Probability 1 – Exercises

Tutorial no. 7 — 26th Oct 2023

7.1 Which constants A and B are needed for the following to be a Cumulative Distribution Function (CDF):

(a)

$$F(x) = A + B \cdot \arctan(x)$$

HW (b) (1 point)

$$F(x) = \exp(-Ae^{-Bx})$$

HW (c) (1 point)

$$F(x) = \begin{cases} 0, & -\infty < x \leq 0 \\ \frac{A+x}{1+Bx}, & 0 < x < \infty \end{cases}$$

7.2 Which constants C are needed for the following to be a Probability Density Function (PDF)? For these cases, calculate the expected value, and the probability that the corresponding random variable is greater than 2.

(a)

$$f(x) = \begin{cases} C \cdot \cos(\pi x), & \frac{-1}{2} \leq x \leq \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

HW (b) (1 point)

$$f(x) = \begin{cases} C \cdot x^{-5}, & 1 \leq x \\ 0, & \text{else} \end{cases}$$

HW (c) (1 point)

$$f(x) = \begin{cases} C \cdot (-\ln(\frac{x}{4})), & 0 < x \leq 4 \\ 0, & \text{else} \end{cases}$$

7.3 A gas station gets their shipment of oil once a week. Their weekly sales (measured in 1000 liters) is a random variable with PDF

$$f(x) = \begin{cases} 5 \cdot (1-x)^4, & 0 < x < 1 \\ 0, & \text{else.} \end{cases}$$

Then how big of a shipment do they need weekly if they want a less than 0.01 probability to run out of oil in a given week?

7.4 If X is uniform on $[0, 1]$, then what is the probability that X, $1 - X$ and $\frac{1}{2}$ form a triangle?

7.5 The buses arrive every hour at :00, :15, :30 and :45. If we arrive at random between 7:00 and 7:30, then what is the probability that

(a) I have to wait less than 4 minutes?

(b) I have to wait more than 7 minutes?

(c) What are the answers to a) and b), if I arrive between 7:08 and 7:38 instead?

(d) What are the answers to a) and b), if I arrive between 7:00 and 7:25 instead?

HW **7.6** (3 points) Trains go towards city A every 15 minutes, starting from 7:00am, while they go towards city B every 15 minutes, starting from 7:05am.

(a) If a passenger arrives at the station uniformly between 7:00am and 8:00am, and board whichever train leaves earlier, then what is the probability that they end up going to B? (Or A?)

(b) What if the passenger arrives between 7:10am and 8:10am?

(c) What if the train that goes towards city B leaves every 10 minutes starting from 7:05am?

7.7 A bus goes between city A and city B, which are 100km from each other. The bus breaks down with equal probability (uniform) on the route. If it breaks down, then they call the closest repair center. Currently, the repair centers are in the 2 cities, and one halfway in between them. It has been suggested, that it would be better to place the repair centers at 25km, 50km and 75km. Do we agree? Why? What would be a better placement? How do we define "better"?

7.8 We choose two points at random on the unit circle. What is the CDF of their distance?

HW **7.9** (3 points) Let's say we buy a pretzel stick with unit length, which has a piece of salt at s, $(0 < s < 1)$. The pretzel breaks into 2 pieces at a uniform random point in our backpack while we go home from the store. What is the expected value of the length of the piece which has the salt on it?

7.10 Choose a point uniformly inside of a regular hexagon with sides of unit length. Let ξ be the distance of the point from the closest side. What is the CDF and PDF of ξ?

♣ **7.11** (3 points) If I arrive at a meeting $s \in \mathbb{R}$ minutes earlier, I have to pay $a \cdot s$ coins. If I'm late s minutes, I have to pay $b \cdot s$ coins. The amount of time it takes me to arrive is random due to the chaotic public transport, but it has a PDF of $f(x)$. How many minutes before the meeting should I leave, if I want to minimise the expected cost? (Hint: write the integral form and differentiate it).