Skorohod reflection of Brownian paths and BES^3

Bálint Vető

Budapest University of Technology and Economics
http://www.math.bme.hu/~vetob

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joint work with Bálint Tóth

Outline

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Skorohod reflection

Let $b, x : [0, T) \to \mathbb{R}$ be continuous functions with $x(0) \geq b(0)$.

1. $\exists\ x_{b\uparrow} : [0, T) \to \mathbb{R}$ with
   - $x_{b\uparrow}(0) = x(0)$,
   - $x_{b\uparrow} - b$ non-negative,
   - $x_{b\uparrow} - x$ non-decreasing,
   - $x_{b\uparrow} - x$ increases only when $x_{b\uparrow} = b$.
2. $t \mapsto x_{b\uparrow}(t)$ is given by

$$x_{b\uparrow}(t) = x(t) + \sup_{0 \leq s \leq t} \big(x(s) - b(s)\big)_{-}.$$

3. The map $(b(\cdot), x(\cdot)) \mapsto (b(\cdot), x_{b\uparrow}(\cdot))$ is continuous in supremum distance.

The function $x_{b\uparrow}$ is the *upwards Skorohod-reflection* of x on b.
Similarly, $y_{b\downarrow}(t) = y(t) - \sup_{0 \leq s \leq t} \big(y(s) - b(s)\big)_{+}$.

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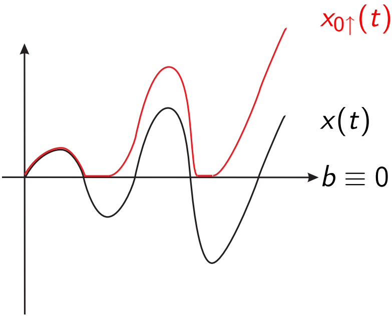
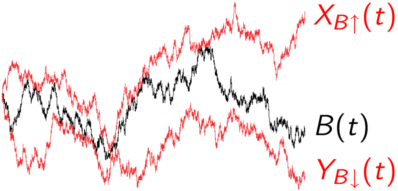
Explanation





Result

Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions starting from 0.
For each realization of the the Brownian motions, $X_{B\uparrow}$ and $Y_{B\downarrow}$ can be defined.

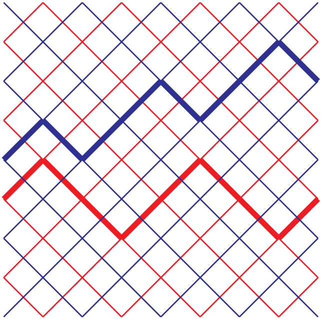
$$Z(t) := X_{B\uparrow}(t) - Y_{B\downarrow}(t).$$

Theorem (B. Tóth, B. V., 2007)

The process $2^{-1/2}Z(t)$ *is* BES^3, *that is a standard 3d Bessel process,*

$$\mathrm{d}Z(t) = 2\frac{1}{Z(t)}\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W(t), \qquad Z(0) = 0.$$

Special case of Jon Warren's theorem (2007)
We have a simpler proof.



Dual square lattices:

$$\mathcal{L} := \{(t,x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is even}\}$$
$$\mathcal{L}^* := \{(t,x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is odd}\}$$

Lattice walk:
$y : [0,T] \cap \mathbb{Z} \to \mathbb{Z}$ if the consecutive points of $(0, y(0)), (1, y(1)), \ldots, (T, y(T))$ are edges in $\mathcal{L}$ or $\mathcal{L}^*$.

Discrete Skorohod reflection

Let $b, x : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ be walks in $\mathcal{L}$ and $\mathcal{L}^*$ with $x(0) \geq b(0)$.

1. $\exists\ x_{b\uparrow} : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ in $\mathcal{L}^*$ with the following properties:
   - $x_{b\uparrow}(0) = x(0)$,
   - $x_{b\uparrow} - b$ non-negative,
   - $x_{b\uparrow} - x$ non-decreasing,
   - $x_{b\uparrow} - x$ increases only when $x_{b\uparrow} = b + 1$.
2. The function $t \mapsto x_{b\uparrow}(t)$ can be expressed as

$$x_{b\uparrow}(t) = x(t) + \sup_{s \in [0,t] \cap \mathbb{Z}} (x(s) - b(s) - 1)_-.$$

The walk $x_{b\uparrow}$ is the *discrete upwards Skorohod reflection* of x on b.

Similarly, $y_{b\downarrow}(t) = y(t) - \sup_{s \in [0,t] \cap \mathbb{Z}} (y(s) - b(s) + 1)_+$.

