Stochastic processes exam

21st Dec 2022

Theoretical part

1. (a) (3 points) What does the detailed balance condition mean for a discrete time Markov chain? Define reversible discrete time Markov chains.

   (b) (5 points) Introduce the simple random walk on simple graphs. Show that the distribution
   $$\pi(x) = \frac{\deg(x)}{2|E|}$$
   satisfies the detailed balance condition where $\deg(x)$ is the degree of the vertex x in the graph and $|E|$ is the total number of edges.

2. (a) (4 points) Define the time-homogeneous and the inhomogeneous Poisson processes.

   (b) (4 points) Consider a time-homogeneous Poisson process with parameter $\lambda > 0$ and let $p : \mathbb{R}_+ \to [0, 1]$ be a continuous function. Independently for each point of the Poisson process, we keep a point at s with probability $p(s)$ and discard it with probability $1 - p(s)$. What can we say about the set of points which are kept after this procedure?

3. (a) (4 points) State the optional stopping theorem for martingales.

   (b) (5 points) Apply the theorem for the fair gambler's ruin problem and compute the ruin probability.

Exercise part

4. (6 points) Find $\lim_{n\to\infty} P^n(i, j)$ for
   $$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/8 & 1/4 & 5/8 & 0 \\ 0 & 1/6 & 0 & 5/6 \end{pmatrix}.$$

5. (5+2 points) A taxi company has three cabs. Calls come in the dispatcher at times of a Poisson process with rate 1 per hour. Suppose that each requires an exponential amount of time with mean 30 minutes, and that callers will hang up if they hear there are no cabs available.

   (a) What is the probability all three cabs are busy when a call comes in?

   (b) In the long run, on the average how many customers are served per hour?

6. (6 points) Let $S_n = X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are i.i.d. with $\mathbf{E}(X_1) = 0$ and $\mathrm{Var}(X_1) = \sigma^2$. Show that $S_n^2 - n\sigma^2$ is a martingale with respect to the natural filtration.

7. (6 points) Let $(B(t), t \geq 0)$ be a Brownian motion. Prove that $\left(\frac{1}{\sqrt{2}}B(2t), t \geq 0\right)$ is also a Brownian motion.