Probability 1 – Exercises

Tutorial no. 12 — 30th Nov 2023

12.1 Let X and Y be two absolutely continuous random variables. Let X have a marginal PDF of $f_X(x) = \lambda^2 x \exp(-\lambda x)$ if $x \geq 0$, and 0 otherwise. Conditional on $X = x$, let Y be uniform on the interval $[0, x]$. What is:

(a) the joint PDF of (X, Y)?

(b) the marginal PDF of Y, and $\mathbb{E}(Y)$?

(c) $\mathbb{E}(X \mid Y = y)$ and $\mathbb{E}(Y \mid X = x)$?

(d) $\mathbb{E}(X)$? (Hint: use previous results)

(e) $\mathrm{Cov}(X, Y)$?

12.2 Let $X_1, \ldots, X_n$ be independent and identically distributed random variables (iid).

What is $\mathbb{E}(X_1 \mid X_1 + \cdots + X_n)$?

HW **12.3** (2 points) Let X and Y be independent and identical, nonnegative random variables.

$$\mathbb{E}\left(\frac{X}{X+Y}\right) = ?$$

12.4 Let Y be a normal variable with mean μ and variance 1, and $X|Y$ be a normal variable with mean Y and variance 1. What is the conditional distribution of $Y|X$?

HW **12.5** (3 points) We take a PQ line segment with length 3 and break it into two parts, at a uniformly chosen point R. Let X denote the length of the PR line segment. The other, $3 - X$ long line segment is broken again, at a uniformly chosen point S. Let Y denote the length of the RS line segment.

(a) What is the joint PDF of (X, Y)?

(b) $\mathbb{E}(Y \mid X = x)$

(c) Calculate $\mathrm{Cov}(X, Y)$

♣ **12.6** (3 points) Let X, Y and Z be independent random variables. Let the CDF of X and Y be $F(x)$ and $G(y)$, respectively. Let Z be a Bernoulli random variable with parameter p. Determine the CDF of the following distributions:

(a) $T := ZX + (1 - Z)Y$,

(b) $U := ZX + (1 - Z)\max\{X, Y\}$,

(b) $U := ZX + (1 - Z)\min\{X, Y\}$.

12.7 We roll a die. If the result is i, then we choose Y uniformly on $[0, i]$. What is $\mathbb{E}(Y)$ and $\mathbb{D}^2(Y)$?

12.8 The amount of time until a lightbulb breaks follows an exponential distribution with 9 months expected time. If the lightbulb breaks, I immediately replace it with the same kind of lightbulb. I use my lamp for T years, where T is a number uniformly chosen between 0 and 9 years. Let Y be the number of times I have to change the lightbulb during this time. $\mathbb{E}(Y) =?$ and $\mathbb{D}^2(Y) =?$

12.9 Let U_1 and U_2 be two independent random variables with uniform distribution on $[0, 1]$. Show that if

$$X = \sqrt{-2 \ln U_1} \cdot \cos(2\pi U_2)$$
$$Y = \sqrt{-2 \ln U_1} \cdot \sin(2\pi U_2)$$

Then (X, Y) have a joint Normal distribution.

HW 12.10 (3 points) Let X and Y be the two coordinates of a point that we choose randomly on the disc with radius 1 around the origin. ($f(x,y) = \frac{1}{\pi}$ if $x^2 + y^2 \leq 1$.) What is the joint density function of $R = \sqrt{X^2 + Y^2}$ and $\Theta = \text{arctg}(Y/X)$?

12.11 Let $X, Y \sim \text{Exp}(\lambda)$ be independent. What is the joint density of $U := X + Y$ and $V := \frac{X}{X+Y}$? Show that U and V are independent.

12.12 Let $X_1, \dots, X_n, X_{n+1}$ be independent, standard normal random variables, and let $Y = \frac{X_1 + \cdots + X_n}{n}$.

(a) What is the covariance matrix of (X_1, Y)?

(b) $\mathbb{P}(|Y| \leq |X_{n+1}|)$

HW 12.13 (2 points) Two different research groups are trying to estimate the weight of an *atom*. The first group takes k samples, while the second group takes l samples. The true weight of the atom is μ, and the samples have a Normal distribution with a variance of σ^2, independently. What is the probability that the first group has a closer estimate than the second group?

12.14 Let X be a standard normal distribution, and I independent from X, with $\mathbb{P}(I = 1) = \mathbb{P}(I = 0) = \frac{1}{2}$. Define Y as

$$Y := \begin{cases} X, & \text{if } I = 1 \\ -X, & \text{if } I = 0. \end{cases}$$

(a) Show that Y is standard normal.

(b) Are I and Y independent?

(c) Are X and Y independent?

(d) Show that $\text{Cov}(X, Y) = 0$.