(\omega) = \omega$. Compute $\mathbf{E}(\mathbf{E}(X|\mathcal{G})|\mathcal{H})$.

7. (6 points) Let η_i be a sequence of i.i.d. random variables with normal distribution of mean μ and variance σ^2. Define $X_i = e^{2\eta_i}$. For what values of μ and σ is $M_n = M_0 \cdot X_1 \dots X_n$ a martingale?