Models of the 'true' self-avoiding walk on $\mathbb{Z}^d$

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Outline of the talk

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   - Stationary distribution
   - Martingale technique
   - Diffusive bound on the variance
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3. Results in one dimension

Introduction of the model

joint work with Illés Horváth and Bálint Tóth

$X(n)$ 'true' or 'myopic' self-avoiding walk = true (nearest neighbour) random walk with self-repulsion on $\mathbb{Z}^d$
$\ell(n,x) := \#\{1 \le m \le n : X(m) = x\}$

Transition probabilities:

$$\begin{aligned} &\mathbf{P}\left(X(n+1) = y \,\middle|\, \mathcal{F}_n, X(n) = x\right) \\ &= \mathbb{1}_{\{|x-y|=1\}} \frac{\exp\{-\beta(\ell(n,y) - \ell(n,x))\}}{\sum_{z:|z-x|=1} \exp\{-\beta(\ell(n,z) - \ell(n,x))\}} \\ &= \mathbb{1}_{\{|x-y|=1\}} \frac{r(\ell(n,x) - \ell(n,y))}{\sum_{z:|z-x|=1} r(\ell(n,x) - \ell(n,z))} \end{aligned}$$

where $r : \mathbb{Z} \to \mathbb{R}_+$ non-decreasing and non-constant

Overview of results

First introduction of the 'true' self-avoiding walk:
D. Amit, G. Parisi and L. Peliti in 1983

Conjectured behaviour:
$\mathbf{E}\left(X(n)^2\right) \sim n^{2\nu}$ i.e. $X(n) \sim n^{\nu}$

1. $d = 1$: $\nu = \frac{2}{3}$ and exotic scaling limit (details at the end of the talk)
2. $d = 2$: critical dimension, $\nu = \frac{1}{2}$ with logaritmic corrections (conjectured)
3. $d \geq 3$: $\nu = \frac{1}{2}$ with Gaussian scaling limit, invariance principle (main subject of the talk)

The high dimensional model

No mathematical results in $d \geq 3$ dimensions until now
lace expansion and rigorous renormalisation group arguments failed

A slightly different model in continuous time:
$X(t)$ nearest neigbour jump walk on $\mathbb{Z}^d$
$\ell(t,x) := \ell(0,x) + |\{0 \leq s \leq t : X(s) = x\}|$

Jump rates:

$$\mathbf{P}\left(X(t+\mathrm{d}t) = y \mid \mathcal{F}_t, X(t) = x\right) = r(\ell(t,x) - \ell(t,y))\,\mathrm{d}t + o(\mathrm{d}t)$$

if $|y-x| = 1$ where $r : \mathbb{R} \to \mathbb{R}_+$ non-decreasing and non-constant rate function
Our particular choice: $r(u) = u_+ + \alpha$ with $\alpha > 0$

Environment as seen by the walker

Theorem (I. Horváth, B. Tóth, B. V., 2009)

$$\limsup_{t\to\infty} \frac{\mathbf{E}\left(X(t)^2\right)}{t} < \infty$$

'true' self-avoiding walk = random walk in a dynamically changing random environment

Consider the environment as seen from the position of the random walker

State space: $\Omega := \{\underline{\omega} = (\omega(x) : x \in \mathbb{Z}^d)\} = \mathbb{R}^{\mathbb{Z}^d}$

The process $\eta(t)$ of the environment as seen by the random walker is a Markov process on Ω.

Stationary distribution

Let π be the *massless free Gaussian field* on Ω, i.e. $\omega(x), x \in \mathbb{Z}^d$ are jointly Gaussian with

$$\mathbf{E}_\pi(\omega(x)) = 0$$

$$\mathbf{E}_\pi(\omega(x)\omega(y)) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{e^{i\theta(y-x)}}{D(\theta)} \mathrm{d}\theta$$

where $D(\theta) = \frac{1}{d} \sum_{j=1}^{d} (1 - \cos\theta_j)$.
The integral exists in dimension $d \geq 3$.
π is a stationary distribution of the Markov process $\eta(t)$.

