Markov processes and martingales exam

13th June 2023

Theoretical part

1. (a) (2 points) Define the multivariate normal distribution.
   (b) (2 points) What is the distribution of the image of a multivariate normal random vector under a non-singular affine transformation? The answer does not have to be proved but the parameters of the resulting distribution have to be given.
   Hint: Use the fact that the covariance matrix of the random variable vector $\underline{X}$ can be given in the form $\mathbf{E}((\underline{X}-\mathbf{E}(\underline{X}))(\underline{X}-\mathbf{E}(\underline{X}))^T)$.
   (c) (3 points) Write down the joint density of the bivariate normal distribution (multivariate normal distribution with two components). Show that the joint density factorizes if the correlation of the two components is 0.
   (d) (2 points) State without proof the general theorem about the independence and uncorrelatedness of marginals for multivariate normal vectors.
2. (a) (2 points) State Lévy's downward theorem without proof.
   (b) (7 points) State and prove the strong law of large numbers using Lévy's downward theorem.
3. (a) (2 points) State the central limit theorem for martingales without proof.
   (b) (7 points) State and prove the central limit theorem for Markov chains using the central limit theorem for martingales.

Exercise part

4. (9 points) Let X, Y be two independent $\text{EXP}(\lambda)$ random variables and let $Z = X + Y$. Show that for any non-negative measurable h we have $\mathbf{E}(h(X)|Z) = \frac{1}{Z}\int_0^Z h(t)\,\mathrm{d}t$.
5. Let $\xi_1, \xi_2, \dots$ be independent standard normal variables. Recall that their moment generating function is $M(\theta) = \mathbf{E}\left[e^{\theta\xi_i}\right] = e^{\theta^2/2}$. Let $a, b \in \mathbb{R}$ and define $S_n = \sum_{k=1}^{n} \xi_k$ and $X_n = e^{aS_n - bn}$. Prove that
   (a) (3 points) $X_n \to 0$ a.s. iff $b > 0$;
   (b) (3 points) $X_n \to 0$ in L^r iff $r < \frac{2b}{a^2}$;
   (c) (3 points) X_n is uniformly integrable iff $2b > a^2$ or $a = b = 0$.
   Hint: If $2b > a^2$ then show by part (b) that the $\sup_n \mathbf{E}(X_n^r) < \infty$ for some $r > 1$. In the complementary case (except for $a = b = 0$) show that X_n cannot converge in L^1.
6. Let Z_n be the number of individuals in the nth generation in a branching process, that is, $Z_n = X_1^{(n)} + \dots + X_{Z_{n-1}}^{(n)}$ where $X_k^{(n)}$ are i.i.d. random variables with distribution satisfying
$$\mu = \mathbf{E}(X_k^{(n)}) < \infty \text{ and } 0 < \sigma^2 = \mathrm{Var}(X_k^{(n)}) < \infty.$$
   (a) (3 points) Prove that $M_n = Z_n/\mu^n$ is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_k^{(m)} : k = 1, 2, \dots; m = 1, \dots, n)$.
   (b) (3 points) Show that $\mathbf{E}(Z_{n+1}^2|\mathcal{F}_n) = \mu^2 Z_n^2 + \sigma^2 Z_n$ and conclude that the martingale M_n is bounded in L^2 if and only if $\mu > 1$.
   (c) (3 points) Assume that $\mu > 1$ which implies that $M_\infty = \lim_{n\to\infty} M_n$ exists in L^2 and a.s. Prove that
$$\mathrm{Var}(M_\infty) = \frac{\sigma^2}{\mu(\mu-1)}.$$