Skorohod-reflection of Brownian Paths and BES^3

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September 8, 2007

Dedicated to Sándor Csörgő on the occasion of his 60th birthday

Abstract

Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions. Define $X^+(t)$ and $Y^-(t)$ as the trajectories of the processes $X(t)$ and $Y(t)$ pushed upwards and, respectively, downwards by $B(t)$, according to Skorohod-reflection. In the recent paper [8] Jon Warren proves inter alia that $Z(t) := X^+(t) - Y^-(t)$ is a three dimensional Bessel-process. In this note we present an alternative, elementary proof of this fact.

1 Introduction

The study of 1d Brownian trajectories pushed up or down by Skorohod-reflection on some other Brownian trajectories (running backwards in time) was initiated in [5] and motivated in [7] by the construction of the object what is today called the Brownian Web, see [3]. It turns out that these Brownian paths, reflected on one another, have very interesting, sometimes surprising properties. For further studies of Skorohod-reflection of Brownian paths on one another see also [6], [1], [8] etc. In particular, in [8], Warren considers two interlaced families of Brownian paths with paths belonging to the second family reflected off paths belonging to the first (in Skorohod's sense) and derives a determinantal formula for the distribution of coalescing Brownian motions.

A particular case of Warren's formula is the following: fix a Brownian path and let two other Brownian paths be pushed upwards and, respectively, downwards by Skororhod-reflection on the trajectory of the first one. The difference of the last two will be a three dimensional Bessel-process. In the present note we give an alternative, elementary proof of this fact.

1.1 Skorohod-reflection

Let $T \in (0, \infty)$ and $b, x : [0, T) \to \mathbb{R}$ be continuous functions. Assume $x(0) \geq b(0)$. The construction of the following proposition is due to Skorohod. Its proof can be found either in [4] (see Lemma 2.1 in Chapter VI) or in [5] (see Lemma 2 in Section 2.1).

Proposition 1. *(1) There exists a unique continuous function $x_{b\uparrow} : [0, T) \to \mathbb{R}$ with the following properties:*

- *The function $x_{b\uparrow} - b$ is non-negative.*
- *The function $x_{b\uparrow} - x$ is non-decreasing.*
- *The function $x_{b\uparrow} - x$ increases only when $x_{b\uparrow} = b$. That is*

$$\int_0^T \mathbb{1}\{x_{b\uparrow}(t) \neq b(t)\} \, \mathrm{d}(x_{b\uparrow}(t) - x(t)) = 0.$$

(2) The function $t \mapsto x_{b\uparrow}(t)$ is given by the construction

$$x_{b\uparrow}(t) = x(t) + \sup_{0 \leq s \leq t} \big(x(s) - b(s)\big)_-.$$

(3) The map $C([0, T)) \times C([0, T)) \ni \big(b(\cdot), x(\cdot)\big) \mapsto \big(b(\cdot), x_{b\uparrow}(\cdot)\big) \in C([0, T)) \times C([0, T))$ is continuous in supremum distance.

We call the function $t \mapsto x_{b\uparrow}(t)$ the *upwards Skorohod-reflection* of $x(\cdot)$ on $b(\cdot)$. As it is remarked in [5], the term *Skorohod-pushup* of $x(\cdot)$ by $b(\cdot)$ would be more adequate. Skorohod-reflection on paths $b(t) = \text{const.}$ plays a fundamental role in the proper formulation and proof of Tanaka's formula, see Chapter VI of [4].

The downwards Skorohod-reflection or Skorohod-pushdown is defined for continuous functions $b, y : [0, T) \mapsto \mathbb{R}$ with $y(0) \leq b(0)$ by

$$y_{b\downarrow} := -\left((-y)_{(-b)\uparrow}\right), \qquad y_{b\downarrow}(t) = y(t) - \sup_{0 \leq s \leq t} \big(y(s) - b(s)\big)_+.$$

Given three continuous trajectories $b, x, y : [0, T) \to \mathbb{R}$ with $y(0) \leq b(0) \leq x(0)$, the map $C([0, T)) \times C([0, T)) \times C([0, T)) \ni (b(\cdot), x(\cdot), y(\cdot)) \mapsto (b(\cdot), x_{b\uparrow}(\cdot), y_{b\downarrow}(\cdot)) \in C([0, T)) \times C([0, T)) \times C([0, T))$ is clearly continuous in supremum distance.

