Probability 1 – Exercises

Tutorial no. 9 — 9th Nov 2023

9.1 Assume that Z is such a random variable that $2Z$ has the same distribution. What is the distribution of Z?

9.2 Let X be a continuous random variable, with CDF F. What is the distribution of $Y = F(X)$?

9.3 Calculate the PDF of the following random variables:

(a) If ξ is exponential with parameter λ, then what is the PDF of $X := 4\xi - 5$, and $Z := \sqrt[8]{\xi}$?

(b) If ξ is uniform on the $[-1;1]$ interval, then what is the PDF of $X := \xi^2$ and $U := cos(\pi\xi)$?

HW (c) (1 point) If ξ is exponential with parameter λ, then what is the PDF of $Y := e^{\xi}$?

HW (d) (2 points) If ξ is uniform on $[-1;2]$, then what is the PDF of $Z := \xi^2$?

9.4 Let X be a standard Cauchy random variable. (Recall that its PDF is $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in (-\infty, \infty)$.) Show that X and $1/X$ have the same distribution.

♣ **9.5** (3 point) Let X be a random variable with PDF $\frac{1}{\ln 2}\frac{1}{1+x}$ on the $[0,1]$ interval, and 0 otherwise. What is the PDF of the fractional part of $\frac{1}{X}$?

9.6 We roll two independent fair dice. What is the joint probability distribution function of X and Y if

(a) X is the maximum throw, Y is their sum;

HW (b) (2 points) X is the first throw, Y is the minimum?

9.7 We break a stick of length L into 3 parts, at 2 independent uniformly random points.

(a) What is the probability that the three parts can form a triangle?

(b) What is the expected value of the lenth of the shortest part?

9.8 From the Leaning Tower of Pisa, I drop a ball from height 40.5 meters. As we know from physics, in t seconds it falls $gt^2/2$ meters, where $g = 9m/s^2$, hence it reaches the ground in exactly 3 seconds. Each of my two friends, Galileo and Isaac, takes a photo of the experiment, at independent random times with distribution Uni$[0;3]$ sec.

(a) In expectation, how high is the ball on Isaac's photo?

(b) In expectation, how high is the ball on the photo that is taken later?

HW **9.9** (3 points) Jack arrives to school with a delay of $X \sim$ Uni$[0, 20]$ minutes. Jill arrives with a delay of $Y \sim$ Uni$[0, 15]$ minutes, independently of Jack.

(a) What is the probability that Jill arrives earlier than Jack?

(b) What is the expectation of the difference $|X - Y|$?

9.10 Let X and Y be random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{4}{5}(x + xy + y) & \text{if } 0 < x, y < 1 \\ 0 & \text{else.} \end{cases}$$

Calculate the marginals. Are the two variables independent?

9.11 Calculate the marginal distributions! Are X and Y independent?

HW (a) (2 points) Let the joint probability density function of X and Y be

$$f(x,y) = \begin{cases} A \cdot (x^2y + y^2x) & \text{if } 0 < x, y < 1 \\ 0 & \text{else} \end{cases}$$

where A is a positive constant that must be calculated first.

(b) Same question when (X, Y) is uniform on the unit circle.

(c) (X, Y) is a uniform distribution on $D = \{(x, y) \in \mathbb{R}^2 : \ 2x^2 + y^2/2 \leq 1\}$.

9.12 A rectangle has sides with length 1 and a. We choose one point on each side of length 1, with independent uniform distributions. Let X be the distance of these two points. What is the PDF of X?