Homeworks in Stochastic processes

2021/22 autumn semester

1. **1st homework assignment (due at 10.15 on 16th Sep):** exercises 1.1, 1.3, 1.6, 1.7, 1.8 d), 1.9 b), 1.11 c), 1.13, 1.14, 1.15 in [D12] (pages 62–65).

   The computation of the stationary distribution is not part of the exercise in 1.14.

   Some matrix operations (multiplication, inversion) can be done by computer in exercises 1.7, 1.8, 1.11, 1.13, 1.15.

2. **2nd homework assignment (due at 10.15 on 30th Sep):** exercises 1.19, 1.26, 1.29, 1.37, 1.38, 1.43, 1.46, 1.48, 1.50, 1.64 in [D12] (pages 65–74).

   Some matrix operations (multiplication, inversion) can be done by computer.

3. **3rd homework assignment (due at 10.15 on 8th Oct):** exercises 1.65, 1.67, 1.68, 1.72, 1.73, 1.74, 1.77 in [D12] (pages 74–76).

   Some matrix operations (multiplication, inversion) can be done by computer.

4. **4th homework assignment (due at 10.15 on 14th Oct):**

   4.A Consider the unit interval $I = [0, 1]$. For every n and $(i_1, \dots, i_n) \in \{0, 1, 2\}^n$ we consider the interval $I_{i_1 \dots i_n} \subset I$ which is the set of those numbers whose base 3 expansion starts with $(i_1 \dots i_n)$. That is

$$I_{i_1 \dots i_n} = \left[\sum_{k=1}^{n} \frac{i_k}{3^k}, \frac{1}{3^n} + \sum_{k=1}^{n} \frac{i_k}{3^k} \right].$$

   Let X_0, X_1, X_2 be independent Bernoulli random variables with parameters p_0, p_1, p_2 respectively. That is $\mathbf{P}(X_i = 1) = p_i$ and $\mathbf{P}(X_i = 0) = 1 - p_i$ for $i = 0, 1, 2$. Moreover for every n and $(i_1, \dots, i_n) \in \{0, 1, 2\}^n$ we are given the random variables $X_{i_1 \dots i_n}$ such that on the one hand $\{X_{i_1 \dots i_n}\}_{n \geq 1, (i_1, \dots, i_n) \in \{0,1,2\}^n}$ are independent and on the other hand $X_{i_1 \dots i_n} \stackrel{d}{=} X_{i_n}$. For every $n \geq 1$ we define the set $E_n \subset [0, 1]$ by

$$E_n = \bigcup_{X_{i_1} \cdot X_{i_1, i_2} \cdots X_{i_1, i_2, \dots, i_n} = 1} I_{i_1 \dots i_n}.$$

   Finally we define the set $E = \cap_{n=1}^{\infty} E_n$. Assume that $p_0 = \frac{2}{3}, p_1 = \frac{3}{4}$ and $p_2 = \frac{1}{2}$. Is it true that $\mathbf{P}(E \neq \emptyset) > 0$?

   Hint: Relate the subset E_n to the nth generation of a branching process.

   4.B Given a branching process with the following offspring distribution determine the extinction probabilities:

   (a) $p_0 = 0.25, p_1 = 0.4, p_2 = 0.35$, $p_n = 0$ if $n \geq 3$,

(b) $p_0 = 0.5, p_1 = 0.1, p_2 = 0, p_3 = 0.4$, $p_n = 0$ if $n \geq 4$.

4.C Consider the branching process with offspring distribution as in part (b) of the previous exercise. What is the probability that the population is extinct in the second generation $X_2 = 0$ given that it did not die out in the first generation?

4.D Consider the branching process with offspring distribution given by $\{p_n\}_{n=0}^{\infty}$. We change this process into an irreducible Markov chain by the following modification: whenever the population dies out, then the next generation has exactly one new individual. That is $\mathbf{P}(X_{n+1} = 1 | X_n = 0) = p(0,1) = 1$. For which $\{p_n\}_{n=0}^{\infty}$ will this chain be null recurrent, recurrent, transient? The finite second moment of the offspring distribution can be assumed, i.e. $\sum_{n=0}^{\infty} n^2 p_n < \infty$.

4.E Let $X_1, X_2, \dots$ be i.i.d. random variables taking values in the integers such that $\mathbf{E}(X_i) = 0$ for all i. Let $S_0 = 0$ and $S_n = X_1 + \dots + X_n$.

(a) Let $G_n(x) = \sum_{j=0}^{n} \mathbf{P}(S_j = x)$. That is $G_n(x)$ is the expected number of visits to x in the first n steps. Show that for all n and x, $G_n(0) \geq G_n(x)$. (Hint: consider the first j with $S_j = x$.)

(b) Note that the law of large numbers implies that for each $\varepsilon > 0$ we have $\lim_{n\to\infty} \mathbf{P}(|S_n| \leq n\varepsilon) = 1$. Using this prove that for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x \in \mathbb{Z}: |x| \leq \varepsilon n} G_n(x) = 1.$$

(c) Using parts (a) and (b) show that for each $M < \infty$ there is an n such that $G_n(0) \geq M$.

(d) Now prove that S_n is a recurrent Markov chain.

