Upper tail decay of KPZ models with Brownian initial conditions

Bálint Vető

Budapest University of Technology and Economics (BME)

10th World Congress in Probability and Statistics
19–23 July 2021

MŰEGYETEM 1782

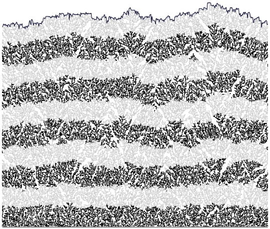
Introduction

joint work with Partik L. Ferrari (Bonn)

P. L. Ferrari, B. Vető: Upper tail decay of KPZ models with Brownian initial conditions, *Electron. Commun. Probab.* **26** (2021), no. 15, 1–14

Motivation: large scale fluctuations of physical phenomena describing surface growth

- crystallization
- interface evolution
- wetting and burning fronts



KPZ equation (Kardar, Parisi, Zhang, 1986):

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left(\partial_x h\right)^2 + \xi$$

where ξ is two-dimensional white noise

Kardar–Parisi–Zhang universality conjecture

Mathematical surface growth models with

- **smoothing effect**
- **slope dependent growth speed**
- **independent noise in space-time**

belong to the Kardar–Parisi–Zhang (KPZ) universality class

Conjectural behaviour: universal limiting fluctuations of the rescaled height

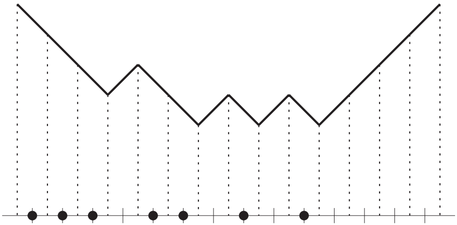
$$\frac{h(Lt, L^{2/3}x) - \mathrm{E}(h(Lt, L^{2/3}x))}{L^{1/3}}$$

which depend on the initial condition

Conjecture: open in general, partial answers in specific (integrable) models

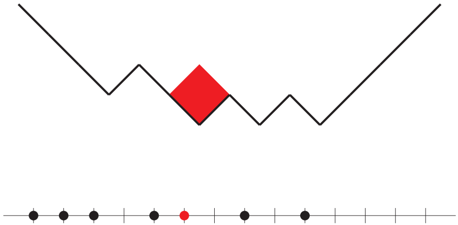
Typical model: totally asymmetric simple exclusion process (TASEP) or corner growth model

TASEP and its height function

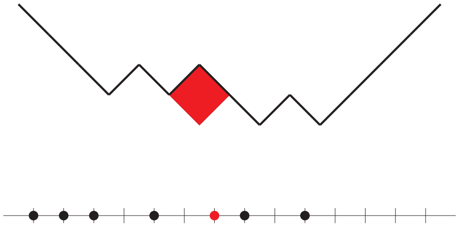


- particles correspond to decreasing segments
- holes correspond to increasing segments

