Markov processes and martingales
2nd midterm test

31st May 2023

1. Let $X_1, X_2, \dots$ be iid. random variables with a continuous distribution function. Let

$$E_n = \{X_n > \max(X_1, \dots, X_{n-1})\}$$

be the event that a record occurs at time n with the convention that $E_1 = \Omega$. Let $Y_n = \mathbb{1}_{E_n}$ and $N_n = Y_1 + \dots + Y_n$ be the number of records until time n.

   (a) (3 points) Prove that $E_1, E_2, \dots$ are independent and $\mathbf{P}(E_n) = 1/n$.
   Hint: For the independence show that for $i_1 < \dots < i_n$ one has $\mathbf{P}(E_{i_1}|E_{i_2}\dots E_{i_n}) = 1/i_1$ and write $\mathbf{P}(E_{i_1}\dots E_{i_n}) = \prod_{j=1}^n \mathbf{P}(E_{i_j}|E_{i_{j+1}}\dots E_{i_n})$.

   (b) (3 points) Prove that $\sum\limits_{n=2}^{\infty} \frac{Y_n - 1/n}{\log n}$ converges almost surely.

   (c) (3 points) Using Kronecker's lemma conclude that $\lim\limits_{n\to\infty} \frac{N_n}{\log n} = 1$ almost surely.
   Hint: Kronecker's lemma states that if b_n is a positive increasing function with $\lim_{n\to\infty} b_n = \infty$ and x_n is any real sequence then the convergence of $\sum_{n=1}^\infty x_n/b_n$ implies that $(x_1 + \dots + x_n)/b_n \to 0$ as $n \to \infty$.

2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space where $\Omega = [0,1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbf{P}$ is the Lebesgue measure. The random variable $U(\omega) = \omega$ has uniform distribution on $[0,1]$. Define the sequence of random variables

$$X_n = \frac{1}{U^{1-1/n}}$$

for all $n = 1, 2, \dots$.

   (a) (3 points) Compute the expectation of X_n. Is the sequence X_n uniformly integrable?

   (b) (3 points) Show that X_n converges almost surely as $n \to \infty$, that is,

$$\mathbf{P}(\{\omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\}) = 1.$$

   What is the almost sure limit of X_n?

   (c) (2 points) Does X_n also converge in L^1?