5. **5th homework assignment (due at 10.15 on 21st Oct):** exercises 2.2, 2.5, 2.6, 2.10, 2.17, 2.20, 2.22, 2.27 in [D12] (pages 92–95). We are interested in the expectation of the waiting time in exercise 2.5.

6. **6th homework assignment (due at 10.15 on 28th Oct):** exercises 2.29, 2.31, 2.32, 2.33, 2.43, 2.46, 2.61 in [D12] (pages 95–99).

7. **7th homework assignment (due at 10.15 on 4th Nov):** exercises 4.1, 4.3, 4.8, 4.10, 4.14, 4.19, 4.22 in [D12] (pages 150–153).

8. **8th homework assignment (due at 10.15 on 18th Nov):**

8.A If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X given that $X + Y = m$.

8.B Let $\Omega = \{-1, 0, +1\}$, $\mathcal{F} = 2^{\Omega}$ and $\mathbf{P}(\{-1\}) = \mathbf{P}(\{0\}) = \mathbf{P}(\{+1\}) = 1/3$. Consider also the sub-$\sigma$-algebras

$$\mathcal{G} = \{\emptyset, \{-1\}, \{0, +1\}, \Omega\}, \quad \mathcal{H} = \{\emptyset, \{-1, 0\}, \{+1\}, \Omega\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(\omega) = \omega$. Compute $\mathbf{E}(\mathbf{E}(X|\mathcal{G})|\mathcal{H})$ and $\mathbf{E}(\mathbf{E}(X|\mathcal{H})|\mathcal{G})$.

8.C A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

8.D Consider n independent trials, each of which results in one of the outcomes $\{1, ..., k\}$, with respective probabilities $\{p_1, ..., p_k\}$, $\sum_{i=1}^k p_i = 1$. Let N_i denote the number of trials that result in outcome i, $i = 1, ..., k$. For $i \neq j$ find $\mathbf{E}(N_i | N_j > 0)$.

8.E Let U be a uniform random variable on $(0, 1)$, and suppose that the conditional distribution of X, given that $U = p$, is binomial with parameters n and p. Find the probability mass function of X. That is find $\mathbf{P}(X = i)$ for all $0 \leq i \leq n$.

Hint: In the solution of this problem you may want to use the following formula:

$$\int_0^1 p^i (1-p)^{n-i} \mathrm{d}p = \frac{i!(n-i)!}{(n+1)!}.$$

8.F The joint density of X and Y is given by $f(x, y) = \frac{e^{-x/y} e^{-y}}{y}$ for $x, y \in (0, \infty)$. Compute $\mathbf{E}(X^2 | Y)$.

9. **9th homework assignment (due at 10.15 on 25th Nov):** exercises 5.2, 5.3, 5.6, 5.7, 5.8, 5.9, 5.10 in [D12] (pages 175–176).

Hint to exercise 5.2 (c): show and use that on the even $\{0 < X_n < N\}$ it holds that

$$N - 1 \leq X_n(N - X_n) \leq \frac{N^2}{4}.$$

Remark to exercise 5.9[1]: for the proof the following version of the optional stopping theorem can be used.

Theorem. *Let M_n be a martingale with bounded increments, that is, $|M_{n+1} - M_n| < c$ for all n with a deterministic $c \in \mathbb{R}$. Assume that T is a stopping time such that $\mathbf{E}(T) < \infty$. Then $\mathbf{E}(M_T) = \mathbf{E}(M_0)$.*

10. **10th homework assignment (due at 10.15 on 9th Dec):**

10.A Let $Z \sim \mathcal{N}(0, 1)$. We define X_t for all $t \geq 0$ by $X_t = \sqrt{t} \cdot Z$. Then the stochastic process $X = \{X_t : t \geq 0\}$ has continuous path and for all $t \geq 0$ and we have $X_i \sim \mathcal{N}(0, t)$. Is X_t a Brownian motion?

Hint: Check the defining conditions of Brownian motion, in particular the variance of the increments.

10.B Let $B(t)$ be the one-dimensional Brownian motion. Show that $\mathrm{Cov}(B(t), B(s)) = \min(s, t)$.

10.C Let $B(t)$ be the one-dimensional Brownian motion. Fix an arbitrary positive number s. Show that the process $B(t + s) - B(s)$ is also Brownian motion.

10.D Let $B(t)$ be the one-dimensional Brownian motion. Show that the process $-B(t)$ is also Brownian motion.

10.E Let $B(t)$ be the one-dimensional Brownian motion. Fix a positive number a. Prove that $a^{-1/2} B(at)$ is also Brownian motion.

10.F Let $B(t)$ be the one-dimensional Brownian motion. Consider the stochastic process $V(0) = 0$ and $V(t) = tB(1/t)$. Prove that $V(t)$ is also a Brownian motion.

10.G Let $B(t)$ and $\widetilde{B}(t)$ be two independent Brownian motions and let $\rho \in (0, 1)$. We define $X(t) = \rho B(t) + \sqrt{1 - \rho^2} \widetilde{B}(t)$. Prove that $X(t)$ is also a Brownian motion.

[1] The general case does not follow from the hint. The statement should rather be proved first when T is replaced by $n \wedge T$ and it should be argued that $S_{n \wedge T}$ is a Cauchy sequence in L^2

References

[D12] R. Durrett: Essentials of Stochastic Processes, Second edition, Springer, 2012