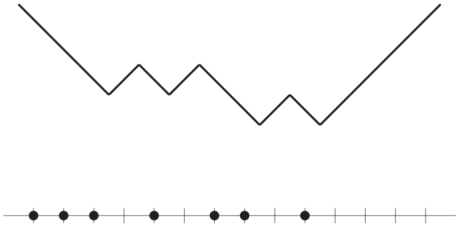
TASEP and its height function



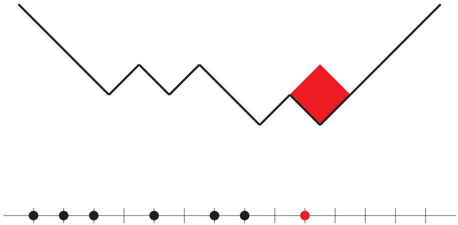
TASEP and its height function



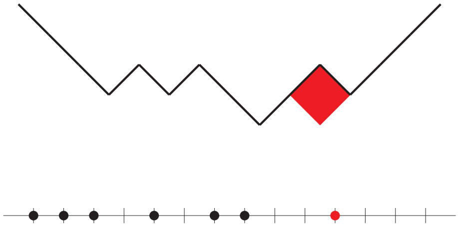
TASEP and its height function



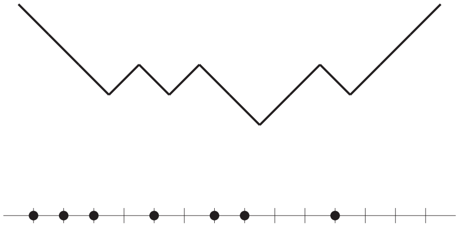
TASEP and its height function



TASEP and its height function



TASEP and its height function



TASEP results

Time evolution: each particle jumps to the right by one at unit rate subject to the exclusion rule (jump is suppressed if target position is occupied)

Step initial condition: negative integer positions are occupied

Theorem (Johansson, 2000)

Let $h(t,x)$ denote the height function corresponding to TASEP with step initial condition. Then

$$\mathrm{P}\left(\frac{h(L,0)-L/2}{L^{1/3}} \geq -s\right) \to F_{\mathrm{GUE}}(s)$$

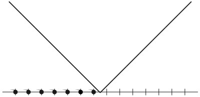
as $L \to \infty$.

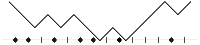
Airy$_2$ process: $\mathcal{A}_2(x) - x^2 = -\lim_{L\to\infty} \frac{h(L,L^{2/3}x)-L/2}{L^{1/3}}$ scaling limit of the height as a function of the position; the Airy$_2$ process $\mathcal{A}_2(x)$ is stationary with F_{GUE} marginal

MŰEGYETEM 1782

Initial conditions

step	periodic	stacionary
$h(0,x)=\lvert x\rvert$	$h(0,x)=\begin{cases}0 & \text{for } x \text{ even}\\ 1 & \text{for } x \text{ odd}\end{cases}$	$(h(0,x), x\in\mathbb{Z})$ two-sided RW





Theorem (Corwin, Liu, Wang, 2016)

Let $h_0(x)=\lim_{L\to\infty}\frac{h(0,L^{2/3}x)}{L^{1/3}}$ *be the rescaled initial height for TASEP. Then*

$$\mathrm{P}\left(\frac{h(L,0)-L/2}{L^{1/3}}\geq -s\right)\to \mathrm{P}\left(\max_{y\in\mathbb{R}}\left(\mathcal{A}_2(y)-y^2-h_0(y)\right)\leq s\right)$$

as $L\to\infty$.

$h_0^{\mathrm{step}}(x)=\begin{cases}0 & \text{if } x=0\\ \infty & \text{if } x\neq 0\end{cases}$ $h_0^{\mathrm{per}}(x)=0$, $h_0^{\mathrm{stat}}(x)$ two-sided BM

The parametric distribution $F^{(\sigma)}$

Let $\sigma \geq 0$ be a parameter. Let $B(x)$ denote a standard two-sided Brownian motion. We assume that for the initial condition of TASEP $\frac{h(0,L^{2/3}x)}{L^{1/3}} \to \sigma B(x)$ holds as $L \to \infty$. Then

$$\mathrm{P}\left(\frac{h(L,0)-L/2}{L^{1/3}} \geq -s\right) \to F^{(\sigma)}(s)$$

where $F^{(\sigma)}(s) = \mathrm{P}(\max_{y\in\mathbb{R}}(\mathcal{A}_2(y) - y^2 + \sqrt{2}\sigma B(y)) \leq s)$ (see Chhita, Ferrari, Spohn, 2018). $\mathcal{A}_2(y)$ and $B(y)$ are independent.
The $\sigma = 0$ case is the periodic initial condition, $\sigma = 1$ corresponds to the stationary initial condition.