1.2 The result

Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions starting from 0 and define

$$X^+(t) := X_{B\uparrow}(t), \qquad \widehat{X}(t) := \quad X^+(t) - B(t), \tag{1}$$

$$Y^-(t) := Y_{B\downarrow}(t), \qquad \widehat{Y}(t) := -Y^-(t) + B(t). \tag{2}$$

We are interested in the difference process

$$Z(t) := X^+(t) - Y^-(t) = \widehat{X}(t) + \widehat{Y}(t). \tag{3}$$

It is straightforward that $2^{-1/2}\widehat{X}(t)$ and $2^{-1/2}\widehat{Y}(t)$ are both standard reflected Brownian motions. They are, of course, strongly dependent.

The following fact is a particular consequence of the main results in [8]:

Theorem. *The process $2^{-1/2}Z(t)$ is BES^3, that is a standard 3d Bessel-process,*

$$\mathrm{d}Z(t) = \frac{1}{2}\frac{1}{Z(t)}\,\mathrm{d}t + \frac{1}{\sqrt{2}}\,\mathrm{d}W(t), \qquad Z(0) = 0.$$

In the next section we present an elementary proof of this fact.

2 Proof

2.1 Discrete Skorohod-reflection

Define the following square lattices embedded in $\mathbb{R} \times \mathbb{R}$:

$$\mathcal{L} := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is even}\}, \qquad \mathcal{L}^* := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is odd}\}. \tag{4}$$

In both of the lattices the points (t_1, x_1) and (t_2, x_2) are connected with an edge if and only if $|t_1 - t_2| = |x_1 - x_2| = 1$. Note that $\mathcal{L}$ and $\mathcal{L}^*$ are Whitney-duals of each other.

We define the discrete analogue of the Skorohod-reflection in $\mathcal{L}$ and $\mathcal{L}^*$. Later on, we say that the function $y : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ is a *walk* in the lattice $\mathcal{L}$ or $\mathcal{L}^*$ if the consecutive elements of the sequence $(0, y(0)), (1, y(1)), \ldots, (T, y(T))$ are edges in $\mathcal{L}$ or $\mathcal{L}^*$.

Let $b : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ and $x : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ be two walks in the lattices $\mathcal{L}$ and $\mathcal{L}^*$, respectively. Assume that $x(0) \geq b(0)$. An analogue of Proposition 1 holds in this case, but the proof is even easier.

Proposition 2. *(1) There is a unique walk $x_{b\uparrow} : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ in $\mathcal{L}^*$ with the following properties:*

- *The function $x_{b\uparrow} - b$ is non-negative.*
- *The function $x_{b\uparrow} - x$ is non-decreasing.*
- *The function $x_{b\uparrow} - x$ increases only when $x_{b\uparrow} = b + 1$, i.e.*

$$\sum_{t=1}^{T} \mathbb{1}\{x_{b\uparrow}(t) - b(t) > 1\} \left[\big(x_{b\uparrow}(t) - x(t)\big)\big(x_{b\uparrow}(t-1) - x(t-1)\big)\right] = 0.$$

(2) The function $t \mapsto x_{b\uparrow}(t)$ can be expressed as

$$x_{b\uparrow}(t) = x(t) + \sup_{s\in[0,t]\cap\mathbb{Z}} (x(s) - b(s) - 1)_-.$$

We call the function $t \mapsto x_{b\uparrow}(t)$ the *discrete upwards Skorohod-reflection* of $x(\cdot)$ on $b(\cdot)$. The discrete downwards Skorohod-reflection is defined similarly. If $y : [0,T]\cap\mathbb{Z} \to \mathbb{Z}$ is a walk in $\mathcal{L}$ and $b : [0,T] \cap \mathbb{Z} \to \mathbb{Z}$ is a walk in $\mathcal{L}^*$ with $y(0) \leq b(0)$, then

$$y_{b\downarrow} := -\left((-y)_{(-b)\uparrow}\right), \qquad y_{b\downarrow}(t) = y(t) - \sup_{s\in[0,t]\cap\mathbb{Z}} (y(s) - b(s) + 1)_+.$$

In this paper, we use the same notation for the discrete Skorohod-reflection and the continuous one (defined as Skorohod-reflection), but it will be always clear from the context which is the adequate one.

2.2 Approximation of reflected Brownian motions

Let $M(t)$ be a random walk on the lattice $\mathcal{L}$ with jumps from (t,x) to $(t+1,x+1)$ or $(t+1,x-1)$ with probability $1/2-1/2$ and $M(0)=0$. We define the random walks $U(t)$ and $L(t)$ on $\mathcal{L}^*$ with the same transition probabilities, which are independent of each other and of $M(t)$. The initial values are $U(0)=1$ and $L(0)=-1$. We extend our walks for non-integral values of t linearly, so the trajectories are continuous.