Gradient space of the massless free Gaussian field

$\widetilde{\omega}_k(x) := \omega(x + e_k) - \omega(x)$. With this notation:

$$\mathbf{E}_\pi\left(\widetilde{\omega}(x)\right) = 0$$
$$\mathbf{E}_\pi\left(\widetilde{\omega}(x)\widetilde{\omega}(y)\right) = C_{k,l}(y - x)$$

where

$$C_{k,l}(z) = \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} \frac{(1 - e^{i\theta e_k})(1 - e^{-i\theta e_l})}{D(\theta)} e^{i\theta z} \mathrm{d}\theta.$$

The integral exists in any dimension.

Natural derivation of the gradient field

$$R(u) := \int_0^u (r(v) - r(-v))\,\mathrm{d}v = R(-u)$$

Consider the probability measure

$$\mu(\mathrm{d}u) = \frac{1}{Z}\exp(-R(u))\,\mathrm{d}u$$

where Z is a normailising constant.
With $r(u) = u_+ + \alpha$, μ is Gaussian.

The distribution of $\{\widetilde{\omega}_k(x) : x \in \mathbb{Z}^d, k = 1, \dots, d\}$ is the product of copies of μ conditioned to be gradient.

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Infinitesimal generator

Space of functions: $L^2(\Omega, \pi)$.

$$\tau_k : \Omega \to \Omega, \qquad (\tau_k \underline{\omega})(x) := \omega(x + e_k)$$
$$T_k : L^2(\Omega, \pi) \to L^2(\Omega, \pi), \qquad (T_k f)(\underline{\omega}) := f(\tau_k \underline{\omega})$$

π is invariant under translations.

Infinitesimal generator of $\eta(t)$:

$$(Gf)(\underline{\omega}) = \frac{\partial f}{\partial \omega(0)}(\underline{\omega})$$
$$+ \sum_{k=1}^{d} \left[r(\omega(0) - \omega(e_k))(T_k - I) + r(\omega(0) - \omega(-e_k))(T_k^* - I) \right] f(\underline{\omega})$$

Adjoint operator

$$(G^* f)(\underline{\omega}) = -\frac{\partial f}{\partial \omega(0)}(\underline{\omega})$$

$$+ \sum_{k=1}^{d} \left[r(\omega(e_k) - \omega(0))(T_k - I) + r(\omega(-e_k) - \omega(0))(T_k^* - I) \right] f(\underline{\omega})$$

Consequences:

1. π is indeed a stationary distribution (ergodic)
2. Yaglom reversibility

$$(Jf)(\underline{\omega}) := f(-\underline{\omega}), \quad J = J^* = J^{-1}$$

$$G^* = JGJ$$

$$\eta(-t) \stackrel{\mathrm{d}}{=} -\eta(t)$$

Conditional speed of the random walker

$\varphi : \Omega \to \mathbb{R}^d$

$$\varphi_k(\underline{\omega}) := r(\omega(0) - \omega(e_k)) - r(\omega(0) - \omega(-e_k))$$

With

$$s(u) := \frac{r(u) + r(-u)}{2} \quad \text{and} \quad a(u) := \frac{r(u) - r(-u)}{2}$$

let

$$\overline{\varphi}_k(\underline{\omega}) := s(\omega(0) - \omega(e_k)) - s(\omega(0) - \omega(-e_k))$$
$$\widetilde{\varphi}_k(\underline{\omega}) := a(\omega(0) - \omega(e_k)) - a(\omega(0) - \omega(-e_k))$$

which gives $\varphi(\underline{\omega}) = \overline{\varphi}(\underline{\omega}) + \widetilde{\varphi}(\underline{\omega})$.
If $r(u) = u_+ + \alpha$, then $s(u) = \frac{|u|}{2}$ and $a(u) = \frac{u}{2}$.

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The displacement

$$M(t) := X(t) - X(0) - \int_0^t \varphi(\eta(u))\,\mathrm{d}u$$

is a martingale with stationary and ergodic increments

$$\mathbf{E}\left((M_k(t) - M_k(s))^2\right) = 2(t-s)\int_\Omega r(\omega(0) - \omega(e_k))\pi(\mathrm{d}\underline{\omega})$$

Lemma

$$\limsup_{t\to\infty} \frac{1}{t}\mathbf{E}\left(\left(\int_0^t \varphi(\eta(u))\,\mathrm{d}u\right)^2\right) < \infty.$$