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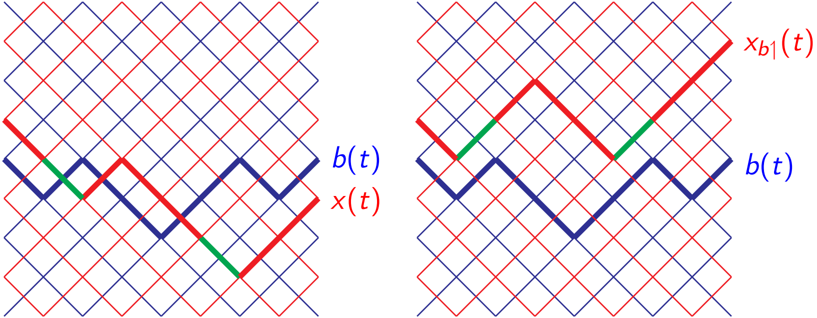
Example



Proof of Theorem

$M(t)$ symmetric random lattice walk on $\mathcal{L}$ with $M(0)=0$
$U(t)$ and $L(t)$ symmetric random lattice walk on $\mathcal{L}^*$ with $U(0)=1$ and $L(0)=-1$ independently

$$\left(\frac{M(nt)}{\sqrt{n}}, \frac{U(nt)}{\sqrt{n}}, \frac{L(nt)}{\sqrt{n}}\right) \stackrel{\mathrm{d}}{\Longrightarrow} (B(t), X(t), Y(t)).$$

By Donsker's invariance principle

$$\left(\frac{M(nt)}{\sqrt{n}}, \frac{U_{M(n\cdot)\uparrow}(nt)}{\sqrt{n}}, \frac{L_{M(n\cdot)\downarrow}(nt)}{\sqrt{n}}\right) \stackrel{\mathrm{d}}{\Longrightarrow} (B(t), X_{B\uparrow}(t), Y_{B\downarrow}(t)).$$

$$D(n) := \frac{1}{2}(U_{M\uparrow}(n) - L_{M\downarrow}(n))$$

Enough to show that

$$\frac{\sqrt{2}D(nt)}{\sqrt{n}} \stackrel{\mathrm{d}}{\Longrightarrow} \mathrm{BES}^3.$$

Markov property of $D(n)$

$D(n)$ is Markovian with transition matrix

$$\widetilde{P}_{xy} = \frac{y}{x} \cdot \begin{cases} \frac{1}{2} & \text{if } y = x \\ \frac{1}{4} & \text{if } |y-x| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Heuristic calculations with $\mathrm{d}t = \frac{1}{A}$ yield

$$\mathbf{E}\left(\frac{\mathrm{d}D(At)}{\sqrt{A}}\right) = \frac{1}{2}\frac{\sqrt{A}}{D(At)}\,\mathrm{d}t \qquad \mathbf{Var}\left(\frac{\mathrm{d}D(At)}{\sqrt{A}}\right) = \left(\frac{1}{2} + o(1)\right)\mathrm{d}t$$

Therefore

$$\frac{2D(At)}{\sqrt{A}} \overset{\mathrm{d}}{\Longrightarrow} Z(t)$$

with

$$\mathrm{d}Z(t) = 2\frac{1}{Z(t)}\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W(t).$$

Remark: given $D(n)$, the position of $M(n)$ is discrete uniform between $L_{M\downharpoonleft}(n)$ and $U_{M\upharpoonleft}$.

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Outlook

Let ξ_n be a birth and death process, which is a Markov chain on $\{0,1,2,\dots\}$ with transition matrix Q.
Suppose that ξ_n is a martingale, i.e. $\sum_y yQ_{xy} = x$ for all x.
Let $\tau_0 := \min\{k \geq 0 : \xi_k = 0\}$. Then

$$\mathbf{P}\left(\xi_{n+1} = y \mid \xi_n = x, \tau_0 = \infty\right) = \widetilde{Q}_{xy} := \frac{y}{x} Q_{xy}.$$

1. If the transition matrix of ξ_n is
$$P_{xy} = \begin{cases} 1/2 & \text{if } y = x \\ 1/4 & \text{if } |y-x| = 1 \end{cases}$$
then $\xi_{At}/\sqrt{A} \stackrel{\mathrm{d}}{\Rightarrow} \mathrm{BES}^1$ and
$\left(\xi_{At}/\sqrt{A} \mid \tau_0 = \infty\right) \stackrel{\mathrm{d}}{=} D(At)/\sqrt{A} \stackrel{\mathrm{d}}{\Rightarrow} \mathrm{BES}^3$. $1 + 3 = 4$
2. If U_n is a critical branching process,
then $U_{At}/\sqrt{A} \stackrel{\mathrm{d}}{\Rightarrow} \mathrm{BESQ}^0$
and $\left(U_{At}/\sqrt{A} \mid \tau_0 = \infty\right) \stackrel{\mathrm{d}}{\Rightarrow} \mathrm{BESQ}^4$. $0 + 4 = 4$

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The end

Thank you for your attention!