Since all these three random walks have steps with mean 0 and variance 1, it follows that

$$\left(\frac{M(nt)}{\sqrt{n}}, \frac{U(nt)}{\sqrt{n}}, \frac{L(nt)}{\sqrt{n}}\right) \overset{\mathrm{d}}{\Longrightarrow} (B(t), X(t), Y(t)) \qquad (n\to\infty). \tag{5}$$

We established earlier that the map $(b(\cdot), x(\cdot), y(\cdot)) \mapsto (b(\cdot), x_{b\uparrow}(\cdot), y_{b\downarrow}(\cdot))$ is continuous in supremum distance. From Donsker's invariance principle (see e.g. Section 7.6 of [2]), we conclude that

$$\left(\frac{M(nt)}{\sqrt{n}}, \frac{U_{M(n\cdot)\uparrow}(nt)}{\sqrt{n}}, \frac{L_{M(n\cdot)\downarrow}(nt)}{\sqrt{n}}\right) \overset{\mathrm{d}}{\Longrightarrow} (B(t), X^+(t), Y^-(t)) \tag{6}$$

in distribution as $n\to\infty$. Note that we can use the discrete Skorohod-reflection to transform U and L, because the difference is only the addition of 1, which vanishes in the limit. At this point, it suffices to show that

$$\frac{U_{M(n\cdot)\uparrow}(nt) - L_{M(n\cdot)\uparrow}(nt)}{\sqrt{n}}$$

converges to a BES^3 process.

For $x, y \in \mathbb{Z}^+$ we define the stochastic matrix

$$\mathbf{P}_{xy} = \frac{y}{x} \cdot \begin{cases} \frac{1}{2} & \text{if } y = x, \\ \frac{1}{4} & \text{if } |y-x| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that if X_n is a homogeneous Markov-chain with transition probabilities $(\mathbf{P}_{xy})_{x,y\in\mathbb{Z}^+}$, then its diffusive limit is BES^3, i.e. for every $T > 0$ the process $(n^{-1/2}X_{nt})_{0\le t\le T}$ converges to a 3d Bessel-process in the Skorohod-topology as $n \to \infty$. So the proof of our theorem relies on the following.

Lemma 1. *$U_{M\uparrow}(t) - L_{M\downarrow}(t)$ is a Markov-chain and its transition matrix is $(\mathbf{P}_{xy})_{x,y\in\mathbb{Z}^+}$, where $U_{M\uparrow}$ and $L_{M\downarrow}$ are discrete Skorohod-reflections.*

2.3 Markov-property of the distance of the two reflected walks

We introduce a different notation for the triple $(M, U_{M\uparrow}, L_{M\downarrow})$, which is just a linear transformation. Let $K_n := L_{M\downarrow}(n)$ be the position of the lower reflected walk. With the definition $D_n := \frac{1}{2}(U_{M\downarrow}(n) - L_{M\uparrow}(n))$, the distance of the two reflected walks is $2D_n$. $P_n := \frac{1}{2}(M(n) - L_{M\downarrow}(n) - 1)$, which means that the position of M related to the lower walk is $2P_n + 1$. The vector (K_n, D_n, P_n) is clearly a Markov-chain.

We are only interested in the coordinate D_n, which turns out to be also Markovian and to have transition matrix $(\mathbf{P}_{xy})_{x,y\in\mathbb{Z}^+}$. To show this, we have to determine the conditional distribution of P_n, because, in certain cases, it modifies the transition rules of D_n.

Lemma 2. *The following identities hold:*

$$\mathbb{P}\left(P_n = x \mid D_0^n\right) = \frac{1}{D_n}\mathbb{1}(x \in \{0, 1, \ldots, D_n - 1\}), \tag{7}$$

$$\mathbb{P}\left(D_{n+1} = y \mid D_0^n\right) = \mathbf{P}_{D_n y}, \tag{8}$$

where D_0^n means the sequence of variables $D_0, \ldots, D_n$.

Proof. The two identities (7), respectively, (8) of the lemma are proved by a common induction on n. Since $D_0 = 1$ and $P_0 = 0$, the case $n = 0$ is trivial.

For the induction step, we have to enumerate the possible transitions of the Markov-chain (K_n, D_n, P_n). For the sake of simplicity, we only prove for $D_n = D_{n-1} - 1$, the other cases are similar. It is easy to check that the transition $(k, d, p) \to (k + 1, d - 1, p)$ has probability $\frac{1}{8}\mathbb{1}(p \in \{0, 1, \ldots, d - 2\})$; this will be called type A events. Type B events are the transitions $(k, d, p) \to (k + 1, d - 1, p - 1)$, which happen with probability $\frac{1}{8}\mathbb{1}(p \in \{1, 2, \ldots, d - 1\})$. No other cases give $d \to d - 1$.