It is the key step of the central limit theorem for the displacement

$$X(t) - X(s) = M(t) - M(s) + \int_s^t \varphi(\eta(u))\,\mathrm{d}u.$$

Abstract upper bound

(Ω, π) probability space
$\eta(t)$ stationary and ergodic Markov process on it
G generator of η defined on $L^2(\Omega, \pi)$
$S := -\frac{1}{2}(G + G^*)$ symmetric part
$\xi(t)$ stationary, ergodic and reversible Markov process on the same space with infinitesimal generator $-S$
$\psi \in L^2(\Omega, \pi)$ with $\mathbf{E}_\pi(\psi) = 0$

Lemma (S. Sethuraman, S. R. S. Varadhan, H.-T. Yau, 2000)

$$\limsup_{t\to\infty} \frac{1}{t}\mathbf{E}\left(\left(\int_0^t \psi(\eta(u))\,\mathrm{d}u\right)^2\right) \leq \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left(\left(\int_0^t \psi(\xi(u))\,\mathrm{d}u\right)^2\right)$$

The reversible Markov process

$$(-Sf)(\underline{\omega}) = \sum_{k=1}^{d} s(\omega(0) - \omega(e_k))(T_k - I)f(\underline{\omega}) + \sum_{k=1}^{d} s(\omega(0) - \omega(-e_k))(T_k^* - I)f(\underline{\omega})$$

In our case, $\xi(t)$ is a symmetric random walk in random environment.

One has to verify

$$\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left(\left(\int_0^t \overline{\varphi}(\xi(u))\,\mathrm{d}u\right)^2\right) < \infty$$

$$\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left(\left(\int_0^t \widetilde{\varphi}(\xi(u))\,\mathrm{d}u\right)^2\right) < \infty$$

where $\overline{\varphi}$ is the symmetric part

and $\widetilde{\varphi}$ is the antisymmetric part of φ

Abstract time reversal symmetry trick

$\xi(t)$ stationary, ergodic and reversible Markov process on (Ω, π)
$\xi^*(t) := \xi(-t)$ the time-revered process
$Y(s,t) := \mathcal{Y}(\xi([s,t]))$ an additive functional of the trajectory of ξ
$Y^*(s,t) := \mathcal{Y}(\xi^*([s,t]))$ the functional observed along the backward trajectory
$N(t) - N(s) = Y(s,t) - \int_s^t \psi(\xi(u))\,\mathrm{d}u$ is a martingale with stationary increments with the compensator $\psi \in L^2(\Omega, \pi)$

Lemma (A. De Masi, P. A. Ferrari, S. Goldstein, D. Wick, 1989)

If Y is odd with respect to time reversal of trajectories, i.e. $Y^*(-t,-s) = -Y(s,t)$, then

$$\mathbf{E}\left((N(t)-N(s))^2\right) = \mathbf{E}\left(Y(s,t)^2\right) + \mathbf{E}\left(\left(\int_s^t \psi(\xi(u))\,\mathrm{d}u\right)^2\right).$$

Bounds on the asymptotic variance of $\overline{\varphi}$ and $\widetilde{\varphi}$

1. For $\overline{\varphi}$, let $Y(s,t)$ be the displacement of the random walk in random environment in $[s,t]$, which can be written an additive functional of $\xi([s,t])$.
2. For $\widetilde{\varphi}$, let

$$Y(s,t) := \sum_{s\le u\le t} \frac{-\xi(u-0)(e_k)}{s(\xi(u-0)(0)-\xi(u-0)(e_k))} \mathbb{1}_{\{\xi(u+0)=\tau_k\xi(u-0)\}} + \sum_{s\le u\le t} \frac{\xi(u-0)(0)}{s(\xi(u-0)(0)-\xi(u-0)(-e_k))} \mathbb{1}_{\{\xi(u+0)=\tau_k^{-1}\xi(u-0)\}}$$

i.e. the asymptotic variance of $\psi(\underline{\omega}) = \omega(0) - \omega(e_k)$ is finite.

Remark: bound on $\overline{\varphi}$ is valid in any dimension
estimate of $\widetilde{\varphi}$, if the massless Gaussian field exists

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Functional analytic condition

$R_\lambda := (\lambda I - G)^{-1}$ resolvent operator
$\psi_\lambda := R_\lambda \varphi$, or the solution of $\lambda \psi_\lambda - G\psi_\lambda = \varphi$

Lemma

Suppose that $\|S^{-1/2}\varphi\| < \infty$. *If* $\lim_{\lambda \to 0} \sqrt{\lambda}\|\psi_\lambda\| = 0$ *and* $\lim_{\lambda,\mu \to 0}(S(\psi_\lambda - \psi_\mu), \psi_\lambda - \psi_\mu) = 0$, *then central limit theorem holds, i.e.*

$$\frac{\int_0^t \varphi(\eta(s))\,\mathrm{d}s}{\sqrt{t}} \Longrightarrow N(0,\sigma)$$

with $\sigma \in (0,\infty)$.

