Markov processes and martingales exam

22nd June 2023

Theoretical part

1. (a) (2 points) State Doob's forward convergence theorem.
   (b) (7 points) Define the upcrossings of a stochastic process. State without proof Doob's upcrossing lemma. Prove the almost sure limit in the convergence theorem using the upcrossing lemma.
2. (a) (2 points) State Doob's submartingale inequality without proof.
   (b) (2 points) State the law of iterated logarithm.
   (c) (5 points) Prove the following inequality using the submartingale inequality which is the first step in the proof of the law of iterated logarithm. For all $c > 0$,

   $$\mathbf{P}\left(\sup_{k \le n} S_k \ge c\right) \le e^{-\frac{c^2}{2n}}$$

   where S_k is the sum of k i.i.d. standard normal random variables.
3. (a) (3 points) Define what is a dynamical system, the invariant σ-algebra and ergodicity.
   (b) (2 points) Describe the rotation of the circle.
   (c) (4 points) State and prove the condition for the ergodicity in the rotation of the circle.

Exercise part

4. (3+3+3 points) Let X_j, $j = 1, 2, \dots$ be i.i.d. random variables with common distribution $\mathbf{P}(X_j = -1) = \mathbf{P}(X_j = +1) = 1/2$ and let $S_n = X_1 + \cdots + X_n$. Compute the conditional expectations $\mathbf{E}(X_1|S_n)$, $\mathbf{E}(S_n|X_1)$ and $\mathbf{E}(S_{n+m}^2|S_n)$.
5. There are n white and n black balls in an urn. We pull out all of them one-by-one without replacement. Whenever we pull
   - a black ball we have to pay 1$,
   - a white ball we receive 1$.

   Let $X_0 := 0$ and X_i be the amount we gained or lost after the ith ball was pulled. We define

   $$Y_i := \frac{X_i}{2n - i} \text{ for } 1 \le i \le 2n - 1, \quad \text{and} \quad Z_i := \frac{X_i^2 - (2n - i)}{(2n - i)(2n - i - 1)} \text{ for } 1 \le i \le 2n - 2.$$

   (a) (3+3 points) Prove that $Y = (Y_i)$ and $Z = (Z_i)$ are martingales.
   (b) (3 points) Find $\text{Var}(X_i)$.
6. (5+4 points) Let $\mathcal{C}$ be a class of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Prove that $\mathcal{C}$ is UI if and only if the following two conditions hold

   (a) $\mathcal{C}$ is L^1-bounded, that is, $\sup\{\mathbf{E}(|X|) : X \in \mathcal{C}\} < \infty$ and
   (b) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

   $$F \in \mathcal{F} \text{ and } \mathbf{P}(F) < \delta \Longrightarrow \mathbf{E}(|X|\mathbb{1}_F) < \varepsilon \text{ for all } X \in \mathcal{C}.$$

   Hint: for the 'if' part use Markov's inequality to see that for all $X \in \mathcal{C}$ and for all $K > 0$ we have

   $$\mathbf{P}(|X| > K) \le \frac{\sup\{\mathbf{E}(|X|) : X \in \mathcal{C}\}}{K}.$$