Proof of (7): Let $x, y \in \mathbb{Z}^+$. We suppose that $y = D_{n-1} - 1$.

$$\begin{aligned}
&\mathbb{P}\left(P_n = x \mid D_n = y, D_0^n\right) = \qquad (9)\\
&\quad = \sum_{z\in\mathbb{Z}} \mathbb{P}\left(P_n = x \mid P_{n-1} = z, D_n = y, D_0^{n-1}\right) \mathbb{P}\left(P_{n-1} = z \mid D_n = y, D_0^{n-1}\right)\\
&\quad = \sum_{z\in\mathbb{Z}} \frac{\mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = z, D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid P_{n-1} = z, D_0^{n-1}\right)} \mathbb{P}\left(P_{n-1} = z \mid D_n = y, D_0^{n-1}\right)\\
&\quad = \sum_{z=x}^{x+1} \mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = z, D_0^{n-1}\right) \frac{\mathbb{P}\left(P_{n-1} = z \mid D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid D_0^{n-1}\right)}\\
&\quad = \mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = x, D_0^{n-1}\right) \frac{\mathbb{P}\left(P_{n-1} = x \mid D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid D_0^{n-1}\right)}\\
&\qquad + \mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = x+1, D_0^{n-1}\right) \frac{\mathbb{P}\left(P_{n-1} = x+1 \mid D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid D_0^{n-1}\right)}\\
&\quad = \frac{1}{8}\mathbb{1}(x \in \{0, 1, \ldots, D_{n-1} - 2\}) \frac{\frac{1}{D_{n-1}}\mathbb{1}(x \in \{0, 1, \ldots, D_{n-1} - 1\})}{\frac{1}{4}\frac{D_{n-1}-1}{D_{n-1}}}\\
&\qquad + \frac{1}{8}\mathbb{1}(x \in \{0, 1, \ldots, D_{n-1} - 2\}) \frac{\frac{1}{D_{n-1}}\mathbb{1}(x \in \{-1, 0, \ldots, D_{n-1} - 2\})}{\frac{1}{4}\frac{D_{n-1}-1}{D_{n-1}}}\\
&\quad = \frac{1}{D_{n-1} - 1}\mathbb{1}(x \in \{0, \ldots, D_{n-1} - 2\}) = \frac{1}{y}\mathbb{1}(x \in \{0, 1, \ldots, y - 1\}).
\end{aligned}$$

First, we used the law of total probability and the definition of conditional probability and the identity $\mathbb{P}(E|F)/\mathbb{P}(F|E) = \mathbb{P}(E)/\mathbb{P}(F)$ on a conditional probability space. As remarked at the beginning of this proof, there are only two cases to reduce the value of D, so the sum has only two terms. Then, we used both inductional hypotheses to evaluate the conditional probabilities. The remaining steps are obvious.

Proof of (8): We spell out the proof for $D_{n+1} = D_n - 1$, the cases $D_{n+1} = D_n$ and $D_{n+1} = D_n + 1$ are similar.

$$\begin{aligned}
&\mathbb{P}\left(D_{n+1} = D_n - 1 \mid D_0^n\right) = \qquad (10)\\
&\quad = \sum_{x=0}^{D_n-1} \mathbb{P}\left(D_{n+1} = D_n - 1 \mid P_n = x, D_0^n\right) \mathbb{P}\left(P_n = x \mid D_0^n\right)\\
&\quad = \sum_{x=0}^{D_n-1} \left(\frac{1}{8}\mathbb{1}(x \in \{0, 1, \ldots, D_n - 2\}) + \frac{1}{8}\mathbb{1}(x \in \{1, 2, \ldots, D_n - 1\})\right) \frac{1}{D_n}\\
&\quad = \frac{1}{4}\frac{D_n - 1}{D_n} = \mathbf{P}_{D_n(D_n-1)}.
\end{aligned}$$

In the second step, only type A and B events can cause the transition $D_{n+1} = D_n - 1$. We applied part (1) of this lemma to evaluate the second conditional probability factor.

□

As a consequence, we see that the distribution of D_{n+1} conditioned on D_0^n depends only on D_n, which means that D_n is a Markov-chain with transition matrix $(\mathbf{P}_{xy})_{x,y\in\mathbb{Z}^+}$. From this, the assertion of the theorem follows.

Acknowledgement: The result was originally proved without knowledge of Jon Warren's work. We thank Wendelin Werner for drawing our attention to the existence of the paper [8]. The authors' research is partially supported by the OTKA (Hungarian National Research Fund) grants K 60708 (for B.T. and B.V.) and TS 49835 (for B.V.).

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