Stochastic processes exam

11th Jan 2023

Theoretical part

1. (a) (3 points) Consider a discrete-time Markov chain on the state space S with transition matrix P. Let $x \in S$ be one of the states. State (without proof) a necessary and sufficient condition for x being recurrent in terms of a sum involving powers of the matrix P.

   (b) (3+3 points) Identify the transition matrix of the simple symmetric random walk on $\mathbb{Z}$, compute $P^{2n}(0,0)$ and apply the above condition to show that 0 is a recurrent state.

   Hint: Use Stirling's approximation: $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$ as $n \to \infty$.

2. (2+6 points) Consider a continuous-time Markov chain on the state space S with infinitesimal generator Q. Let π be a probability distribution on S. State and prove a necessary and sufficient condition for the distribution π being stationary in terms of the generator Q.

   Hint: Kolmogorov's forward and backward equations $\frac{\mathrm{d}}{\mathrm{d}t} P_t = P_t Q = Q P_t$ can be used.

3. Let X be a random variable on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ which satisfies $\mathbf{E}(|X|) < \infty$. Let $\mathcal{F} \subset \mathcal{A}$ be a sub σ-algebra.

   (a) (2 points) Define the conditional expectation $\mathbf{E}(X|\mathcal{F})$.

   (b) (6 points) State (without proof) the following properties of the conditional expectation: monotonicity, dominated convergence, tower property.

Exercise part

4. (2+2+2 points) Suppose that the probability that it rains today is 1/3 if neither of the last two days was rainy, but 2/3 if at least one of the last two days was rainy. Let the weather on day n denoted by W_n be R for rain and S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$.

   (a) Compute its transition probabilities.

   (b) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday?

   (c) What is the fraction of rainy days on the long run? (The answer to the last question may involve the inversion of a matrix which does not have to be computed, but the matrix to be inverted should be given explicitly.)

5. (2+2+2 poinst) Consider a bank with two tellers. Three people, Alice, Betty, and Carol enter the bank at almost the same time and in that order. Alice and Betty go directly into service while Carol waits for the first available teller. Suppose that the service times for each customer are exponentially distributed with mean 4 minutes.

   (a) What is the expected total amount of time for Carol to complete her businesses?

   (b) What is the expected total time until the last of the three customers leaves?

   (c) What is the probability Carol is the last one to leave?

6. (4+3 points) Let $S_n = S_0 + X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are i.i.d. with $\mathbf{P}(X_i = 1) = 2/5$ and $\mathbf{P}(X_i = -1) = 3/5$. Let $V = \min\{n \geq 0 : S_n = 0\}$. Show that $\mathbf{E}_1(V) = \mathbf{E}(V|S_0 = 1) = 5$ in the following steps.

   (a) Use Wald's equation for the stopping time $V \wedge n = \min(V, n)$ and use the lower bound $\mathbf{E}_1(S_{V\wedge n}) \geq 0$ to conclude that $\mathbf{E}_1(V \wedge n) \leq 5$ for all n which implies $\mathbf{E}_1(V) \leq 5$.

   (b) Use Wald's equation for the stopping time V to prove $\mathbf{E}_1(V) = 5$.

   Hint: Wald's equation states that for any stopping time T with respect to the filtration $\mathcal{F}_n = \sigma(\{X_1, \ldots, X_n\})$ which satisfies $\mathbf{E}(T) < \infty$ it follows that $\mathbf{E}(S_T - S_0) = \mathbf{E}(X_1)\mathbf{E}(T)$.

7. (6 points) Let $(B(t), t \geq 0)$ be a Brownian motion. Show that the process $(B(t+1) - B(1), t \geq 0)$ is also a Brownian motion.