$M_\lambda(t) := \psi_\lambda(\eta(t)) - \psi_\lambda(\eta(0)) - \int_0^t (G\psi_\lambda)(\eta(s))\,\mathrm{d}s$ martingales
Conditions imply that $\exists M_0(t)$ limit martingale.
$M_0(t)$ approximates $\int_0^t \varphi(\eta(s))\,\mathrm{d}s$.

Resolvent analysis

$$R_\lambda = (\lambda - G)^{-1} = (\lambda + S - A)^{-1} = (\lambda + S)^{-1/2} M_\lambda (\lambda + S)^{-1/2}$$

where

$$M_\lambda = \left(I - (\lambda + S)^{-1/2} A (\lambda + S)^{-1/2} \right)^{-1}$$

Note that $\|M_\lambda\| \leq 1$.

$$\sqrt{\lambda}\psi_\lambda = \sqrt{\lambda}(\lambda + S)^{-1/2} M_\lambda (\lambda + S)^{-1/2} \varphi$$

$(\lambda + S)^{-1/2}\varphi \to S^{-1/2}\varphi \in L^2(\Omega, \pi)$ because of our first condition. Suppose that $S^{-1/2} A S^{-1/2}$ exists as closed and densely defined unbounded operator. If

$$M_\lambda \overset{\text{st. op.}}{\longrightarrow} M_0 := (I - S^{-1/2} A S^{-1/2})^{-1},$$

then $\sqrt{\lambda}\|\psi_\lambda\| \to 0$.

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One dimensional continuous time model

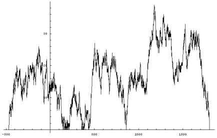
the same model with $d = 1$ and $\ell(0, x) = 0$ for all $x \in \mathbb{Z}$

Based on the behaviour of the one dimensional discrete time self-avoiding model with edge repulsion (B. Tóth, 1995):

Theorem (B. Tóth, B. V., 2008)

$$\frac{X(At)}{A^{2/3}} \Longrightarrow \mathcal{X}(t)$$

in terms of the finite dimensional marginals where $\mathcal{X}(t)$ is the true self-repelling motion (B. Tóth, W. Werner, 1998).

Local times: 

One dimensional discrete time walk with oriented edges

$\widetilde{X}(n)$ discrete time nearest neigbour random walk on $\mathbb{Z}$
$\ell^{\pm}(n,k) := \#\{0 \le j \le n-1 : \widetilde{X}(j) = k, \widetilde{X}(j+1) = k \pm 1\}$
$w : \mathbb{Z} \to \mathbb{R}_+$ non-increasing and non-constant weight function

$$\mathbf{P}\left(\widetilde{X}(n+1) = x \pm 1 \,\middle|\, \mathcal{F}_n, \widetilde{X}(n) = x\right) = \frac{w(\mp(\ell^+(n,x) - \ell^-(n,x)))}{w(\ell^+(n,x) - \ell^-(n,x)) + w(\ell^-(n,x) - \ell^+(n,x))}$$

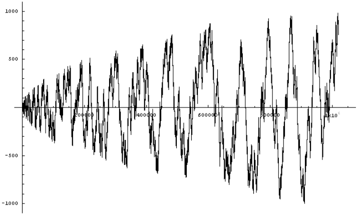
Theorem (B. Tóth, B. V., 2008)

$$\frac{\widetilde{X}([At])}{A^{1/2}} \Longrightarrow \mathrm{UNI}(-\sqrt{t}, \sqrt{t})$$

without continuous limit process.

Simulation results

Trajectory:



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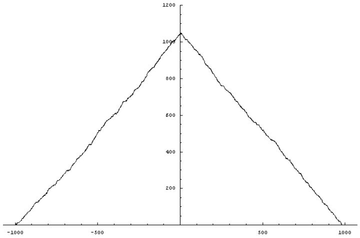

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Thank you for your attention!