Markov processes and martingales
1st midterm test

5th Apr 2023

1. Let L, H and W denote the length, height and width of a certain type of ant in millimeters. Assume that the vector (L, H, W) has a multivariate normal distribution with mean $(8, 3, 4)$ and covariance matrix

$$\Sigma = \begin{pmatrix} 3 & 1/2 & 1 \\ 1/2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

   (a) (1 point) What is the correlation of the length and the width?

   (b) (2 points) What is the distribution of the sum of the three dimensions?

   (c) (2 points) Conditionally given that the width of an ant is 6 mm, what is the joint distribution of its length and its height? Recall from the lecture that the parameters of the conditional normal distribution are $\mu_{X|Y} = \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(Y - \mu_Y)$ and $\Sigma_{X|Y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$.

   (d) (2 points) Write down the joint density of the conditional distribution above and show that it factorizes hence the two marginals are (conditionally) independent.

2. Consider a casino where a player's winning per unit stake on game n is ξ_n where $\xi_1, \xi_2, \dots$ are i.i.d. random variables with $\mathbf{P}(\xi_n = +2) = p$ and $\mathbf{P}(\xi_n = -1) = 1 - p$ where $p \geq 1/3$. Let C_n be the player's stake on game n which is previsible, that is $C_{n+1} \in \mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ for all n. We assume that there is an $\varepsilon > 0$ such that $0 \leq C_n \leq (1-\varepsilon)Y_{n-1}$. Let Y_n denote the wealth of the player after the nth round.

   (a) (3 points) Using the notation $x_n = C_n/Y_{n-1}$ for the fraction of the player's wealth betted in the nth round (which is $\mathcal{F}_{n-1}$ measurable) compute the conditional expectation $\mathbf{E}(\log(1 + x_n\xi_n)|\mathcal{F}_{n-1})$ as a function of p and x_n and denote it by $f(p, x_n)$.

   (b) (2 points) For any fixed $p \geq 1/3$, find the maximal value $\alpha(p) = \max_{x\in(0,1)} f(p, x)$ and determine $x^{(p)}$ for which $x \mapsto f(p, x)$ is maximal, that is $\alpha(p) = f(p, x^{(p)})$.

   (c) (3 points) Prove that for any choice of the sequence x_n, the process $Z_n = \log Y_n - n\alpha(p)$ is a supermartingale and it is a martingale for $x_n = x^{(p)}$.

   Hint: Observe that $Y_{n+1} = Y_n \cdot (1 + x_{n+1}\xi_{n+1})$ and use the previous calculations when computing $\mathbf{E}(\log(Y_{n+1})|\mathcal{F}_n)$.