Theorem (Ferrari, V, 2021, conjecture: Meerson, Schmidt, 2017)

Let $\sigma \geq 0$ be fixed. Then there are positive real constants C_1, C_2 such that

$$C_1\, s^{-3/4} e^{-\frac{4}{3}\frac{1}{\sqrt{1+3\sigma^4}}s^{3/2}} \leq 1 - F^{(\sigma)}(s) \leq C_2\, s^{3/4} \ln(s)\, e^{-\frac{4}{3}\frac{1}{\sqrt{1+3\sigma^4}}s^{3/2}}$$

as $s \to \infty$.

Tail decay of the distribution $F^{(\sigma)}$

Idea: write

$\mathcal{A}_2(y) - y^2 + \sqrt{2}\sigma B(y) = \left(\mathcal{A}_2(y) - (1-c)y^2\right) + \left(\sqrt{2}\sigma B(y) - cy^2\right)$

Known asymptotics:

- $1 - F_{\mathrm{GUE}}(s) \simeq e^{-\frac{4}{3}s^{3/2}}$
- for any $c \in (0, 3/4)$, $\mathrm{P}(\max_{y\in\mathbb{R}}(\mathcal{A}_2(y) - (1-c)y^2) > s) \simeq e^{-\frac{4}{3}s^{3/2}}$
- for any $c \in (0,1)$ the density of $\max_{y\in\mathbb{R}}(\sqrt{2}\sigma B(y) - cy^2)$ at s is $\simeq e^{-\frac{4}{3\sqrt{3}}\frac{\sqrt{c}}{\sigma^2}s^{3/2}}$

Theorem (Ferrari, V, 2021)

There is a constant $C > 0$ such that for all $c \in (0,1)$

$$1 - F_{\mathrm{GUE}}(s) \leq \mathrm{P}\left(\max_{y\in\mathbb{R}}\left(\mathcal{A}_2(y) - (1-c)y^2\right) > s\right) \leq C\frac{\ln(s/(1-c))}{s^{3/4}\sqrt{1-c}}e^{-\frac{4}{3}s^{3/2}}$$

holds as $s \to \infty$.

Tail decay of the distribution $F^{(\sigma)}$

Upper bound: for any $c \in (0,1)$

$$\begin{aligned}
&\mathrm{P}\left(\max_{y\in\mathbb{R}}(\mathcal{A}_2(y) - y^2 + \sqrt{2}\sigma B(y)) > s\right) \\
&\leq \mathrm{P}\left(\max_{y\in\mathbb{R}}(\sqrt{2}\sigma B(y) - cy^2) + \max_{y\in\mathbb{R}}(\mathcal{A}_2(y) - (1-c)y^2) > s\right) \\
&\simeq \int_0^1 e^{-\frac{4}{3\sqrt{3}}\frac{\sqrt{c}}{\sigma^2}(\mu s)^{3/2}} \mathrm{P}\left(\max_{y\in\mathbb{R}}(\mathcal{A}_2(y) - (1-c)y^2) > (1-\mu)s\right) \mathrm{d}\mu \\
&\simeq \int_0^1 e^{g(\mu)s^{3/2}} \mathrm{d}\mu
\end{aligned}$$

where $g(\mu) = -\frac{4}{3\sqrt{3}}\frac{\sqrt{c}}{\sigma^2}\mu^{3/2} - \frac{4}{3}(1-\mu)^{3/2}$.

The maximum of $g(\mu)$ at $\mu_0 = \frac{3\sigma^4}{c+3\sigma^4}$ is

$g(\mu_0) = -\frac{4}{3}\frac{\sqrt{c}}{\sqrt{c+3\sigma^4}} = -\frac{4}{3}\frac{1}{\sqrt{1+3\sigma^4}} + C(1-c) + \mathcal{O}((1-c)^2)$ as $c \to 1$. By choosing $1-c = s^{-3/2}$, we have $g(\mu_0)s^{3/2} = -\frac{4}{3}\frac{1}{\sqrt{1+3\sigma^4}}s^{3/2} + C + o(1)$.

Lower bound: same integral with $c = 1$ for the argmax of $\sqrt{2}\sigma B(y) - y^2$

The end

Thank you for your attention!