Probability 1 – Exercises

Tutorial no. 13 　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　7th Dec 2023

13.1 We throw a fair die 10 times. Let X denote the number of times an even number follows an odd number. What is the expected value and variance of X?

13.2 Type i light bulbs function for a random amount of time having mean μ_i and standard deviation σ_i for $i = 1, 2$. A light bulb randomly chosen from a bin of bulbs is a type 1 bulb with probability p and a type 2 bulb with probability $1 - p$. Let X denote the lifetime of this bulb. Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

13.3 Recall the random variable Y from Exercise 12.5 about iterated stick-breaking. Determine $\mathbb{E}(Y)$ and $\mathbb{D}^2(Y)$ using the tower rules.

13.4 Let A and B be two points uniformly selected on the $[0, 1]$ interval, and let X be their distance. What is the CDF and PDF of X?

13.5 We are rolling a die repeatedly until the sum of the numbers rolled exceeds 300. Estimate the probability that we need more than 80 rolls.

13.6 We have 50 real numbers. We round each of them to the closest integer, then sum these integers. Assume that the rounding errors are independent $\mathrm{Uni}[-1/2; 1/2]$ random variables. Estimate the probability that the sum of the rounded numbers differs from the real sum by more than 3.

13.7 Flip a fair coin 60 times, and let $X \sim \mathrm{Bin}(60, 1/2)$ be the number of heads. Using Markov's inequality for e^{tX} with the best possible t, which can be found by minimizing the convex function $f(t) = \log(1 + e^t) - \frac{5}{6}t$, show that

$$\mathbb{P}(|X - 30| \geq 20) \leq 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}.$$

13.8 Above the village where my Grandpa lives, two types of angels fly by occasionally: Exterminating Angels and Blessing Angels, according to two independent Poisson processes of intensity 1 per 120 years. Grandpa maintains a very healthy life: he is now 147 years old, and will die only when the next Exterminating Angel appears.

(a) What is the probability that he will live his 200th birthday?

(b) In expectation, how many Blessing Angels will he see before he dies?

(c) The exemplary life of my Grandpa drew the attention of the Devil, who offers the following deal for his soul: if he dies at age X (counting continuously), and during his last year of life K Blessing Angels flew by, the Devil donates X^K ducats to a charitable cause chosen by Grandpa. If Granda accepts this deal, what is the expected amount of the charity he would get when he dies?